An SOS-based Control Lyapunov Function Design for Polynomial Fuzzy Control of Nonlinear Systems

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***Abstract*—This paper deals with a sum-of-squares (SOS) based control Lyapunov function (CLF) design for polynomial fuzzy control of nonlinear systems. The design starts with exactly replacing (smooth) nonlinear systems dynamics with polynomial fuzzy models which are known as universal approximators. Next, global stabilization conditions represented in terms of SOS are provided in the framework of the CLF design, i.e., a stabilizing controller with non parallel distributed compensation form is explicitly designed by applying Sontag's control law once a CLF for a given nonlinear system is constructed. Furthermore, semi-global stabilization conditions on operation domains are derived in the same fashion as in the global stabilization conditions. Both global and semi-global stabilization problems are formulated as SOS optimization problems which reduce to numerical feasibility problems. Five design examples are given to show the effectiveness of our proposed approach over the existing linear matrix inequality (LMI) and SOS approaches.**

***Index Terms*—Control Lyapunov function, global stabilization, polynomial fuzzy system, operation domain, semi-global stabilization, sum-of-squares (SOS).**

I. Introduction

NONLINEAR systems analysis and design using the Takagi-Sugeno (T-S) fuzzy model [1] based control methodology [2], [3] have received much attention as a powerful tool to deal with complex nonlinear control systems in the last two decades. The T-S fuzzy model provides a convenient platform that can represent any smooth nonlinear systems by fuzzily blending linear sub-systems and its stabilization conditions based on Lyapunov stability theory [4] can be represented in terms of linear matrix inequalities (LMIs) [2], [5]. Thus, the designs have been carried out using LMI optimization techniques [6]. In the T-S fuzzy model based control, the parallel distributed compensation (PDC) concept [2], [5] based on a common quadratic Lyapunov function has been mainly employed to design a fuzzy controller for the system. Nowadays there are a large number of researches [7]–[21] conducted to obtain more relaxed stability (stabilization) conditions of nonlinear systems expressed as the T-S fuzzy model.

A more general version of the T-S fuzzy model called the polynomial fuzzy model was introduced recently in [22]. Unlike the T-S fuzzy model which only deals with constants in the system matrices, the polynomial fuzzy model allows us to deal also with polynomials in the system matrices. Therefore, the nonlinear system representation can be conducted more efficiently, especially when there are a number of polynomial terms contained in the system. However, when we handle the polynomial fuzzy model, LMI optimization techniques cannot be utilized to solve stability analysis and control design problems directly. Hence, the paper [22] introduced a sum-of-squares (SOS) optimization technique to perform stability analysis and control design for the polynomial fuzzy model. The problems represented in terms of SOS can be numerically solved by free third-party MATLAB toolboxes such as SOSTOOLS [23] and SOSOPT [24], etc. In recent years, there exist some extended works on relaxing stability (stabilization) conditions presented in [22] such as [25]–[30], and so on.

As another methodology to nonlinear systems control, control Lyapunov function (CLF) approaches [31], [32] have been discussed in the literature. In general, the CLF concept is similar to the Lyapunov stability concept, but the main CLF idea is that the derivative of a candidate Lyapunov function can be made negative pointwise by selecting control values [33]. It is known that if a system is continuous and there exists a continuous feedback law that stabilizes the system, then there must exist a CLF for the system. Once a CLF is found, a stabilizing controller is designed using the Sontag's formula [32]. Although the CLF method is straight-forward, the construction of CLF itself is still a tough problem. The paper [34] presents a method to construct a CLF for polynomial nonlinear systems using an SOS optimization technique. However, the approach in the paper [34] does not work for other types of nonlinear systems with non-polynomial nonlinear terms, e.g., trigonometric functions such as $\sin x$, $\cos x$, etc. Thus, a CLF construction for a general nonlinear system is still an open problem.

This paper deals with an SOS based CLF design for polynomial fuzzy control of nonlinear systems. By using the so-called sector nonlinearity concept [2], a nonlinear system even with non-polynomial nonlinear terms is exactly replaced with a polynomial fuzzy model which is known as an universal

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approximator. It should be noted in the polynomial fuzzy model construction that all the non-polynomial nonlinear terms can be captured by membership functions [2], [22]. Next, global stabilization conditions represented in terms of SOS are provided in the framework of the CLF design. In other words, a stabilizing controller with non-PDC form is explicitly designed by applying the Sontag's control law [32] once a CLF for a given nonlinear system is constructed. It has been reported in the previous SOS approaches that polynomial Lyapunov function constructions in the fuzzy controller designs are restrictive on the input matrices. The SOS design conditions derived in this paper provide a better way that no longer requires the restriction. Furthermore, for complicated systems that are not globally stabilizable, semi-global stabilization conditions on considered operation domains are derived in the same fashion as in the global stabilization conditions. Both global and semi-global stabilization problems are formulated as SOS optimization problems which reduce to numerical feasibility problems. Five design examples are given to show the effectiveness of our proposed approach over the existing LMI and SOS approaches. There is a work [37] on applying the CLF framework to the polynomial fuzzy model. However, the design method in [37] deals only with single input cases and cannot be applied to multiple input cases. The design approaches proposed in this paper are available even for multiple input cases. In addition, this paper newly derives semi-global stabilization condition (Section IV) for the domain of attraction.

This paper is organized as follows: Section II presents the definitions and lemmas as preliminaries for our proposed approach. In Section III, construction of global CLFs for polynomial fuzzy systems with the design examples are provided. Section IV shows a CLF construction method in semi-global stabilization and its examples. Finally, a conclusion is drawn in Section V. We will assume throughout the paper that all the matrices and vectors have appropriate dimensions.

II. PRELIMINARIES

A. Notation and Definitions

Throughout this paper, the following standard notations and definitions are used as defined in some literature, e.g., [34], [35].

