A Sum of Squares Approach to Modeling and Control of Nonlinear Dynamical Systems with Polynomial Fuzzy Systems

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***Abstract*— This paper presents a sum of squares (SOS) approach for modeling and control of nonlinear dynamical systems using polynomial fuzzy systems. The proposed SOS-based framework provides a number of innovations and improvements over the existing LMI-based approaches to Takagi-Sugeno fuzzy model and control. First, we propose a polynomial fuzzy modeling and control framework that is more general and effective than the well-known Takagi-Sugeno fuzzy model and control. Secondly, we obtain stability and stabilizability conditions of the polynomial fuzzy systems based on polynomial Lyapunov functions that contain quadratic Lyapunov functions as a special case. Hence, the stability and stabilizability conditions presented in this paper are more general and relaxed than those of the existing LMI-based approaches to Takagi-Sugeno fuzzy model and control. Moreover, the derived stability and stabilizability conditions are represented in terms of SOS and can be numerically (partially symbolically) solved via the recently developed SOSTOOLS. To illustrate the validity and applicability of the proposed approach, a number of analysis and design examples are provided. The first example shows that the SOS approach renders more relaxed stability results than those of both the LMI-based approaches and a polynomial system approach. The second example presents an extensive application of the SOS approach in comparison with a piecewise Lyapunov function approach. The last example is a design exercise that demonstrates the viability of the SOS-based approach to synthesize a stabilizing controller.**

***Index Terms*— polynomial fuzzy system, sum of squares, polynomial Lyapunov function, polynomial fuzzy controller, stability, stabilizability.**

I. Introduction

THE history of Takagi-Sugeno (T-S) fuzzy model based control goes back more than two decades. The idea began in 1985, when Takagi and Sugeno published their seminal work [1] introducing a new type of fuzzy model representation. In the beginning of 1990's, the issue of stability [2]-[6] for T-S fuzzy control systems has been investigated extensively within the framework of nonlinear system stability. Today, there exists a large body of literature (e.g., [7]-[20]) on stability analysis and design of T-S fuzzy control systems. In particular, there has been a flurry of research activities on linear matrix inequality (LMI) (e.g.,[21]-[31]) based approaches. These results range from elegant stable, optimal or robust control to more recent advanced nonlinear control paradigms. In LMI-based design approaches, a numerical solution is obtained by convex optimization methods such as interior point method. Though LMI-based approaches have enjoyed great success and popularity, there still exists a large number of design problems that either can not be represented in terms of LMIs or the results obtained through LMIs are too conservative. In this paper, we seek to provide a post-LMI framework for fuzzy modeling and control of nonlinear systems. In other words, we formulate and solve a class of polynomial design problems that can not be represented in terms of LMIs, i.e., that can not be solved by convex optimization methods.

Specifically, this paper presents a sum of squares (SOS) approach for modeling and control of nonlinear systems using polynomial fuzzy systems. The proposed SOS-based approach provides two innovative extensions over the existing LMI-based approaches to Takagi-Sugeno fuzzy model and control. First, we propose a polynomial fuzzy modeling and control framework that is a generalization of the Takagi-Sugeno fuzzy model and is more effective in representing nonlinear control systems. Secondly, the stability and stabilizability conditions of the fuzzy polynomial systems are derived based on polynomial Lyapunov functions that contain quadratic Lyapunov functions as a special case. Hence, the stability and stabilizability conditions obtained in this paper are more general and relaxed than those based on the existing LMI-based approaches to Takagi-Sugeno fuzzy model and control. Central to the problem of stability analysis and control design, the derived stability and stabilizability conditions are represented in terms of SOS and can be numerically (partially symbolically) solved via the recently developed SOSTOOLS [34]. These SOS conditions can not be generally solved via convex optimization methods. To the best of our knowledge, this paper represents the first attempt to apply SOS techniques to fuzzy control systems.

SOSTOOLS [34] is a free, third party MATLAB[1] toolbox for solving sum of squares problems. The techniques behind it are based on the sum of squares decomposition for multivariate polynomials, which can be efficiently computed using semidefinite programming. SOSTOOLS is developed as a consequence of the recent interest in sum of squares poly-

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nomials, partly due to the fact that these techniques provide convex relaxations for many hard problems such as global, constrained, and boolean optimization. For more details, please refer to the manual of SOSTOOLS [34].

The rest of the paper is organized as follows. Section II recalls the Takagi-Sugeno fuzzy model and associated stability results. Section III introduces a polynomial fuzzy model and a polynomial Lyapunov function to facilitate SOS-based techniques for analysis and design of fuzzy control systems. Sections IV and V present stability analysis via SOS and two modeling and analysis examples, respectively. The first example shows that our approach provides more relaxed stability results than those of both the existing LMI approaches and a polynomial system approach. The second example carries out an extensive application of the SOS approach in comparison with a piecewise Lyapunov function approach [35]. Section VI introduces new types of polynomial-based fuzzy controller and presents a stable control design based on polynomial Lyapunov functions. Section VII entails a design example to demonstrate the viability of our SOS design approach. It is followed by concluding remarks.

II. Takagi-Sugeno fuzzy model and quadratic Lyapunov function-based stability analysis

The fuzzy model-based control methodology [21] provides a natural, simple and effective design approach to complement other nonlinear control techniques (e.g., [32]) that require special and rather involved knowledge. Moreover, there is no loss of generality in adopting the T-S fuzzy model based control design framework as it has been established that any smooth nonlinear control systems can be approximated by the T-S fuzzy models (with liner model consequence) [33].

In this section, we recall the Takagi-Sugeno fuzzy model and its associated stability analysis based on quadratic Lyapunov functions [21]. The Takagi-Sugeno fuzzy model is described by fuzzy IF-THEN rules which represent local linear input-output relations of a nonlinear system. The main feature of this model is to express the local dynamics of each fuzzy implication (rule) by a linear system model. The overall fuzzy model of the system is achieved by fuzzy blending of the linear system models.

Consider the following nonlinear system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \tag{1}$$

where f is a nonlinear function. $\boldsymbol{x}(t) = [x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)]^T$ is the state vector and $\boldsymbol{u}(t) = [u_1(t) \ u_2(t) \ \cdots \ u_m(t)]^T$ is the input vector. Based on the sector nonlinearity concept [21], we can exactly represent (1) with the following Takagi-Sugeno fuzzy model (globally or at least semi-globally).

Model Rule i:

$$\begin{aligned} &If\ z_1(t)\ is\ M_{i1}\ and \cdots\ and\ \ z_p(t)\ is\ M_{ip} \\ &then\ \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) \quad i = 1, 2, \cdots, r, \end{aligned} \tag{2}$$

where $z_j(t)$ $(j = 1, 2, \cdots, p)$ is the premise variable. The membership function associated with the ith $Model\ Rule$ and jth premise variable component is denoted by M_{ij}. r denotes the number of $Model\ Rules$. Each $z_j(t)$ is a measurable time-varying quantity that may be states, measurable external variables and/or time. The defuzzification process of the model (2) can be represented as

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \frac{\sum_{i=1}^{r} w_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)\}}{\sum_{i=1}^{r} w_i(\boldsymbol{z}(t))} \\ &= \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)\}, \end{aligned} \tag{3}$$

where

$$\boldsymbol{z}(t) = [z_1(t) \cdots z_p(t)]$$

and

$$w_i(\boldsymbol{z}(t)) = \prod_{j=1}^{p} M_{ij}(z_j(t)).$$

It should be noted from the properties of membership functions that the following relations hold.

$$\sum_{i=1}^{r} w_i(\boldsymbol{z}(t)) > 0, \quad w_i(\boldsymbol{z}(t)) \geq 0 \quad i = 1, 2, \cdots, r$$

Hence,

$$h_i(\boldsymbol{z}(t)) = \frac{w_i(\boldsymbol{z}(t))}{\sum_{i=1}^{r} w_i(\boldsymbol{z}(t))} \geq 0, \quad \sum_{i=1}^{r} h_i(\boldsymbol{z}(t)) = 1.$$

By employing the quadratic Lyapunov function $\boldsymbol{x}^T(t)\boldsymbol{P}\boldsymbol{x}(t)$, stability conditions of the open-loop system (3) with $\boldsymbol{u}(t) = 0$ are obtained as

$$\boldsymbol{P} > \boldsymbol{0}, \tag{4}$$

$$-\boldsymbol{A}_i^T\boldsymbol{P} - \boldsymbol{P}\boldsymbol{A}_i > \boldsymbol{0}. \tag{5}$$

The conditions (4) and (5) are represented in terms of LMIs. Thus, the stability conditions can be efficiently solved numerically by interior point algorithms.

III. Polynomial Fuzzy Model and Polynomial Lyapunov function

Section II summarized Takagi-Sugeno fuzzy model and stability analysis based on quadratic Lyapunov functions. In this section, we will introduce a new type of fuzzy model with polynomial rule consequence, i.e., a fuzzy model whose consequent parts are represented by polynomials.