Definition 1. *A monomial in $\boldsymbol{x} = [x_1\ x_2\ \cdots\ x_n]$ is a function of the form $x_1^{d_1}\ x_2^{d_2}\ \cdots\ x_n^{d_n}$, where d_1, d_2, $\cdots$, d_n are non-negative integers. The degree of a monomial is defined as $d = \sum_{i=1}^{n} d_i$.*

Definition 2. *Define $\mathcal{P}$ as the universe of polynomials. A polynomial $\omega(\boldsymbol{x})$ is defined as a finite linear combination of monomials with real coefficients. $\omega(\boldsymbol{x}) \in \mathcal{P}$ is positive semi-definite if $\omega(\boldsymbol{x}) \geq 0$. The set of positive semi-definite polynomials is denoted as $\mathcal{P}^{0+}$. $(q_i(\boldsymbol{x}))_{i=1,\cdots,r} \in \mathcal{P}$ denotes $q_1(\boldsymbol{x}), q_2(\boldsymbol{x}), \cdots, q_r(\boldsymbol{x})$ are polynomials.*

Definition 3. *Define $\mathcal{S}$ as the space of SOS polynomials, where $\mathcal{S} \subset \mathcal{P}$. A polynomial $\omega(\boldsymbol{x})$ is an SOS if it can be written $\omega(\boldsymbol{x}) = \sum_{i=1}^{n} f_i^2(\boldsymbol{x})$, where $(f_i(\boldsymbol{x}))_{i=1,\cdots,n} \in \mathcal{P}$. Obviously, if $\omega(\boldsymbol{x}) \in \mathcal{S}$, then $\omega(\boldsymbol{x}) \in \mathcal{P}^{0+}$. An SOS polynomial $\omega(\boldsymbol{x})$ is positive definite if $\omega(\boldsymbol{x}) > 0$ when $\boldsymbol{x} \neq 0$. The set of positive definite SOS polynomials is denoted as $\mathcal{S}^{+}$.*

Definition 4. *Given $(g_j(\boldsymbol{x}))_{j=1,\cdots,c} \in \mathcal{P}$, the multiplicative monoid $\mathcal{M}(g_1(\boldsymbol{x}), \cdots, g_v(\boldsymbol{x}))$ is the set of all finite products of $g_j(\boldsymbol{x})$'s including 1 (i.e. the empty product).*

Definition 5. *Given $(f_i(\boldsymbol{x}))_{i=1,\cdots,\iota} \in \mathcal{P}$, the cone is defined as*

$$\mathcal{C}(f_1(\boldsymbol{x}), \cdots, f_\iota(\boldsymbol{x})) := s_0(\boldsymbol{x}) + \textstyle\sum_{l=1}^{w} s_l(\boldsymbol{x}) e_l(\boldsymbol{x}),$$

where w is a positive integer, $s_l(\boldsymbol{x}) \in \mathcal{S}$, and $e_l(\boldsymbol{x}) \in \mathcal{M}$ $(f_1(\boldsymbol{x}),\cdots,f_\iota(\boldsymbol{x}))$.

Definition 6. *Given $(h_\varsigma(\boldsymbol{x}))_{\varsigma=1,\cdots,\varrho} \in \mathcal{P}$, using the $h_\varsigma(\boldsymbol{x})$'s, the Ideal is defined as $\mathcal{I}(h_1(\boldsymbol{x}),\cdots,h_\varrho(\boldsymbol{x})) := \sum_{k=1}^{\varrho} h_k(\boldsymbol{x})$ $p_\varsigma(\boldsymbol{x})$ where $p_\varsigma(\boldsymbol{x}) \in \mathcal{P}$ and ϱ is a positive integer.*

B. Polynomial Fuzzy Model

Consider a nonlinear control system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{g}(\boldsymbol{x}(t))\boldsymbol{u}(t), \tag{1}$$

where $\boldsymbol{x}(t) = [x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)]^T$ is the state vector and $\boldsymbol{u}(t) = [u_1(t) \quad u_2(t) \quad \cdots \quad u_q(t)]^T$ is the input vector. The system (1) considered in this paper belongs to a class of nonlinear systems affine in $\boldsymbol{u}(t)$. $\boldsymbol{f}(\boldsymbol{x}(t))$ and $\boldsymbol{g}(\boldsymbol{x}(t))$ are smooth nonlinear functions, where $\boldsymbol{f}(0) = 0$. According to the sector nonlinearity concept [2], the nonlinear system (1) can be exactly represented by the polynomial fuzzy model as in [22]:

Model Rule i:
IF $z_1(t)$ is M_{i1} and $\cdots$ and $z_o(t)$ is M_{io},
THEN $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)$,
$i = 1, 2, \cdots, r,$ (2)

where r is the number of fuzzy rules, $z_j(t)$ $(j = 1, 2, \cdots, o)$ is the known premise variable, and M_{ij} is the fuzzy set that is associated with ith model rule and jth premise variable component. $\boldsymbol{A}_i(\boldsymbol{x}(t)) \in \mathbb{R}^{n\times N}$ and $\boldsymbol{B}_i(\boldsymbol{x}(t)) \in \mathbb{R}^{n\times q}$ are the known polynomial system and input matrices in $\boldsymbol{x}(t)$ respectively. $\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) \in \mathbb{R}^N$ denotes a monomial vector in $\boldsymbol{x}(t)$ with assumption that $\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) = 0$ if and only if $\boldsymbol{x}(t) = 0$. The system dynamics of (2) is described as follows:

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)\}, \tag{3}$$

where $\boldsymbol{z}(t) = [z_1(t) \quad z_2(t) \cdots \quad z_o(t)]^T \in \mathbb{R}^o$,

$$h_i(\boldsymbol{z}(t)) = \frac{\prod_{j=1}^{o} M_{ij}(z_j(t))}{\sum_{k=1}^{r}\prod_{j=1}^{o} M_{kj}(z_j(t))},$$
$$h_i(\boldsymbol{z}(t)) \geq 0, \quad \forall i,$$
$$\sum_{i=1}^{r} h_i(\boldsymbol{z}(t)) = 1.$$

For the sake of saving the space, the t notations are omitted for the rest of the paper, i.e., $\boldsymbol{x}$ is used instead of $\boldsymbol{x}(t)$.