As shown in Section II, the stability conditions (4) and (5) for the T-S fuzzy system and the quadratic Lyapunov function reduce to LMIs. As a result, the stability conditions can be efficiently solved numerically by interior point algorithms such as the LMI toolbox of MATLAB. In this paper, we will show that the stability conditions for polynomial fuzzy systems based on polynomial Lyapunov functions can be reduced to SOS problems. Therefore, instead of the LMI toolbox, these problems can be solved via SOSTOOLS [34].

A. Polynomial fuzzy model

Using the sector nonlinearity concept, we introduce a so-called polynomial fuzzy model to exactly represent (1). The main difference between (2) and a polynomial fuzzy model lies in the consequent part representation. The fuzzy model of (2) features linear model consequence, whereas the proposed polynomial fuzzy model has polynomial model consequence as shown below.

Model Rule i**:**

$$If\ z_1(t)\ is\ M_{i1}\ and \cdots\ and\ \ z_p(t)\ is\ M_{ip}$$

$$then\ \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t) \qquad i = 1, 2, \cdots, r, \tag{6}$$

where $\boldsymbol{A}_i(\boldsymbol{x}(t))$ and $\boldsymbol{B}_i(\boldsymbol{x}(t))$ are polynomial matrices in $\boldsymbol{x}(t)$. $\hat{\boldsymbol{x}}(\boldsymbol{x}(t))$ is a column vector whose entries are all monomials in $\boldsymbol{x}(t)$. That is, $\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) \in \boldsymbol{R}^N$ is an $N \times 1$ vector of monomials in $\boldsymbol{x}(t)$. A monomial in $\boldsymbol{x}(t)$ is a function of the form $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where α_1, α_2, $\cdots$, α_n are nonnegative integers. Therefore, $\boldsymbol{A}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)$ is a polynomial vector. Thus, the polynomial fuzzy model (6) has a polynomial in each consequent part. We assume that

$$\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) = 0 \text{ iff } \boldsymbol{x}(t) = 0$$

throughout this paper.

The defuzzification process of the model (6) can be represented as

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \frac{\sum_{i=1}^{r} w_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)\}}{\sum_{i=1}^{r} w_i(\boldsymbol{z}(t))} \\ &= \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)\}. \end{aligned} \tag{7}$$

Thus, the overall fuzzy model is achieved by fuzzy blending of the polynomial system models.

If $\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) = \boldsymbol{x}(t)$ and $\boldsymbol{A}_i(\boldsymbol{x}(t))$ and $\boldsymbol{B}_i(\boldsymbol{x}(t))$ are constant matrices for all i, then $\boldsymbol{A}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)$ reduces to $\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)$, that is, then (7) reduces to (3). Therefore, (7) is a more general representation.

Remark 1: As shown in Section III-C and Section V, the number of rules in a polynomial fuzzy model is generally fewer than that in a T-S fuzzy model. Furthermore, the proposed SOS approach to polynomial fuzzy models provides significantly more relaxed stability results than the existing LMI approaches to T-S fuzzy models.

B. Polynomial Lyapunov function

To obtain more relaxed stability results, we employ a polynomial Lyapunov function represented by

$$\hat{\boldsymbol{x}}^T(\boldsymbol{x}(t))\boldsymbol{P}(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)), \tag{8}$$

where $\boldsymbol{P}(\boldsymbol{x}(t))$ is a polynomial matrix in $\boldsymbol{x}(t)$. If $\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) = \boldsymbol{x}(t)$ and $\boldsymbol{P}(\boldsymbol{x}(t))$ is a constant matrix, then (8) reduces to the quadratic Lyapunov function $\boldsymbol{x}^T(t)\boldsymbol{P}\boldsymbol{x}(t)$. Therefore, (8) is a more general representation.

C. Nonlinear Modeling Example

Consider the following nonlinear system

$$\begin{aligned} \dot{x}_1(t) &= x_2^{\ell}(t), \\ \dot{x}_2(t) &= -2x_1(t) - x_2(t) - g(t)x_1(t), \end{aligned} \tag{9}$$

where $g(t) \in [0, k]$ for all t. ℓ is a positive integer. Without loss of generality, assume that $\ell = 2$ in this example. For other positive integer values of ℓ, we can construct a fuzzy model in the same way as below.

First, we construct a T-S fuzzy model to represent the system. To begin with, we assume that $x_2(t) \in [-d, d]$ for all t where $d > 0$. Note that we can always choose a sufficiently large d to cover the nonlocal dynamics. Using the concept of sector nonlinearity, we arrive at the following T-S fuzzy model that can exactly represent the dynamics under $x_2(t) \in [-d, d]$ and $g(t) \in [0, k]$ for all t.

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{4} h_i(\boldsymbol{z}(t))\boldsymbol{A}_i\boldsymbol{x}(t), \tag{10}$$

where $\boldsymbol{x}(t) = [x_1(t)\ x_2(t)]^T$ and $\boldsymbol{z}(t) = [g(t)\ x_2(t)]^T$

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & -d \\ -2 & -1 \end{bmatrix},\ \boldsymbol{A}_2 = \begin{bmatrix} 0 & -d \\ -2-k & -1 \end{bmatrix},$$

$$\boldsymbol{A}_3 = \begin{bmatrix} 0 & d \\ -2 & -1 \end{bmatrix},\ \boldsymbol{A}_4 = \begin{bmatrix} 0 & d \\ -2-k & -1 \end{bmatrix}.$$

The membership functions are obtained as

$$h_1(\boldsymbol{z}(t)) = \frac{k-g(t)}{k} \cdot (-\frac{x_2(t)-d}{2d}),$$

$$h_2(\boldsymbol{z}(t)) = \frac{g(t)}{k} \cdot (-\frac{x_2(t)-d}{2d}),$$

$$h_3(\boldsymbol{z}(t)) = \frac{k-g(t)}{k} \cdot \frac{x_2(t)+d}{2d},$$

$$h_4(\boldsymbol{z}(t)) = \frac{g(t)}{k} \cdot \frac{x_2(t)+d}{2d}.$$

As a comparison, we proceed to derive the following polynomial fuzzy model that can also exactly represent the dynamics under $g(t) \in [0, k]$ for all t.

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{2} h_i(z(t))\boldsymbol{A}_i\boldsymbol{x}(t), \tag{11}$$

where $z(t) = g(t)$ and

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} 0 & x_2 \\ -2 & -1 \end{bmatrix},\ \boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} 0 & x_2 \\ -2-k & -1 \end{bmatrix}.$$

The membership functions are obtained as

$$h_1(z(t)) = \frac{k-g(t)}{k},\ h_2(z(t)) = \frac{g(t)}{k}.$$

Note that the assumption of $x_2(t) \in [-d, d]$ for all t is not needed in the construction of the polynomial fuzzy models. Note also the number of rules for the polynomial fuzzy model is less than that of the T-S fuzzy model.

IV. Stability Analysis via SOS

A. Sum of Squares

The computational method used in this paper relies on the sum of squares decomposition of multivariate polynomials. A multivariate polynomial $f(\boldsymbol{x}(t))$ (where $\boldsymbol{x}(t) \in R^n$) is a sum of squares (SOS) if there exist polynomials $f_1(\boldsymbol{x}(t)), \cdots, f_m(\boldsymbol{x}(t))$ such that $f(\boldsymbol{x}(t)) = \sum_{i=1}^m f_i^2(\boldsymbol{x}(t))$. It is clear that $f(\boldsymbol{x}(t))$ being an SOS naturally implies $f(\boldsymbol{x}(t)) > 0$ for all $\boldsymbol{x}(t) \in R^n$. This can be shown equivalent to the existence of a special quadric form stated in the following proposition [36].

Proposition 1: [37] Let $f(\boldsymbol{x}(t))$ be a polynomial in $\boldsymbol{x}(t) \in R^n$ of degree $2d$. In addition, let $\hat{\boldsymbol{x}}(\boldsymbol{x}(t))$ be a column vector whose entries are all monomials in $\boldsymbol{x}(t)$ with degree no greater than d. Then $f(\boldsymbol{x}(t))$ is a sum of squares iff there exists a positive semidefinite matrix $\boldsymbol{P}$ such that

$$f(\boldsymbol{x}(t)) = \hat{\boldsymbol{x}}^T(\boldsymbol{x}(t))\boldsymbol{P}\hat{\boldsymbol{x}}(\boldsymbol{x}(t)). \tag{12}$$

Expressing an SOS polynomial using a quadratic form as in (12) has also been referred to as the Gram matrix method.

Recall that a monomial in $\boldsymbol{x}(t)$ is a function of the form $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$, where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are nonnegative integers. In this case, the degree of the monomial is given by $\alpha_1 + \alpha_1 + \cdots + \alpha_n$.

A sum of squares decomposition for $f(\boldsymbol{x}(t))$ can be computed using semidefinite programming, since it amounts to searching for an element $\boldsymbol{P}$ in the intersection of the cone of positive semidefinite matrices and a set defined by some affine constraints that arise from (12). Note in particular that the polynomial $f(\boldsymbol{x}(t))$ is globally nonnegative if it can be decomposed as a sum of squares. Hence the sum of squares decomposition in conjunction with semidefinite programming provides a polynomial-time computational relaxation for proving global nonnegativity of multivariate polynomials [37], [38], which belongs to the class of NP-hard problems. Even though the sum of squares condition is not necessary for nonnegativity, numerical experiments seem to indicate that the gap between sum of squares and nonnegativity is small [36].

B. Stability Conditions

To lighten the notation, in this subsection we drop the notation with respect to time t. For instance, we will use $\boldsymbol{x}$, $\hat{\boldsymbol{x}}(\boldsymbol{x})$ instead of $\boldsymbol{x}(t)$, $\hat{\boldsymbol{x}}(\boldsymbol{x}(t))$, respectively. Though the reference to time t is not explicitly denoted, it should be kept in mind that $\boldsymbol{x}$ means $\boldsymbol{x}(t)$. Also note that $\boldsymbol{A}_i^k(\boldsymbol{x})$ denotes the k-th row of $\boldsymbol{A}_i(\boldsymbol{x})$.

Theorem 1: The zero equilibrium of the system (7) with $\boldsymbol{u} = 0$ is stable if there exists a symmetric polynomial matrix $\boldsymbol{P}(\boldsymbol{x}) \in \boldsymbol{R}^{N\times N}$ such that (13) and (14) are satisfied, where $\epsilon_1(\boldsymbol{x})$ and $\epsilon_{2i}(\boldsymbol{x})$ are non negative polynomials such that $\epsilon_1(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq 0$ and $\epsilon_{2i}(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x}$.

$$\hat{\boldsymbol{x}}^T(\boldsymbol{x})(\boldsymbol{P}(\boldsymbol{x}) - \epsilon_1(\boldsymbol{x})\boldsymbol{I})\hat{\boldsymbol{x}}(\boldsymbol{x}) \ is\ SOS \tag{13}$$

$$-\hat{\boldsymbol{x}}^T(\boldsymbol{x})\Big(\boldsymbol{P}(\boldsymbol{x})\boldsymbol{T}(\boldsymbol{x})\boldsymbol{A}_i(\boldsymbol{x}) + \boldsymbol{A}_i^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x})\boldsymbol{P}(\boldsymbol{x}) + \sum_{k=1}^n \frac{\partial \boldsymbol{P}}{\partial x_k}(\boldsymbol{x})\boldsymbol{A}_i^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) + \epsilon_{2i}(\boldsymbol{x})\boldsymbol{I}\Big)\hat{\boldsymbol{x}}(\boldsymbol{x}) \quad is\ SOS \quad \forall i, \tag{14}$$

where $\boldsymbol{T}(\boldsymbol{x}) \in \boldsymbol{R}^{N\times n}$ is a polynomial matrix whose (i, j)-th entry is given by

$$T^{ij}(\boldsymbol{x}) = \frac{\partial \hat{x}_i}{\partial x_j}(\boldsymbol{x}). \tag{15}$$

In addition, if (14) holds with $\epsilon_{2i}(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq 0$, then the zero equilibrium is asymptotically stable. If $\boldsymbol{P}(\boldsymbol{x})$ is a constant matrix, then the stability holds globally.

Proof: Consider a candidate of polynomial Lyapunov function.

$$V(\boldsymbol{x}) = \hat{\boldsymbol{x}}^T(\boldsymbol{x})\boldsymbol{P}(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}), \tag{16}$$

where $\boldsymbol{P}(\boldsymbol{x}) \in R^{N\times N}$ is a symmetric polynomial matrix. The condition (13) implies that $\boldsymbol{P}(\boldsymbol{x})$ is positive definite for all $\boldsymbol{x}$, and $V(\boldsymbol{x})$ is a positive definite function of $\boldsymbol{x}$.

The time derivative of $V(\boldsymbol{x})$ along the open-loop trajectory (7) with $\boldsymbol{u} = 0$ is given by

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &= \hat{\boldsymbol{x}}^T(\boldsymbol{x})\boldsymbol{P}(\boldsymbol{x})\dot{\hat{\boldsymbol{x}}}(\boldsymbol{x}) + \dot{\hat{\boldsymbol{x}}}^T(\boldsymbol{x})\boldsymbol{P}(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \\ &\quad + \hat{\boldsymbol{x}}^T(\boldsymbol{x})\dot{\boldsymbol{P}}(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \\ &= \hat{\boldsymbol{x}}^T(\boldsymbol{x})\boldsymbol{P}(\boldsymbol{x})\boldsymbol{T}(\boldsymbol{x})\dot{\boldsymbol{x}}(\boldsymbol{x}) + \dot{\boldsymbol{x}}^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x})\boldsymbol{P}(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \\ &\quad + \hat{\boldsymbol{x}}^T(\boldsymbol{x})\left(\sum_{k=1}^n \frac{\partial \boldsymbol{P}}{\partial x_k}(\boldsymbol{x})\dot{x}_k\right)\hat{\boldsymbol{x}}(\boldsymbol{x}).\end{aligned} \tag{17}$$

We have

$$\dot{x}_k = \sum_{i=1}^r h_i(\boldsymbol{z})\boldsymbol{A}_i^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}). \tag{18}$$

From (7), (17) and (18), $\dot{V}(\boldsymbol{x})$ becomes

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &= \sum_{i=1}^r h_i(\boldsymbol{z})\times \\ &\quad \hat{\boldsymbol{x}}^T(\boldsymbol{x})\Big\{\boldsymbol{P}(\boldsymbol{x})\boldsymbol{T}(\boldsymbol{x})\boldsymbol{A}_i(\boldsymbol{x}) + \boldsymbol{A}_i^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x})\boldsymbol{P}(\boldsymbol{x}) \\ &\quad + \sum_{k=1}^n \frac{\partial \boldsymbol{P}}{\partial x_k}(\boldsymbol{x})\boldsymbol{A}_i^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x})\Big\}\hat{\boldsymbol{x}}(\boldsymbol{x}).\end{aligned}$$

Therefore, if (14) holds, then $\dot{V}(\boldsymbol{x}) \leq 0$. Furthermore, if (14) holds with $\epsilon_{2i}(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq 0$, then $\dot{V}(\boldsymbol{x}) < 0$ at $\boldsymbol{x} \neq 0$. Then, the zero equilibrium is asymptotically stable. Finally, if $\boldsymbol{P}(\boldsymbol{x})$ is a constant matrix, then $V(\boldsymbol{x})$ is radially unbounded, and the stability holds globally. ■

Remark 2: When $\boldsymbol{A}_i(\boldsymbol{x})$, $\boldsymbol{B}_i(\boldsymbol{x})$ and $\boldsymbol{P}(\boldsymbol{x})$ are constant matrices and $\hat{\boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{x}$, the system representation (7) and the polynomial Lyapunov function (8) are the same as the Takagi-Sugeno fuzzy model and the quadratic Lyapunov function used in a large body of existing literature, e.g., [21]. Thus, the propsed SOS approach to polynomial fuzzy models contains the existing LMI approaches to Takagi-Sugeno fuzzy models as a special case. Therefore, the SOS-based polynomial fuzzy models provide significantly more relaxed stability results than the existing LMI approaches to Takagi-Sugeno fuzzy models. We will see the extent of the relaxation in Section V.

V. Modeling and Stability Analysis Examples

This section demonstrates the utility of our proposed approach through two examples.

A. Example 1

Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1(t) &= -(\frac{7}{2} + \frac{3}{2}\sin x_1(t))x_1(t) - 4x_2(t), \\ \dot{x}_2(t) &= (\frac{19}{2} - \frac{21}{2}\sin x_1(t))x_1(t) - 2x_2(t). \end{aligned} \tag{19}$$

By employing the concept of sector nonlinearity, the dynamics of (19) can be exactly converted into the following Takagi-Sugeno fuzzy model:

Model Rule 1: If $x_1(t)$ is $h_1(x_1(t))$ then $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_1\boldsymbol{x}(t)$,
Model Rule 2: If $x_1(t)$ is $h_2(x_1(t))$ then $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_2\boldsymbol{x}(t)$,

where

$$h_1(x_1(t)) = \frac{1+\sin x_1(t)}{2},\ h_2(x_1(t)) = \frac{1-\sin x_1(t)}{2},$$

$$\boldsymbol{A}_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \qquad \boldsymbol{A}_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}.$$

No quadratic Lyapunov functions for the above fuzzy model exists. However, all the trajectories converge to zero as shown in Figure 1 which depicts the response for the initial condition $\boldsymbol{x}(0) = [0.15\ 0.15]^T$. In contrast, our SOS approach can find polynomial Lyapunov functions of orders six, eight and ten satisfying (13) and (14) under $\epsilon_{2i}(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq 0$. It is easy to see that the polynomial Lyapunov functions constructed below are radially unbounded. Therefore, the system is globally asymptotically stable.