C. Control Lyapunov Function

The feedback stabilizability of a nonlinear system is guaranteed if there exists a CLF for the system [31]. The most popular control design method using the CLF of a system is the Sontag's formula [32].

A smooth, radially unbounded, positive definite function $V(\boldsymbol{x})$ is a CLF for system (1) when it satisfies

$$\inf_{u_i \in \mathbb{R}} \Big\{ \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) + \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \sum_{i=1}^{q} \boldsymbol{g}_i(\boldsymbol{x}) u_i \Big\} < 0, \ \forall \boldsymbol{x} \neq 0, \tag{4}$$

where $\boldsymbol{g}_i(\boldsymbol{x})$ and u_i denote the ith column of $\boldsymbol{g}(\boldsymbol{x})$ and $\boldsymbol{u}$, respectively.

When $V(\boldsymbol{x})$ that satisfies (4) is found, the closed loop system for (1) is globally asymptotically stabilizable at the origin, i.e., there exists a feedback controller to stabilize the system (1). There are several ways to design the feedback controller, one of them is proposed by [32] (the Sontag's formula):

$$\boldsymbol{u} = [u_1 \ \cdots \ u_q], \tag{5}$$

where

$$u_i = \begin{cases} -\frac{u_f(\boldsymbol{x}) + \sqrt{u_f^2(\boldsymbol{x}) + u_G^2(\boldsymbol{x})}}{u_G(\boldsymbol{x})} u_{gi}(\boldsymbol{x}) & , \quad u_G(\boldsymbol{x}) \neq 0, \\ 0 & , \quad u_G(\boldsymbol{x}) = 0, \end{cases}$$

for $i = 1, 2, \cdots, q$, and

$$u_f(\boldsymbol{x}) = \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}), \quad u_{gi}(\boldsymbol{x}) = \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{g}_i(\boldsymbol{x}),$$
$$u_G(\boldsymbol{x}) = \sum_{\kappa=1}^{q} u_{g\kappa}^2(\boldsymbol{x}).$$

Therefore, after the CLF is found, the stabilizing controller design is straight-forward.

The following Positivstellensatz (P-satz) lemma [35] and $\mathcal{S}$-Procedure lemma [36] play important roles in our design approach.

D. The Positivstellensatz

The following Positivstellensatz (P-satz) lemma [35] is used to transform empty sets conditions to SOS conditions.

Lemma 1. *Given polynomials* $(f_i(\boldsymbol{x}))_{i=1,\cdots,\iota}$, $(g_j(\boldsymbol{x}))_{j=1,\cdots,v}$, $(h_\varsigma(\boldsymbol{x}))_{\varsigma=1,\cdots,\varrho}$, *the set*

$$\{\boldsymbol{x} \in \mathbb{R}^n | f_1(\boldsymbol{x}) \geq 0, \cdots, f_\iota(\boldsymbol{x}) \geq 0, g_1(\boldsymbol{x}) \neq 0, \cdots, g_v(\boldsymbol{x}) \neq 0, h_1(\boldsymbol{x}) = 0, \cdots, h_\varrho(\boldsymbol{x}) = 0\} \tag{6}$$

is empty if and only if there exist $f \in \mathcal{C}(f_1(\boldsymbol{x}), \cdots, f_\iota(\boldsymbol{x}))$, $g \in \mathcal{M}(g_1(\boldsymbol{x}), \cdots, g_v(\boldsymbol{x}))$, $h \in \mathcal{I}(h_1(\boldsymbol{x}), \cdots, h_\varrho(\boldsymbol{x}))$ *such that*

$$f + g^2 + h = 0. \tag{7}$$

E. $\mathcal{S}$-Procedure

The following $\mathcal{S}$-Procedure lemma [36] is used for the semi-global stabilization on operation domain in Section IV.

Lemma 2. *Given polynomials* $f_1(\boldsymbol{x})$ *and* $f_2(\boldsymbol{x})$, *define sets* Q_1 *and* Q_2:

$$Q_1 := \{\boldsymbol{x} \in \mathbb{R}^n : f_1(\boldsymbol{x}) \leq 0\},$$
$$Q_2 := \{\boldsymbol{x} \in \mathbb{R}^n : f_2(\boldsymbol{x}) \leq 0\}.$$

If there exists a polynomial $\lambda(\boldsymbol{x}) \in \mathcal{P}^{0+}, \forall x$ *such that* $-f_1(\boldsymbol{x}) + \lambda(\boldsymbol{x}) f_2(\boldsymbol{x}) \in \mathcal{P}^{0+}, \forall \boldsymbol{x}$ *then* $Q_2 \subseteq Q_1$.

III. Global Stabilization

This section provides global stabilization SOS conditions to construct a CLF for the polynomial fuzzy control system. Once a CLF is found, the Sontag's formula [32] is applied to design a stabilizing controller.

A. Stabilization Condition

Theorem 1 shows a set of SOS stabilization conditions. CLF construction is carried out by solving the SOS conditions.

Theorem 1. *A nonlinear system that is represented by the polynomial fuzzy model (3) is feedback stabilizable if there exist a smooth and radially unbounded function* $V(\boldsymbol{x})$, *SOS polynomials* $s_m(\boldsymbol{x})$ *and* $v_i(\boldsymbol{x})$, *positive definite SOS polynomials* $l_{2im}(\boldsymbol{x})$, *and polynomials* $p_{1jm}(\boldsymbol{x})$ *and* $p_{2m}(\boldsymbol{x})$ *that satisfy (8) and (9) for a non-positive* γ.