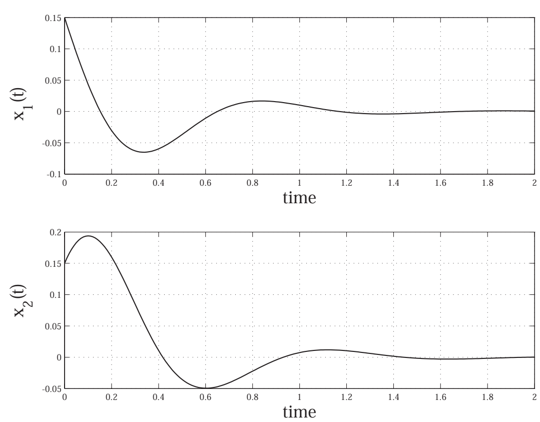

Fig. 1. Time response.

1) Sixth order polynomial Lyapunov function: The sixth order polynomial Lyapunov function obtained in our approach is

$$\begin{aligned} V(\boldsymbol{x}) &= 1.22x_1^6 + 0.586x_1^5x_2 + 0.855x_1^4x_2^2 \\ &\quad +0.321x_1^3x_2^3 + 0.380x_1^2x_2^4 \\ &\quad -0.0439x_1x_2^5 + 0.0517x_2^6. \end{aligned} \tag{20}$$

2) Eighth order polynomial Lyapunov function: The eightth order polynomial Lyapunov function obtained in our approach is

$$\begin{aligned} V(\boldsymbol{x}) &= 2.52x_1^8 + 1.39x_1^7x_2 + 2.61x_1^6x_2^2 \\ &\quad +1.28x_1^5x_2^3 + 1.09x_1^4x_2^4 \\ &\quad +0.238x_1^3x_2^5 + 0.393x_1^2x_2^6 \\ &\quad -0.0513x_1x_2^7 + 0.0378x_2^8. \end{aligned} \tag{21}$$

3) Tenth order polynomial Lyapunov function: The tenth order polynomial Lyapunov function obtained in our approach is

$$\begin{aligned} V(\boldsymbol{x}) &= 6.36x_1^{10} + 3.52x_1^9x_2 + 8.36x_1^8x_2^2 \\ &\quad +4.67x_1^7x_2^3 + 4.83x_1^6x_2^4 \\ &\quad +1.84x_1^5x_2^5 + 1.58x_1^4x_2^6 \\ &\quad +0.0795x_1^3x_2^7 + 0.480x_1^2x_2^8 \\ &\quad -0.0693x_1x_2^9 + 0.0341x_2^{10}. \end{aligned} \tag{22}$$

We now compare the proposed SOS approach to a (non-fuzzy) polynomial system approach. It is known that a non-polynomial (nonlinear) system can be converted into a polynomial system by a variable transformation. We introduce new variables $x_3(t) = \sin x_1(t)$ and $x_4(t) = \cos x_1(t) - 1$. Then, the time derivative of these new variables are

$$\begin{aligned} \dot{x}_3(t) &= \dot{x}_1(t)\cos x_1(t) = \dot{x}_1(t)(1 + x_4(t)) \\ &= -\frac{7}{2}x_1(t) - \frac{3}{2}x_1(t)x_3(t) - 4x_2(t) \\ &\quad -\frac{7}{2}x_1(t)x_4(t) - \frac{3}{2}x_1(t)x_3(t)x_4(t) - 4x_2(t)x_4(t), \end{aligned} \tag{23}$$

$$\begin{aligned} \dot{x}_4(t) &= -\dot{x}_1(t)\sin x_1(t) = -\dot{x}_1(t)x_3(t) \\ &= \frac{7}{2}x_1(t)x_3(t) + \frac{3}{2}x_1(t)x_3^2(t) + 4x_2(t)x_3(t). \end{aligned} \tag{24}$$

Hence, the system (19) can be converted into the following polynomial system.

$$\begin{aligned} \dot{x_1}(t) &= -\frac{7}{2}x_1(t) - \frac{3}{2}x_1(t)x_3(t) - 4x_2(t) \\ \dot{x_2}(t) &= \frac{19}{2}x_1(t) - \frac{21}{2}x_1(t)x_3(t) - 2x_2(t) \\ \dot{x}_3(t) &= -\frac{7}{2}x_1(t) - \frac{3}{2}x_1(t)x_3(t) - 4x_2(t) \\ &\quad -\frac{7}{2}x_1(t)x_4(t) - \frac{3}{2}x_1(t)x_3(t)x_4(t) \\ &\quad -4x_2(t)x_4(t) \\ \dot{x}_4(t) &= \frac{7}{2}x_1(t)x_3(t) + \frac{3}{2}x_1(t)x_3^2(t) + 4x_2(t)x_3(t) \end{aligned}$$

For this example, no polynomial Lyapunov functions with the same structures as in (20), (21) and (22) can be found using SOSTOOLS. This result shows that our approach can be superior to the existing (non-fuzzy) polynomial system approach.

B. Example 2

This example shows comparative results between the proposed polynomial Lyapunov function (8) and a number of well-know piecewise Lyapunov functions.

Recall the nonlinear system (9) in Section III-C, for simplicity assume that $\ell = 1$. Then, the dynamics can be described as

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -2x_1(t) - x_2(t) - g(t)x_1(t), \end{aligned} \tag{25}$$

where $g(t) \in [0, k]$ for all t.

Similarly as before, we can exactly represent the nonlinear dynamics with the following fuzzy model

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{2} h_i(z(t))\boldsymbol{A}_i\boldsymbol{x}(t), \tag{26}$$

where $z(t) = g(t)$ and

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \ \boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ -2-k & -1 \end{bmatrix}.$$

The membership functions are obtained as

$$h_1(z(t)) = \frac{k - g(t)}{k}, \ h_2(z(t)) = \frac{g(t)}{k}.$$

First the quadratic Lyapunov function approach can guarantees the stability for $k \le 3.82$.

The piecewise Lyapunov functions used in this example are

$$\begin{aligned} V(\boldsymbol{x}(t)) = max\{\boldsymbol{x}^T(t)\boldsymbol{P}_1\boldsymbol{x}(t), \ \boldsymbol{x}^T(t)\boldsymbol{P}_2\boldsymbol{x}(t)\} \\ \boldsymbol{P}_1 > \boldsymbol{0}, \ \boldsymbol{P}_2 > \boldsymbol{0} \end{aligned} \tag{27}$$

and

$$\begin{aligned} V(\boldsymbol{x}(t)) = min\{\boldsymbol{x}^T(t)\boldsymbol{P}_1\boldsymbol{x}(t), \ \boldsymbol{x}^T(t)\boldsymbol{P}_2\boldsymbol{x}(t)\} \\ \boldsymbol{P}_1 > \boldsymbol{0}, \ \boldsymbol{P}_2 > \boldsymbol{0}. \end{aligned} \tag{28}$$

Stability conditions [35] based on (27) and (28) are obtained for the system (25). The condition based on (27) guarantees the stability for $k \le 4.7$. The condition based on (28) guarantees the stability for $k \le 4.4$. Both are improvement over the quadratic Lyapunov function approach.

We now apply the proposed SOS approach to the system (26), where analytical results from second order to tenth order polynomial Lyapunov functions are considered. It can be seen that higher order polynomial Lyapunov functions achieve more relaxed stability results.

1) Second order polynomial Lyapunov function: Consider a second order polynomial Lyapunov function $V(\boldsymbol{x})$. Note that it is equivalent to the quadratic Lyapunov function. The conditions (13) and (14) are feasible. We have the following Lyapunov function

$$V(\boldsymbol{x}) = 27.4x_1^2 + 6.97x_1x_2 + 7.02x_2^2 \tag{29}$$

The conditions (13) and (14) guarantee the stability for $k \le 3.82$. As expected, the result is same as that of the quadratic Lyapunov function approach.

2) Fourth order polynomial Lyapunov function: Consider a fourth order polynomial Lyapunov function $V(\boldsymbol{x})$. Then, the conditions (13) and (14) are feasible. We have the following Lyapunov function

$$\begin{aligned} V(\boldsymbol{x}) &= 271.0x_1^4 + 83.5x_1^3x_2 + 157.0x_1^2x_2^2 \\ &\quad +38.7x_1x_2^3 + 12.6x_2^4 \end{aligned} \tag{30}$$

The conditions (13) and (14) guarantee the stability for $k \le 5.73$. The result is better than that of the second order polynomial Lyapunov function and those of the two piecewise Lyapunov functions.

3) Sixth order polynomial Lyapunov function: Consider a sixth order polynomial Lyapunov function $V(\boldsymbol{x})$. Then, the conditions (13) and (14) are feasible. We have the following Lyapunov function

$$\begin{aligned} V(\boldsymbol{x}) &= 2330.0x_1^6 + 713.0x_1^5x_2 + 1920.0x_1^4x_2^2 \\ &\quad +889.0x_1^3x_2^3 + 553.0x_1^2x_2^4 + 108.0x_1x_2^5 \\ &\quad +23.1x_2^6 \end{aligned} \tag{31}$$

The conditions (13) and (14) guarantee the stability for $k \le 6.21$. The result is better than the fourth order polynomial Lyapunov function result.

4) Eighth order polynomial Lyapunov function: Consider an eighth order polynomial Lyapunov function $V(\boldsymbol{x})$. Then, the conditions (13) and (14) are feasible. We have the following Lyapunov function

$$\begin{aligned} V(\boldsymbol{x}) &= 3990.0x_1^8 + 1580.0x_1^7x_2 \\ &\quad +4680.0x_1^6x_2^2 + 2560.0x_1^5x_2^3 \\ &\quad +1850.0x_1^4x_2^4 + 675.0x_1^3x_2^5 \\ &\quad +284.0x_1^2x_2^6 + 49.2x_1x_2^7 + 7.44x_2^8 \end{aligned} \tag{32}$$

The conditions (13) and (14) guarantee the stability for $k \le 6.39$. The result is better than the sixth order polynomial Lyapunov function result.

5) Tenth order polynomial Lyapunov function: Consider a tenth order polynomial Lyapunov function $V(\boldsymbol{x})$. Then, the conditions (13) and (14) are feasible. We have the following Lyapunov function

$$\begin{aligned} V(\boldsymbol{x}) &= 28100.0x_1^{10} + 10200.0x_1^9x_2 \\ &\quad +40100.0x_1^8x_2^2 + 29100.0x_1^7x_2^3 \\ &\quad +24900.0x_1^6x_2^4 + 10400.0x_1^5x_2^5 \\ &\quad +5630.0x_1^4x_2^6 + 1990.0x_1^3x_2^7 \\ &\quad +609.0x_1^2x_2^8 + 89.0x_1x_2^9 + 10.4x_2^{10} \end{aligned} \tag{33}$$

The conditions (13) and (14) guarantee the stability for $k \le 6.64$. The result is better than the eighth order polynomial Lyapunov function result.

Fig. 2 summarizes the comparative results among the quadratic Lyapunov function approach ($k_{max} = 3.82$), the piecewise Lyapunov function approach ($k_{max} = 4.7$) and our approach. As mentioned above, higher order polynomial Lyapunov functions achieve more relaxed stability results. In particular, the tenth order polynomial Lyapunov function result ($k_{max} = 6.64$) is much better than the piecewise Lyapunov function approach ($k_{max} = 4.7$).

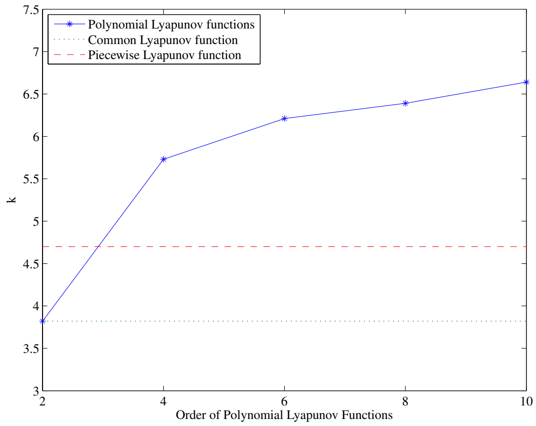

Fig. 2. Comparison results among quadratic Lyapunov function, piecewise Lyapunov function and polynomial Lyapunov functions.

Fig. 3 shows time transients of the tenth order polynomial Lyapunov function constructed in our approach, where the initial states $\boldsymbol{x}(0) = [8 \quad 0]^T$ and $g(t) = \frac{k_{max}}{2}(\sin 10t + 1)$. It can be found from the figure that the time transient of the polynomial Lyapunov function monotonically decreases.

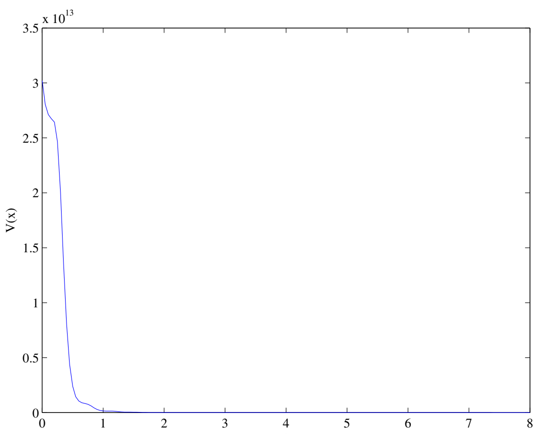

Fig. 3. Time transient of tenth order polynomial Lyapunov function ($k_{max} = 6.64$).

VI. Stabilization via SOS

A. Parallel Distributed Compensation and LMI Design Condition

The parallel distributed compensation (PDC) [21] offers a procedure to design a fuzzy controller from the given Takagi-Sugeno fuzzy model (2):

Control Rule i**:**

$$\begin{aligned} &If \ z_1(t) \ is \ M_{i1} \ and \cdots \ and \ \ z_p(t) \ is \ M_{ip} \\ &then \ \boldsymbol{u}(t) = -\boldsymbol{F}_i\boldsymbol{x}(t) \quad i = 1, 2, \cdots, r \end{aligned} \tag{34}$$

The overall fuzzy controller can be calculated by

$$\boldsymbol{u}(t) = -\sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{F}_i\boldsymbol{x}(t). \tag{35}$$

The well-known LMI conditions [21] for the stability of the feedback system consisting of (3) and (35) are given as follows:

$$\boldsymbol{X} > \boldsymbol{0} \tag{36}$$

$$-\boldsymbol{X}\boldsymbol{A}_i^T - \boldsymbol{A}_i\boldsymbol{X} + \boldsymbol{M}_i^T\boldsymbol{B}_i^T + \boldsymbol{B}_i\boldsymbol{M}_i > \boldsymbol{0} \tag{37}$$

$$\begin{aligned} &-\boldsymbol{X}\boldsymbol{A}_i^T - \boldsymbol{A}_i\boldsymbol{X} - \boldsymbol{X}\boldsymbol{A}_j^T - \boldsymbol{A}_j\boldsymbol{X} \\ &+\boldsymbol{M}_j^T\boldsymbol{B}_i^T + \boldsymbol{B}_i\boldsymbol{M}_j + \boldsymbol{M}_i^T\boldsymbol{B}_j^T + \boldsymbol{B}_j\boldsymbol{M}_i \geq \boldsymbol{0} \quad i < j \end{aligned} \tag{38}$$

where $\boldsymbol{M}_i = \boldsymbol{F}_i\boldsymbol{X}$.

B. Polynomial Fuzzy Controller

Since the PDC mirrors the structure of the fuzzy model of a system, a fuzzy controller with polynomial rule consequence can be constructed from the given polynomial fuzzy model (6).

Control Rule i**:**

$$\begin{aligned} &If \ z_1(t) \ is \ M_{i1} \ and \cdots \ and \ \ z_p(t) \ is \ M_{ip} \\ &then \ \boldsymbol{u}(t) = -\boldsymbol{F}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) \quad i = 1, 2, \cdots, r \end{aligned} \tag{39}$$

The overall fuzzy controller can be calculated by

$$\boldsymbol{u}(t) = -\sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{F}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)). \tag{40}$$

From (7) and (40), the closed-loop system can be represented as

$$\begin{aligned} \dot{\boldsymbol{x}}(t) \quad = \quad &\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z}(t))h_j(\boldsymbol{z}(t)) \times \\ &\{\boldsymbol{A}_i(\boldsymbol{x}(t)) - \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{F}_j(\boldsymbol{x}(t))\}\hat{\boldsymbol{x}}(\boldsymbol{x}(t)). \end{aligned} \tag{41}$$

If $\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) = \boldsymbol{x}(t)$ and $\boldsymbol{A}_i(\boldsymbol{x}(t))$, $\boldsymbol{B}_i(\boldsymbol{x}(t))$ and $\boldsymbol{F}_j(\boldsymbol{x}(t))$ are constant matrices for all i and j, then (7) and (40) reduce to (3) and (35), respectively. Therefore, (7) and (40) are more genral representations.

We provide another important proposition with respect to the relaxation of polynomial conditions.

Proposition 2: [36] Let $F(\boldsymbol{x}(t))$ be an $N \times N$ symmetric polynomial matrix of degree $2d$ in $\boldsymbol{x}(t) \in R^n$. Furthermore, let $\hat{\boldsymbol{x}}(\boldsymbol{x}(t))$ be a column vector whose entries are all monomials in $\boldsymbol{x}(t)$ with degree no greater than d, and consider the following conditions.

(1) $F(\boldsymbol{x}(t)) \geq 0$ for all $\boldsymbol{x}(t) \in R^n$.
(2) $\boldsymbol{v}^T(t)F(\boldsymbol{x}(t))\boldsymbol{v}(t)$ is a sum of squares, where $\boldsymbol{v}(t) \in R^N$.
(3) There exists a positive semidefinite matrix $\boldsymbol{Q}$ such that $\boldsymbol{v}^T(t)F(\boldsymbol{x}(t))\boldsymbol{v}(t) = (\boldsymbol{v}(t)\otimes\hat{\boldsymbol{x}}(\boldsymbol{x}(t)))^T\boldsymbol{Q}(\boldsymbol{v}\otimes\hat{\boldsymbol{x}}(\boldsymbol{x}(t)))$, where $\otimes$ denotes the Kronecker product.