$$V(\boldsymbol{x}) - l_1(\boldsymbol{x}) \in \mathcal{S}, \tag{8}$$

$$-\sum_{i=1}^{r}\sum_{m=1}^{r} \hat{h}_i^2 \hat{h}_m^2 \Big\{ s_m(\boldsymbol{x}) \Big[\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{A}_i(\boldsymbol{x}) \hat{\boldsymbol{x}}(\boldsymbol{x}) - \gamma v_i(\boldsymbol{x}) \Big] + \sum_{j=1}^{q} p_{1jm}(\boldsymbol{x}) \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{B}_{ij}(\boldsymbol{x}) + p_{2m}(\boldsymbol{x}) + l_{2im}(\boldsymbol{x}) \Big\} + \sum_{m=1}^{r} \hat{h}_m^2 p_{2m}(\boldsymbol{x}) \in \mathcal{S}, \tag{9}$$

where $\boldsymbol{B}_{ij}(\boldsymbol{x})$ *denotes* j*th column of the input matrix* $\boldsymbol{B}_i(\boldsymbol{x})$. $\hat{h}_i^2$ *is a non-negative value as will be addressed in Remark 1.* $l_1(\boldsymbol{x}) \in \mathcal{S}^+$ *is given (not a decision variable) and is a slack variable to keep the positivity of* $V(\boldsymbol{x})$. *If we can find* $V(\boldsymbol{x})$ *that satisfies (8) and (9), based on Sontag's formula [32], the feedback controller* $\mathbf{u}$ *can be constructed as follows:*

$$\boldsymbol{u} = [u_1 \ \cdots \ u_q], \tag{10}$$

where

$$u_j = \begin{cases} -\frac{u_a(\boldsymbol{x}) + \sqrt{u_a^2(\boldsymbol{x}) + u_B^2(\boldsymbol{x})}}{u_B(\boldsymbol{x})} u_{bj}(\boldsymbol{x}) & , \quad u_B(\boldsymbol{x}) \neq 0, \\ 0 & , \quad u_B(\boldsymbol{x}) = 0, \end{cases}$$

for $j = 1, 2, \cdots, q$, *and*

$$u_a(\boldsymbol{x}) := \textstyle\sum_{i=1}^{r} h_i(\boldsymbol{z}) \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{A}_i(\boldsymbol{x}) \hat{\boldsymbol{x}}(\boldsymbol{x}),$$

$$u_{bj}(\boldsymbol{x}) := \textstyle\sum_{i=1}^{r} h_i(\boldsymbol{z}) \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{B}_{ij}(\boldsymbol{x}),$$

$$u_B(\boldsymbol{x}) := \textstyle\sum_{\kappa=1}^{q} u_{b\kappa}^2(\boldsymbol{x}).$$

Proof. The CLF condition (4) can be implemented to system (3), i.e., a smooth positive definite function $V(\boldsymbol{x})$ is a CLF for (3) when the function is radially unbounded and satisfies

$$\inf_{u_j\in\mathbb{R}}\Big\{\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}h_i(\boldsymbol{z})\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) + \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}\sum_{j=1}^{q}h_i(\boldsymbol{z})\boldsymbol{B}_{ij}(\boldsymbol{x})u_j\Big\} < 0, \quad \forall \boldsymbol{x}\neq 0, \quad (11)$$

where $\boldsymbol{B}_{ij}(\boldsymbol{x})$ denotes jth column of the input matrix $\boldsymbol{B}_i(\boldsymbol{x})$.

First, the condition for $V(\boldsymbol{x})$ to be a positive definite function can be written in an SOS form as (8), where $l_1(\boldsymbol{x})$ is a positive definite SOS polynomial.

Next, (11) shows that for all non-zero $\boldsymbol{x}$

$$\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}h_i(\boldsymbol{z})\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) < 0, \quad \forall \boldsymbol{x}\in\mathbb{R}^n \text{ when}$$
$$\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}h_i(\boldsymbol{z})\boldsymbol{B}_{ij}(\boldsymbol{x}) = 0, \quad j=1,2,\cdots,q. \quad (12)$$

Let us introduce non-negative variables (quadratic variables) $\hat{h}_i^2$ instead of $h_i(\boldsymbol{z})$ in (12), under the assumption that $\sum_{i=1}^{r}\hat{h}_i^2 = 1$. Then, (12) is satisfied if the following condition (13) is hold. This replacement leads to the successful result of applying the P-satz (Lemma 1).

For $\forall \boldsymbol{x}\neq 0$,

$$\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}\hat{h}_i^2\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) < 0, \quad \forall \boldsymbol{x}\in\mathbb{R}^n \text{ when}$$
$$\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}\hat{h}_i^2\boldsymbol{B}_{ij}(\boldsymbol{x}) = 0, \quad j=1,2,\cdots,q, \quad (13)$$

where

$$\sum_{i=1}^{r}\hat{h}_i^2 = 1.$$

Here, the condition (13) is rewritten as an empty set condition to find CLF:

$$\Big\{\boldsymbol{x}\in\mathbb{R}^n \mid \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}\hat{h}_i^2\boldsymbol{B}_{ij}(\boldsymbol{x}) = 0, \quad j=1,2,\cdots,q, \quad \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}\hat{h}_i^2\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \geq 0, \quad \boldsymbol{x}\neq 0, \quad \sum_{i=1}^{r}\hat{h}_i^2 = 1\Big\} = \varnothing, \quad (14)$$

where $V(\boldsymbol{x})$ is a smooth and radially unbounded positive definite function.

In order to use the SOS optimization technique, let $\boldsymbol{h} = [\hat{h}_1^2 \cdots \hat{h}_r^2]^T$, a positive definite polynomial $l_x(\boldsymbol{h},\boldsymbol{x})$ is used to replace the constraint $\boldsymbol{x}\neq 0$ in (14) where $l_x(\boldsymbol{h},\boldsymbol{x})\neq 0$ iff $\boldsymbol{x}\neq 0$. Let us define $l_x(\boldsymbol{h},\boldsymbol{x})$ as $l_x(\boldsymbol{h},\boldsymbol{x}) = \sum_{i=1}^{r}\hat{h}_i^2 l_{xi}(\boldsymbol{x})$, where $l_{xi}(\boldsymbol{x})\neq 0$ iff $\boldsymbol{x}\neq 0$. Moreover, with the purpose of making the condition to be an optimization problem, a non-positive real number γ (to be minimized) and an SOS polynomial $v(\boldsymbol{h},\boldsymbol{x}) = \sum_{i=1}^{r}\hat{h}_i^2 v_i(\boldsymbol{x})$ are introduced for (14), where $v_i(\boldsymbol{x})\in\mathcal{S}$. A sufficient condition for satisfying (14) can be expressed as

$$\Big\{\boldsymbol{x}\in\mathbb{R}^n \mid \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}\hat{h}_i^2\boldsymbol{B}_{ij}(\boldsymbol{x}) = 0, \quad j=1,2,\cdots,q, \quad \sum_{i=1}^{r}\hat{h}_i^2 - 1 = 0, \sum_{i=1}^{r}\hat{h}_i^2\Big[\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) - \gamma v_i(\boldsymbol{x}))\Big] \geq 0, \quad \sum_{i=1}^{r}\hat{h}_i^2 l_{xi}(\boldsymbol{x}) \neq 0\Big\} = \varnothing. \quad (15)$$

An SOS condition can be derived from (15) by applying Lemma 1 to the condition, where $f_1(\boldsymbol{x})$, $g_1(\boldsymbol{x})$, $(h_j(\boldsymbol{x}))_{j=1,\cdots,q}$, $h_{q+1}(\boldsymbol{x})$ of (6) in the P-satz correspond to

$$\sum_{i=1}^{r}\hat{h}_i^2\Big[\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x} - \gamma v_i(\boldsymbol{x}))\Big],$$
$$\sum_{i=1}^{r}\hat{h}_i^2 l_{xi}(\boldsymbol{x}),$$
$$\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}\hat{h}_i^2\boldsymbol{B}_{ij}(\boldsymbol{x}),$$
$$\sum_{i=1}^{r}\hat{h}_i^2 - 1,$$

in (15), respectively. Accordingly, by applying Lemma 1 to (15), the condition can be rewritten as:

$$s_1(\boldsymbol{h},\boldsymbol{x}) + s_2(\boldsymbol{h},\boldsymbol{x})\sum_{i=1}^{r}\hat{h}_i^2\Big[\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) - \gamma v_i(\boldsymbol{x})\Big] + \sum_{j=1}^{q}p_{1j}(\boldsymbol{h},\boldsymbol{x})\Big[\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}\hat{h}_i^2\boldsymbol{B}_{ij}(\boldsymbol{x})\Big] + p_2(\boldsymbol{h},\boldsymbol{x})\Big[\sum_{i=1}^{r}\hat{h}_i^2 - 1\Big] + \Big[\sum_{i=1}^{r}l_{xi}(\boldsymbol{x})\Big]^2 = 0, \quad (16)$$

where $s_1(\boldsymbol{h},\boldsymbol{x})$, $s_2(\boldsymbol{h},\boldsymbol{x})$, $v_i(\boldsymbol{x})\in\mathcal{S}$, $p_{1j}(\boldsymbol{h},\boldsymbol{x})$, $p_2(\boldsymbol{h},\boldsymbol{x})\in\mathcal{P}$, $l_{xi}(\boldsymbol{x})\in\mathcal{S}^+$, and γ is a non-positive real number. In order to simplify (16), $\Big[\sum_{i=1}^{r}l_{xi}(\boldsymbol{x})\Big]^2$ is written as $\sum_{i=1}^{r}\sum_{m=1}^{r}\hat{h}_i^2\hat{h}_m^2 l_{xi}(\boldsymbol{x})l_{xm}(\boldsymbol{x})$, then we represent $l_{xi}(\boldsymbol{x})l_{xm}(\boldsymbol{x}) = l_{2im}(\boldsymbol{x})$, where note that $l_{2im}(\boldsymbol{x})\in\mathcal{S}^+$. Moreover, we choose $p_{1j}(\boldsymbol{h},\boldsymbol{x}) = \sum_{m=1}^{r}\hat{h}_m^2 p_{1jm}(\boldsymbol{x})$, $p_2(\boldsymbol{h},\boldsymbol{x}) = \sum_{m=1}^{r}\hat{h}_m^2 p_{2m}(\boldsymbol{x})$, and $s_2(\boldsymbol{h},\boldsymbol{x}) = \sum_{m=1}^{r}\hat{h}_m^2 s_m(\boldsymbol{x})$, where $s_m(\boldsymbol{x})\in\mathcal{S}$, $p_{1jm}(\boldsymbol{x})$, $p_{2m}(\boldsymbol{x})\in\mathcal{P}$. Consequently, the sufficient condition of (16) is expressed as follows:

$$-\sum_{i=1}^{r}\sum_{m=1}^{r}\hat{h}_i^2\hat{h}_m^2\Big\{s_m(\boldsymbol{x})\Big[\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) - \gamma v_i(\boldsymbol{x})\Big] + \sum_{j=1}^{q}p_{1jm}(\boldsymbol{x})\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{B}_{ij}(\boldsymbol{x}) + p_{2m}(\boldsymbol{x}) + l_{2im}(\boldsymbol{x})\Big\} + \sum_{m=1}^{r}\hat{h}_m^2 p_{2m}(\boldsymbol{x}) \in \mathcal{S}. \quad (17)$$

From (16), note that (17) is equivalent to $s_1(\boldsymbol{h},\boldsymbol{x})\in\mathcal{S}$, i.e., (17) should be an SOS.