Then (1) $\Leftarrow$ (2) and (2) $\Longleftrightarrow$ (3).

The property that (2) $\Rightarrow$ (1) in Proposition 2 will be utilized in the proof of Theorem 2.

C. SOS Design Conditions

This subsection presents stabilizing control design conditions rendered via SOS. The design of a stabilizing polynomial fuzzy controller is numerically a feasibility problem.

From now on, to lighten the notation, we will drop the notation with respect to time t as in Section IV-B. For instance, we will emply $\boldsymbol{x}$, $\hat{\boldsymbol{x}}$ instead of $\boldsymbol{x}(t)$, $\hat{\boldsymbol{x}}(\boldsymbol{x}(t))$, respectively. Again it should be kept in mind that $\boldsymbol{x}$ means $\boldsymbol{x}(t)$.

Let $\boldsymbol{A}_i^k(\boldsymbol{x})$ denote the k-th row of $\boldsymbol{A}_i(\boldsymbol{x})$, $\boldsymbol{K} = \{k_1, k_2, \cdots k_m\}$ denote the row indices of $\boldsymbol{B}_i(\boldsymbol{x})$ whose corresponding row is equal to zero, and define $\tilde{\boldsymbol{x}} = (x_{k_1}, x_{k_2}, \cdots x_{k_m})$.

Theorem 2: The control system consisting of (7) and (40) is stable if there exist a symetric polynomial matrix $\boldsymbol{X}(\tilde{\boldsymbol{x}}) \in \boldsymbol{R}^{N\times N}$ and a polynomial matrix $\boldsymbol{M}_i(\boldsymbol{x}) \in \boldsymbol{R}^{m\times N}$ such that (42) and (43) are staisifed, where $\epsilon_1(\boldsymbol{x})$ and $\epsilon_{2ij}(\boldsymbol{x})$ are non negative polynomials such that $\epsilon_1(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq 0$ and $\epsilon_{2ij}(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x}$.

$$\boldsymbol{v}^T(\boldsymbol{X}(\tilde{\boldsymbol{x}}) - \epsilon_1(\boldsymbol{x})\boldsymbol{I})\boldsymbol{v} \ \textit{is SOS} \tag{42}$$

$$\begin{aligned}
-\boldsymbol{v}^T\Big(&\boldsymbol{T}(\boldsymbol{x})\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{X}(\tilde{\boldsymbol{x}}) - \boldsymbol{T}(\boldsymbol{x})\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{M}_j(\boldsymbol{x}) \\
&+\boldsymbol{X}(\tilde{\boldsymbol{x}})\boldsymbol{A}_i^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x}) - \boldsymbol{M}_j^T(\boldsymbol{x})\boldsymbol{B}_i^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x}) \\
&+\boldsymbol{T}(\boldsymbol{x})\boldsymbol{A}_j(\boldsymbol{x})\boldsymbol{X}(\tilde{\boldsymbol{x}}) - \boldsymbol{T}(\boldsymbol{x})\boldsymbol{B}_j(\boldsymbol{x})\boldsymbol{M}_i(\boldsymbol{x}) \\
&+\boldsymbol{X}(\tilde{\boldsymbol{x}})\boldsymbol{A}_j^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x}) - \boldsymbol{M}_i^T(\boldsymbol{x})\boldsymbol{B}_j^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x}) \\
&-\sum_{k\in\boldsymbol{K}}\frac{\partial \boldsymbol{X}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{A}_i^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \\
&-\sum_{k\in\boldsymbol{K}}\frac{\partial \boldsymbol{X}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{A}_j^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) + \epsilon_{2ij}(\boldsymbol{x})\boldsymbol{I}\Big)\boldsymbol{v} \\
&\textit{is SOS} \quad i \leq j,
\end{aligned} \tag{43}$$

where $\boldsymbol{v} \in R^N$is a vector that is independent of $\boldsymbol{x}$. $\boldsymbol{T}(\boldsymbol{x}) \in \boldsymbol{R}^{N\times n}$ is a polynomial matrix whose (i, j)-th entry is given by

$$T^{ij}(\boldsymbol{x}) = \frac{\partial \hat{x}_i}{\partial x_j}(\boldsymbol{x}). \tag{44}$$

In addition, if (43) holds with $\epsilon_{2ij}(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq 0$, then the zero equilibirium is asymptotically stable. If $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ is a constant matrix, then the stability holds globally. A stablizing feedback gain $\boldsymbol{F}_i(\boldsymbol{x})$ can be obtained from $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ and $\boldsymbol{M}_i(\boldsymbol{x})$ as

$$\boldsymbol{F}_i(\boldsymbol{x}) = \boldsymbol{M}_i(\boldsymbol{x})\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}}). \tag{45}$$

Proof: Consider a candidate of Lyapunov function.

$$\boldsymbol{V}(\boldsymbol{x}) = \hat{\boldsymbol{x}}^T(\boldsymbol{x})\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})\hat{\boldsymbol{x}}(\boldsymbol{x}), \tag{46}$$

where $\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}}) \in R^{N\times N}$ is a symmetric polynomial matrix. The condition (42) implies that both $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ and $\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})$ are positive definite for all $\boldsymbol{x}$, and $V(\boldsymbol{x})$ is a positive definite function of $\boldsymbol{x}$.

The time derivative of $\boldsymbol{V}(\boldsymbol{x})$ along the closed-loop trajectory (41) is given by

$$\begin{aligned}
\dot{\boldsymbol{V}}(\boldsymbol{x}) &= \hat{\boldsymbol{x}}^T(\boldsymbol{x})\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})\dot{\hat{\boldsymbol{x}}}(\boldsymbol{x}) + \dot{\hat{\boldsymbol{x}}}^T(\boldsymbol{x})\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})\hat{\boldsymbol{x}}(\boldsymbol{x}) \\
&\quad +\hat{\boldsymbol{x}}^T(\boldsymbol{x})\dot{\boldsymbol{X}}^{-1}(\tilde{\boldsymbol{x}})\hat{\boldsymbol{x}}(\boldsymbol{x}) \\
&= \hat{\boldsymbol{x}}^T(\boldsymbol{x})\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})\boldsymbol{T}(\boldsymbol{x})\dot{\boldsymbol{x}}(\boldsymbol{x}) \\
&\quad +\dot{\boldsymbol{x}}^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x})\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})\hat{\boldsymbol{x}}(\boldsymbol{x}) \\
&\quad +\hat{\boldsymbol{x}}^T(\boldsymbol{x})\left(\sum_{k=1}^{n}\frac{\partial \boldsymbol{X}^{-1}}{\partial x_k}(\tilde{\boldsymbol{x}})\dot{x}_k\right)\hat{\boldsymbol{x}}(\boldsymbol{x}).
\end{aligned} \tag{47}$$

Since $\boldsymbol{B}_i^k(\boldsymbol{x}) = \boldsymbol{0}$ for $k \in \boldsymbol{K}$, we have

$$\dot{x}_k = \sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{A}_i^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \tag{48}$$

for $k \in \boldsymbol{K}$. On the other hand,

$$\frac{\partial \boldsymbol{X}^{-1}}{\partial x_i}(\tilde{\boldsymbol{x}}) = \boldsymbol{0} \tag{49}$$

for $i \notin \boldsymbol{K}$. From (41), (48) and (49), $\dot{\boldsymbol{V}}(\boldsymbol{x})$ becomes

$$\begin{aligned}
\dot{\boldsymbol{V}}(\boldsymbol{x}) &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\times \\
&\quad \hat{\boldsymbol{x}}^T(\boldsymbol{x})\Big\{\Omega_{ij}(\boldsymbol{x}) \\
&\quad + \sum_{k\in\boldsymbol{K}}\frac{\partial \boldsymbol{X}^{-1}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{A}_i^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x})\Big\}\hat{\boldsymbol{x}}(\boldsymbol{x}) \\
&= \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\times \\
&\quad \hat{\boldsymbol{x}}^T(\boldsymbol{x})\Big\{\Omega_{ij}(\boldsymbol{x}) + \Omega_{ji}(\boldsymbol{x}) \\
&\quad + \sum_{k\in\boldsymbol{K}}\frac{\partial \boldsymbol{X}^{-1}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{A}_i^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \\
&\quad + \sum_{k\in\boldsymbol{K}}\frac{\partial \boldsymbol{X}^{-1}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{A}_j^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x})\Big\}\hat{\boldsymbol{x}}(\boldsymbol{x}),
\end{aligned} \tag{50}$$

where $\Omega_{ij}(\boldsymbol{x}) = \boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})\boldsymbol{T}(\boldsymbol{x})\{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x})\} + \{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x})\}^T\boldsymbol{T}^T(\boldsymbol{x})\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})$. The condition (43) implies that