Therefore, if there exist a smooth radially unbounded function $V(\boldsymbol{x})$, SOS polynomials $s_m(\boldsymbol{x})$, $v_i(\boldsymbol{x})$, positive definite SOS polynomials $l_{2im}(\boldsymbol{x})$ and polynomials $p_{1jm}(\boldsymbol{x})$, $p_{2m}(\boldsymbol{x})$ that satisfy (8) and (9) for a non-positive γ, then (11) is satisfied, that is, the $V(\boldsymbol{x})$ is a CLF and (3) is feedback stabilizable. □

Remark 1. *A key idea in the proof of Theorem 1 is that we guarantees (12) by ensuring (13). In other words, (12) is satisfied if (13) is satisfied. The empty set condition (14) (to find a CLF) derived from (13) is more suitable for applying the P-satz framework and for deriving the SOS condition (17) (,i.e., (9)). In the SOS design problem given in Theorem 1 (, i.e., (9)), the non-negative variables $\hat{h}_i^2$ and $\hat{h}_m^2$ are treated as just variables like $\boldsymbol{x}$. If we do not utilize the replacement, the empty set condition (to find a CLF) derived from (12) cannot be transformed into SOS conditions by applying the P-satz framework since the membership functions h_i are not strictly polynomials.*

A control Lyapunov function construction, that is, solving the stabilization conditions in Theorem 1, can be efficiently carried out by using the recent-developed frameworks as in a number of literature [13], [34], [36], [38]–[42], [44], [45], [47], and so on. Since the algorithm itself is not a main contribution of this paper, it will not be repeatedly presented and, for more details, see them. The design examples in this paper utilize the recent-developed algorithm presented in [45]. To obtain more relaxed stability results, the co-positive relaxation technique is utilized in [13], [38], [44], [45]. However, we note that the co-positive relaxation technique is not useful for (17) since (17) itself is required to be an SOS.

B. Design Example I

Subsection III-B shows the results of applying the stabilization conditions to a benchmark design example. The benchmark design example utilizes the following three-rules T-S fuzzy model in the form of (2) with constant system matrices and $\hat{\boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{x}$:

$$\begin{aligned}&\textit{Model Rule } i\textit{:}\\&\text{IF } x_1 \text{ is } M_{i1},\\&\text{THEN} \quad \dot{\boldsymbol{x}} = \boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{B}_i u, \quad i = 1,2,3,\end{aligned} \tag{18}$$

where $\boldsymbol{x}^T = [x_1 \ x_2]$,

$$\boldsymbol{A}_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\boldsymbol{A}_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix},$$
$$\boldsymbol{A}_3 = \begin{bmatrix} -a & -4.33 \\ 0 & 0.05 \end{bmatrix}, \quad \boldsymbol{B}_3 = \begin{bmatrix} -b+6 \\ -1 \end{bmatrix}.$$

The membership functions are given as

$$h_1(x_1) = \frac{\cos(10x_1)+1}{4},$$
$$h_2(x_1) = \frac{\sin(10x_1)+1}{4},$$
$$h_3(x_1) = \frac{-\cos(10x_1)-\sin(10x_1)+2}{4}.$$

TABLE I: Comparison of b_{max}.

Method	b_{max}
Theorem 1 (8th order CLF)	8.5
Theorem 1 (6th order CLF)	8.5
Theorem 1 (4th order CLF)	8
Theorem 1 (2nd order CLF)	6.5
Y. Chen, et al. [13]	6.5
A. Sala, et al. [14]	6.5
V. F. Montagner, et al. [18]	6.5
F. Delmotte et, al. [16]	6
C. H. Fang, et al. [11]	6
M. C. M. Teixeira, et al. [9]	6
X. Liu, et al. [10]	2.5
E. Kim, et al. [7]	1

In most of cases [7], [9]–[11], [13], [14], [16], [18], [46], $a = 2$ is set to this benchmark model and b_{max}, i.e., the maximum value of b such that the proposed design conditions are feasible, is compared with other results. In this example, we also set $a = 2$ and find the maximum value of b (b_{max}) in the discrete range $0 \le b \le 9$ with interval 0.5. As a result of solving the stabilization conditions of Theorem 1 for 2nd, 4th, 6th, and 8th order CLF ($\deg[V(\boldsymbol{x})] = 2,4,6,8$), we obtain $b_{max} = 6.5$ for $\deg[V(\boldsymbol{x})] = 2$, $b_{max} = 8$ for $\deg[V(\boldsymbol{x})] = 4$, and $b_{max} = 8.5$ for $\deg[V(\boldsymbol{x})] = 6,8$. The result shows that increasing the order of CLF improves the stabilizability range that can be guaranteed by the derived conditions. Table I summarizes the comparison of b_{max} obtained by Theorem 1 with the other methods when $a = 2$. The results show the utility of our approach.

Next, we investigate the feasible region for Theorem 1 and compare our feasible region with that obtained in [14]. Fig. 1 shows the comparison result. Since the plot marks (×, +, ∘, □) cannot be simultaneously plotted for the overlapped regions, note that the plot mark of the smaller region is plotted in Fig. 1. Hence, the region plot in Fig. 1 means that

× (2nd order CLF) ⊂ + (4th order CLF) ⊂ ∘ (6th order CLF) ⊂ □ (8th order CLF),

where the (inclusion) symbol ⊂ in set theory is used to intuitively compare the feasible regions, for instance, '× (2nd order CLF) ⊂ + (4th order CLF)' means that the feasible region for the 4th order CLF includes that of the 2nd order CLF, that is, the feasible region for the 4th order CLF is larger than that of the 2nd order CLF.

The proposed approach (Theorem 1) with 2nd order CLF has the same feasible region with [14], and as we increase the order of CLF, we obtain larger feasible regions than that of the 2nd order CLF.

For $\deg[V(\boldsymbol{x})] = 6$, $a = 2$, and $b = 8.5$, by solving SOS conditions in Theorem 1, we obtain

$$\begin{aligned}V(\boldsymbol{x}) = {} & 0.0018x_1^6 - 0.013x_1^5x_2 + 0.084x_1^4x_2^2 - 0.22x_1^3x_2^3 \\ & + 0.96x_1^2x_2^4 + 1.5x_1x_2^5 + 3.0x_2^6\end{aligned}$$

Fig. 2 shows the behavior in $x_1 - x_2$ plane with $u = 0$. Note that the system is not globally and asymptotically stable. Fig. 3 shows the behavior of the closed-loop system when

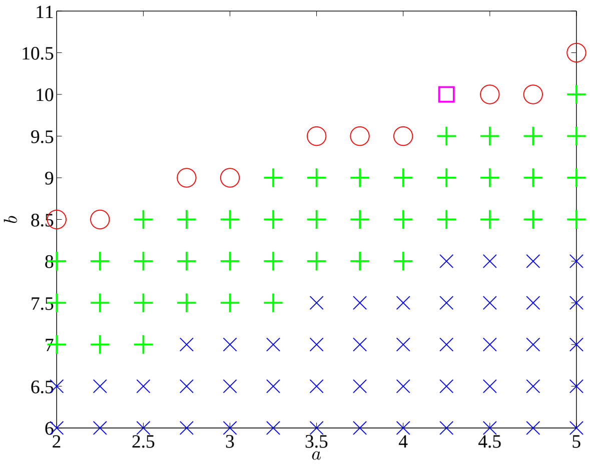

Fig. 1: Feasible region comparison ($\times$ for 2nd order CLF and [14], $+$ for 4th order CLF, $\circ$ for 6th order CLF and $\square$ for 8th order CLF).

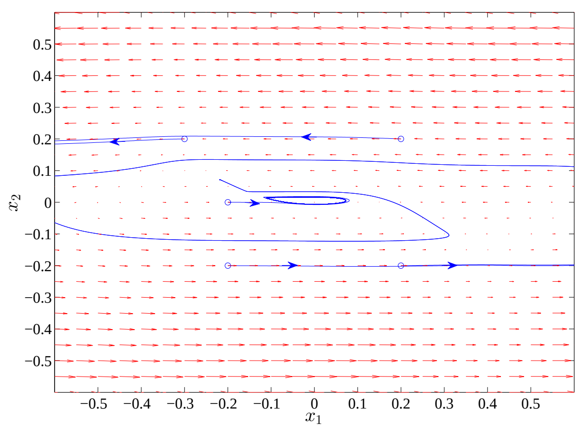

Fig. 2: Behavior in $x_1 - x_2$ plane ($a = 2$ and $b = 8.5$).

the obtained CLF is applied to the Sontag's formula (10) to design the controller. We can see that the designed controller stabilizes the system even in $a = 2$ and $b = 8.5$ that is an infeasible point for other methods [7], [9]–[11], [13], [14], [16], [18], [46]. Thus, the closed-loop system (Fig. 3) is globally and asymptotically stable although the open-loop system (Fig. 2) is not globally and asymptotically stable.

C. Design Example II

Consider the following three-rule polynomial fuzzy model with $\hat{\boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{x}$ [26]:

Model Rule i:
IF x_1 is M_{i1}
THEN $\dot{\boldsymbol{x}} = \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{x})u, \quad i = 1, 2, 3,$ (19)

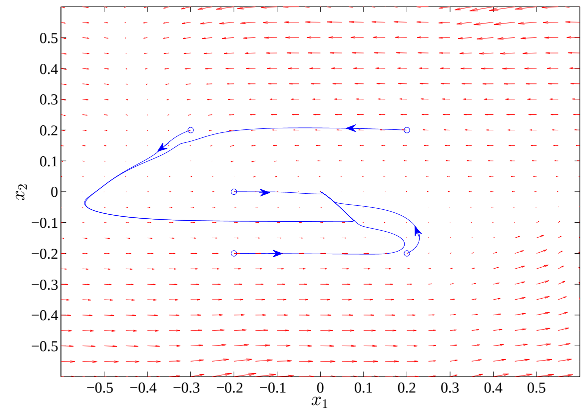

Fig. 3: Behavior in $x_1 - x_2$ plane with feedback (6th order CLF) ($a = 2$ and $b = 8.5$).

where $\boldsymbol{x}^T = [x_1 \ x_2]$,

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} 1.59 + x_1^2 - 2x_2^2 - x_1x_2 & -7.29 + 2x_1x_2 \\ 0.01 & -x_1^2 - x_2^2 \end{bmatrix},$$

$$\boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} 0.02 + x_1^2 - 2x_2^2 - x_1x_2 & -4.64 + 2x_1x_2 \\ 0.35 & 0.21 - x_1^2 - x_2^2 \end{bmatrix},$$

$$\boldsymbol{A}_3(\boldsymbol{x}) = \begin{bmatrix} -a + x_1^2 - 2x_2^2 - x_1x_2 & -4.33 + 2x_1x_2 \\ 0 & 0.05 - x_1^2 - x_2^2 \end{bmatrix},$$

$$\boldsymbol{B}_1(\boldsymbol{x}) = \begin{bmatrix} 1 + x_1 + x_1^2 \\ 0 \end{bmatrix},$$

$$\boldsymbol{B}_2(\boldsymbol{x}) = \begin{bmatrix} 8 + x_1 + x_1^2 \\ 0 \end{bmatrix},$$

$$\boldsymbol{B}_3(\boldsymbol{x}) = \begin{bmatrix} -b + 6 + x_1 + x_1^2 \\ -1 \end{bmatrix},$$

with a and b are constant parameters, and the membership functions are given as

$$h_1(x_1) = \frac{1}{1 + e^{(125x_1+12)/2}},$$
$$h_2(x_1) = \frac{1}{1 + e^{-(125x_1-12)/2}},$$
$$h_3(x_1) = 1 - h_1(x_1) - h_2(x_1).$$

In order to compare feasible regions, we apply Theorem 1 to all points at $2 \le a \le 9$ and $0 \le b \le 9$, with 4th, 6th, and 8th order CLF ($\deg[V(\boldsymbol{x})] = 2, 4, 6, 8$). Fig. 4 shows the comparison of feasible regions of Theorem 1 and the existing SOS approach [26].

As explained in Design Example I, since the plot marks ($\times$, $+$, $\circ$, $\square$, $\diamond$) cannot be simultaneously plotted for the overlapped regions, note that the plot mark of the smaller region is plotted in Fig. 4. Hence, the region plot in Fig. 4 means that

$\times$ [26] $\subset$ $+$ (2nd order CLF) $\subset$ $\circ$ (4th order CLF) $\subset$ $\square$ (6th order CLF) $\subset$ $\diamond$ (8th order CLF).