$$\begin{aligned}
-\Big(&\boldsymbol{T}(\boldsymbol{x})\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{X}(\tilde{\boldsymbol{x}}) - \boldsymbol{T}(\boldsymbol{x})\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{M}_j(\boldsymbol{x}) \\
&+\boldsymbol{X}(\tilde{\boldsymbol{x}})\boldsymbol{A}_i^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x}) - \boldsymbol{M}_j^T(\boldsymbol{x})\boldsymbol{B}_i^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x}) \\
&+\boldsymbol{T}(\boldsymbol{x})\boldsymbol{A}_j(\boldsymbol{x})\boldsymbol{X}(\tilde{\boldsymbol{x}}) - \boldsymbol{T}(\boldsymbol{x})\boldsymbol{B}_j(\boldsymbol{x})\boldsymbol{M}_i(\boldsymbol{x}) \\
&+\boldsymbol{X}(\tilde{\boldsymbol{x}})\boldsymbol{A}_j^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x}) - \boldsymbol{M}_i^T(\boldsymbol{x})\boldsymbol{B}_j^T(\boldsymbol{x})\boldsymbol{T}^T(\boldsymbol{x}) \\
&-\sum_{k\in\boldsymbol{K}}\frac{\partial \boldsymbol{X}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{A}_i^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \\
&-\sum_{k\in\boldsymbol{K}}\frac{\partial \boldsymbol{X}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{A}_j^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x})\Big) \quad i \leq j
\end{aligned}$$

is positive semidefinite for all $\boldsymbol{x}$. Note that $\boldsymbol{A}_i^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x})$ is scalar. Multiplying the last expression from the left and right by $\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})$, we have

$$-\Big\{\Omega_{ij}(\boldsymbol{x})+\Omega_{ji}(\boldsymbol{x}) - \sum_{k\in\boldsymbol{K}}\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})\frac{\partial\boldsymbol{X}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})\boldsymbol{A}_i^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) - \sum_{k\in\boldsymbol{K}}\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})\frac{\partial\boldsymbol{X}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})\boldsymbol{A}_j^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x})\Big\} \geq 0 \quad i\leq j \tag{51}$$

for $\boldsymbol{x}\neq\boldsymbol{0}$.

Next, we rewrite the last two terms in (51). Since $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ is invertible, we have $\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})\boldsymbol{X}(\tilde{\boldsymbol{x}})=\boldsymbol{I}$. Differentiating both sides with respect to x_k yields

$$\frac{\partial\boldsymbol{X}^{-1}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{X}(\tilde{\boldsymbol{x}})+\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})\frac{\partial\boldsymbol{X}}{\partial x_k}(\tilde{\boldsymbol{x}})=\boldsymbol{0}. \tag{52}$$

Hence the following relation holds.

$$\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})\frac{\partial\boldsymbol{X}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}})=-\frac{\partial\boldsymbol{X}^{-1}}{\partial x_k}(\tilde{\boldsymbol{x}}) \tag{53}$$

Define $\boldsymbol{M}_i(\boldsymbol{x})=\boldsymbol{F}_i(\boldsymbol{x})\boldsymbol{X}(\tilde{\boldsymbol{x}})$. The inequality (51) can be reritten by using the relation (53).

$$-\Big\{\Omega_{ij}(\boldsymbol{x})+\Omega_{ji}(\boldsymbol{x}) + \sum_{k\in\boldsymbol{K}}\frac{\partial\boldsymbol{X}^{-1}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{A}_i^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) + \sum_{k\in\boldsymbol{K}}\frac{\partial\boldsymbol{X}^{-1}}{\partial x_k}(\tilde{\boldsymbol{x}})\boldsymbol{A}_j^k(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x})\Big\} \geq 0 \quad i\leq j \tag{54}$$

for $\boldsymbol{x}\neq\boldsymbol{0}$. Therefore, if (54) holds, then we have $\dot{V}(\boldsymbol{x})\leq 0$ from (50). Furthermore, $\epsilon_{2ij}(\boldsymbol{x})>0$ for $\boldsymbol{x}\neq 0$, then $\dot{V}(\boldsymbol{x})<0$ at $\boldsymbol{x}\neq 0$. Then, the zero equilibrium is asymptotically stable. Finally, if $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ is a constant matrix, then $V(\boldsymbol{x})$ is radially unbounded, and the stability holds globally. ∎

Remark 3: Note that $\boldsymbol{v}\in R^N$ is a vector that is independent of $\boldsymbol{x}$, because $\boldsymbol{L}(\boldsymbol{x}(t))$ is not always a positive semi-definite matrix for all $\boldsymbol{x}(t)$ even if $\boldsymbol{x}^T(\boldsymbol{x}(t))\boldsymbol{L}(\boldsymbol{x}(t))\boldsymbol{x}(\boldsymbol{x}(t))$ is an SOS, where $\boldsymbol{L}(\boldsymbol{x}(t))$ is a symetric polynomial matrix in $\boldsymbol{x}(t)$. However, it is guaranteed from Proposition 2 that if $\boldsymbol{v}^T\boldsymbol{L}(\boldsymbol{x}(t))\boldsymbol{v}$ is an SOS, then $\boldsymbol{L}(\boldsymbol{x}(t))\geq 0$ for all $\boldsymbol{x}$. The proof of Theorem 2 utilizes this property (i.e., (2) ⇒ (1) in Proposition 2).

Remark 4: To avoid introducing non-convex condition, we assume that $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ only depends on states $\tilde{\boldsymbol{x}}$ whose dynamics is not directly affected by the control input, namely states whose corresponding rows in $\boldsymbol{B}_i(\boldsymbol{x})$ are zero. In relation to this, it may be advantageous to employ an initial state transformation to introduce as many zero rows as possible in $\boldsymbol{B}_i(\boldsymbol{x})$.

VII. Design Example

To illustrate the viability and validity of the SOS design approach, this section provides a design example.

Consider the following nonlinear system:

$$\dot{x}_1 = -x_1+x_1^2+x_1^3+x_1^2x_2 -x_1x_2^2+x_2+x_1u, \tag{55}$$

$$\dot{x}_2 = -\sin x_1 - x_2. \tag{56}$$

Fig. 4 shows the behavior of the nonlinear system with $u=0$. The nonlinear system is unstable.

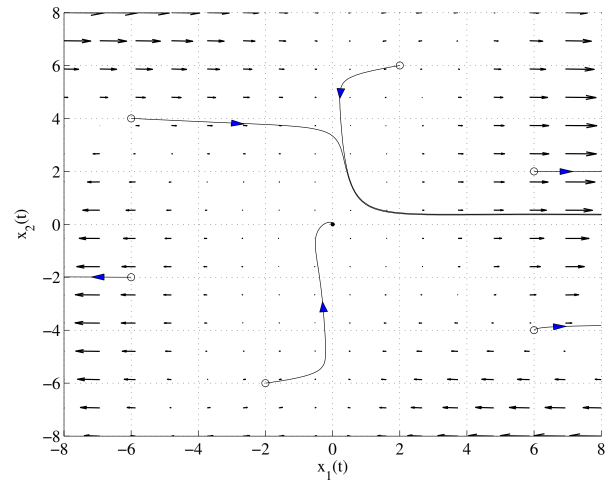

Fig. 4. Behaviors in x_1-x_2 plane.

Based on the concept of sector nonlinearity [21], the nonlinear system can be exactly represented by a Takagi-Sugeno fuzzy model for $x_1\in[-d_1\ d_1]$ and $x_2\in[-d_2\ d_2]$, where d_1 and d_2 are constants satisfying $0<d_1<\infty$ and $0<d_2<\infty$.

The Takagi-Sugeno fuzzy model is obtained as

$$\dot{\boldsymbol{x}}=\sum_{i=1}^{8}h_i(\boldsymbol{z})\{\boldsymbol{A}_i\boldsymbol{x}+\boldsymbol{B}_iu\}, \tag{57}$$

where $\boldsymbol{x}=[x_1\ x_2]^T$ and $\boldsymbol{z}=[x_1\ x_2]^T$,

$$\boldsymbol{A}_1=\begin{bmatrix}k_{\max} & 1\\ -1 & -1\end{bmatrix}, \boldsymbol{A}_2=\begin{bmatrix}k_{\max} & 1\\ -1 & -1\end{bmatrix}, \tag{58}$$

$$\boldsymbol{A}_3=\begin{bmatrix}k_{\max} & 1\\ -\frac{\sin d_1}{d_1} & -1\end{bmatrix}, \boldsymbol{A}_4=\begin{bmatrix}k_{\max} & 1\\ -\frac{\sin d_1}{d_1} & -1\end{bmatrix}, \tag{59}$$

$$\boldsymbol{A}_5=\begin{bmatrix}k_{\min} & 1\\ -1 & -1\end{bmatrix}, \boldsymbol{A}_6=\begin{bmatrix}k_{\min} & 1\\ -1 & -1\end{bmatrix}, \tag{60}$$

$$\boldsymbol{A}_7=\begin{bmatrix}k_{\min} & 1\\ -\frac{\sin d_1}{d_1} & -1\end{bmatrix}, \boldsymbol{A}_8=\begin{bmatrix}k_{\min} & 1\\ -\frac{\sin d_1}{d_1} & -1\end{bmatrix}, \tag{61}$$

$$\boldsymbol{B}_1=\boldsymbol{B}_3=\boldsymbol{B}_5=\boldsymbol{B}_7=\begin{bmatrix}d_1\\ 0\end{bmatrix}, \tag{62}$$

$$\boldsymbol{B}_2=\boldsymbol{B}_4=\boldsymbol{B}_6=\boldsymbol{B}_8=\begin{bmatrix}-d_1\\ 0\end{bmatrix}, \tag{63}$$

$$k_{\min}=\min_{|x_1|<d_1,|x_2|<d_2}(-1+x_1+x_1^2+x_1^1x_2-x_2^2), \tag{64}$$

$$k_{\max}=\max_{|x_1|<d_1,|x_2|<d_2}(-1+x_1+x_1^2+x_1^1x_2-x_2^2). \tag{65}$$

The membership functions are given as

$$h_1(\boldsymbol{z}) = \frac{k - k_{\min}}{k_{\max} - k_{\min}} \cdot \frac{\sin x_1 - \frac{\sin d_1}{d_1} x_1}{(1 - \frac{\sin d_1}{d_1}) x_1} \cdot \frac{x_1 + d_1}{2d_1}, \quad (66)$$

$$h_2(\boldsymbol{z}) = \frac{k - k_{\min}}{k_{\max} - k_{\min}} \cdot \frac{\sin x_1 - \frac{\sin d_1}{d_1} x_1}{(1 - \frac{\sin d_1}{d_1}) x_1} \cdot \frac{d_1 - x_1}{2d_1}, \quad (67)$$

$$h_3(\boldsymbol{z}) = \frac{k - k_{\min}}{k_{\max} - k_{\min}} \cdot \frac{x_1 - \sin x_1}{(1 - \frac{\sin d_1}{d_1}) x_1} \cdot \frac{x_1 + d_1}{2d_1}, \quad (68)$$

$$h_4(\boldsymbol{z}) = \frac{k - k_{\min}}{k_{\max} - k_{\min}} \cdot \frac{x_1 - \sin x_1}{(1 - \frac{\sin d_1}{d_1}) x_1} \cdot \frac{d_1 - x_1}{2d_1}, \quad (69)$$

$$h_5(\boldsymbol{z}) = \frac{k_{\max} - k}{k_{\max} - k_{\min}} \cdot \frac{\sin x_1 - \frac{\sin d_1}{d_1} x_1}{(1 - \frac{\sin d_1}{d_1}) x_1} \cdot \frac{x_1 + d_1}{2d_1}, \quad (70)$$

$$h_6(\boldsymbol{z}) = \frac{k_{\max} - k}{k_{\max} - k_{\min}} \cdot \frac{\sin x_1 - \frac{\sin d_1}{d_1} x_1}{(1 - \frac{\sin d_1}{d_1}) x_1} \cdot \frac{d_1 - x_1}{2d_1}, \quad (71)$$

$$h_7(\boldsymbol{z}) = \frac{k_{\max} - k}{k_{\max} - k_{\min}} \cdot \frac{x_1 - \sin x_1}{(1 - \frac{\sin d_1}{d_1}) x_1} \cdot \frac{x_1 + d_1}{2d_1}, \quad (72)$$

$$h_8(\boldsymbol{z}) = \frac{k_{\max} - k}{k_{\max} - k_{\min}} \cdot \frac{x_1 - \sin x_1}{(1 - \frac{\sin d_1}{d_1}) x_1} \cdot \frac{d_1 - x_1}{2d_1}. \quad (73)$$

Fig. 5 shows the feasible area of the LMI design conditions (36)-(38) consisting of the Takagi-Sugeno fuzzy model (57) and the corresponding PDC fuzzy controller [21]. As shown in Fig. 5, for large d_1, the LMI conditions are infeasible. In addition, the Takagi-Sugeno fuzzy model has eight rules to represent the nonlinear system. In contrast, we will see below that the polynomial fuzzy system (that can exactly and globally represent the nonlinear system) has only two rules.

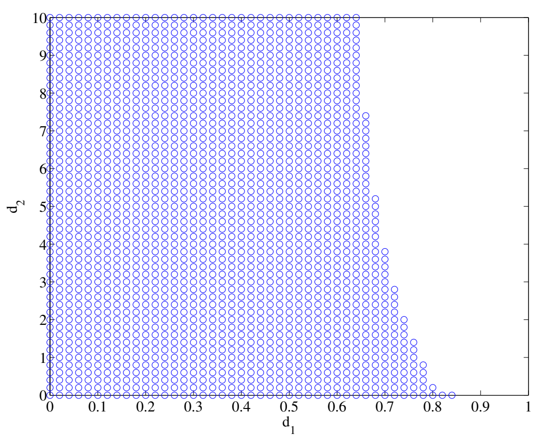

Fig. 5. Feasible area of LMI design conditions (36)-(38) for Takagi-Sugeno fuzzy model (57).

To represent the nonlinear system under consideration, we have the following polynomial fuzzy system that can exactly represent the dynamics of the nonlinear system for $x_1 \in (-\infty\ \infty)$ and $x_2 \in (-\infty\ \infty)$, i.e., globally.

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{2} h_i(z)\{\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}} + \boldsymbol{B}_i(\boldsymbol{x})u\} \quad (74)$$

where $\boldsymbol{x} = \hat{\boldsymbol{x}} = [x_1\ x_2]$ and $z = x_1$,

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ -1 & -1 \end{bmatrix},$$

$$\boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ 0.2172 & -1 \end{bmatrix},$$

$$\boldsymbol{B}_1(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \ \boldsymbol{B}_2(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

The menbership functions are given as

$$h_1(z) = \frac{\sin x_1 + 0.2172x_1}{1.2172x_1}, \ h_2(z) = \frac{x_1 - \sin x_1}{1.2172x_1}.$$

The SOS design conditions in Theorems 2 are feasible.

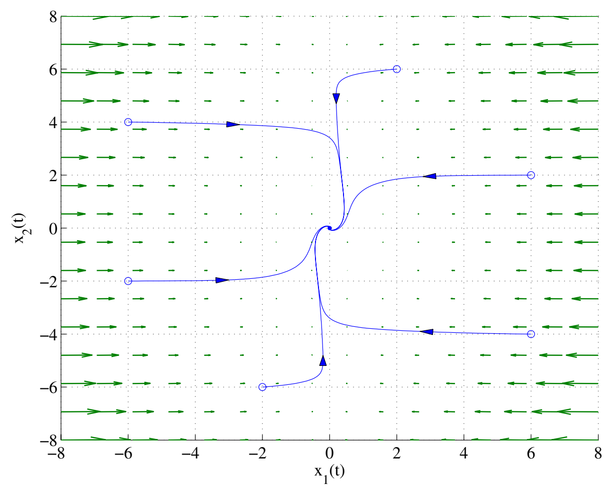

Fig. 6. Behaviors in x_1-x_2 plane (with feedback).

Fig.6 shows the control result via the designed stabilizing controller. The unstable system is stabilized via the SOS designed controller. In fact, the controller guarantees the global asymptotic stability of the controlled system.

It can be seen that a main difference between the Takagi-Sugeno fuzzy model based control and the polynomial fuzzy model based control is that $\boldsymbol{A}_i$, $\boldsymbol{B}_i$ and $\boldsymbol{F}_i$ are permitted to be polynomial matrices in $\boldsymbol{x}$. In addition, our approach utilizes a more general Lyapunov function (namely, a polynomial Lyapunov function). With a more general framework for both modeling and control, our SOS-based approach indeed provides more relaxed analysis and design conditions than the existing LMI approach.

VIII. Conclusions

This paper has presented a sum of squares (SOS) approach for modeling and control of nonlinear dynamical systems in terms of polynomial fuzzy systems. The proposed SOS-based framework is an attempt to provide a post-LMI framework for fuzzy modeling and control of nonlinear systems. First, we have introduced a polynomial fuzzy modeling and control framework that is more general and effective than the Takagi-Sugeno fuzzy model and control. Secondly, stability and stabilizability conditions of the fuzzy polynomial systems have been obtained based on polynomial Lyapunov functions that contain quadratic Lyapunov functions as a special case. The stability and stabilizability conditions presented in this paper are more general and relaxed than those of the existing LMI-based approaches to Takagi-Sugeno fuzzy model and control. The salient feature of the derived stability and stabilizability conditions is that they can be represented in terms of SOS and hence can be numerically (partially symbolically) solved via the recently developed SOSTOOLS. A number of modeling, analysis and design examples have been included to illustrate the validity and applicability of the proposed approach.

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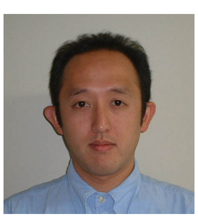

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