As seen in Fig. 4, the proposed approach has larger feasible regions than the feasible region in [26], and higher order CLF provides more relax results (, that is, larger feasible regions).

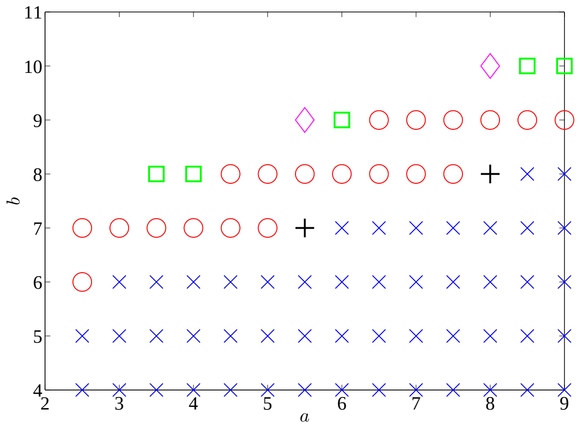

Fig. 4: Feasible regions for Theorem 1 represented by + (2nd order CLF), ○ (4th order CLF), □ (6th order CLF), ◇ (8th order CLF) and [26] represented by ×.

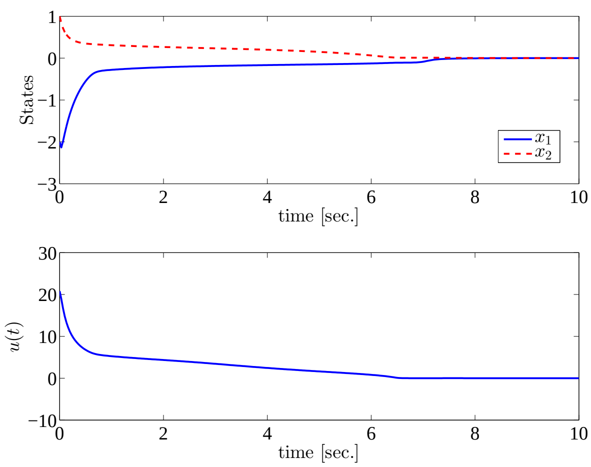

Fig. 5: Control result ($a = 5.5$ and $b = 7$).

For $a = 5.5$ and $b = 7$ that is an infeasible point of the existing SOS approach [26], by solving Theorem 1 with even 2nd order CLF, we obtain

$$V(\boldsymbol{x}) = 0.35x_1^2 + 0.42x_1x_2 + 2.6x_2^2.$$

Fig. 5 shows the control result by the proposed approach, where $x(0) = [-2 \quad 1]^T$. The designed controller stabilizes the system.

The stabilization condition is also feasible in the case of 8th order CLF for a=5.5 and b=9. The Lyapunov function is obtained as

$$\begin{aligned} V(\boldsymbol{x}) = {} & 0.00078x_1^8 - 0.0085x_1^7x_2 + 0.059x_1^6x_2^2 \\ & - 0.27x_1^5x_2^3 + 1.1x_1^4x_2^4 - 1.9x_1^3x_2^5 + 5.3x_1^2x_2^6 \\ & + 9.6x_1x_2^7 + 17x_2^8. \end{aligned} \tag{20}$$

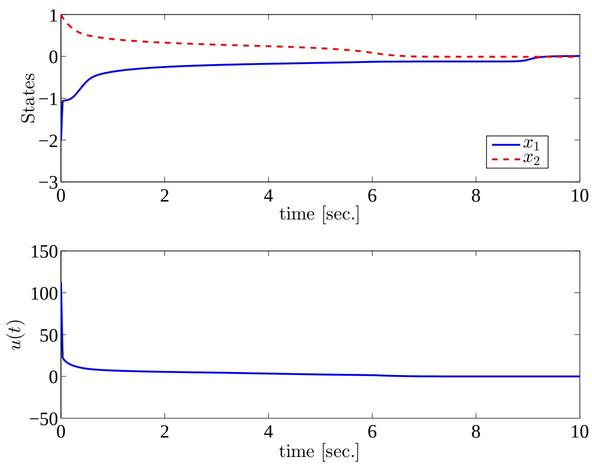

Fig. 6: Control result ($a = 5.5$ and $b = 9$).

Fig. 6 shows the control result by the proposed approach, where $x(0) = [-2 \quad 1]^T$ for $a = 5.5$ and $b = 9$.

D. Design Example III

Consider the following polynomial fuzzy model [25]:

Model Rule i:
IF x_1 is M_{i1}
THEN $\dot{\boldsymbol{x}} = \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{x})u, \quad i = 1, 2, 3,$ (21)

where $\boldsymbol{x}^T = [x_1 \ x_2]$,

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ -a & -6 \end{bmatrix},$$
$$\boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ 0 & -6 \end{bmatrix},$$
$$\boldsymbol{A}_3(\boldsymbol{x}) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ 0.2172a & -6 \end{bmatrix},$$
$$\boldsymbol{B}_1(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ b \end{bmatrix},$$
$$\boldsymbol{B}_2(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ b \end{bmatrix},$$
$$\boldsymbol{B}_3(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ b \end{bmatrix},$$

with a and b are constant parameters, and the membership functions are

$$h_1(x_1) = \frac{1}{1 + e^{(x_1+4)/2}},$$
$$h_3(x_1) = \frac{1}{1 + e^{-(x_1-4)/2}},$$
$$h_2(x_1) = 1 - h_1(x_1) - h_3(x_1).$$

In order to compare feasible regions with the existing SOS approaches, the SOS stabilization conditions in [22] and [25] are employed for $2 \le a \le 9$ and $7.5 \le b \le 15$. Fig. 7 shows the feasibility regions of the proposed approach (Theorem 1) with 2nd order CLF and the existing approaches [22] and [25].

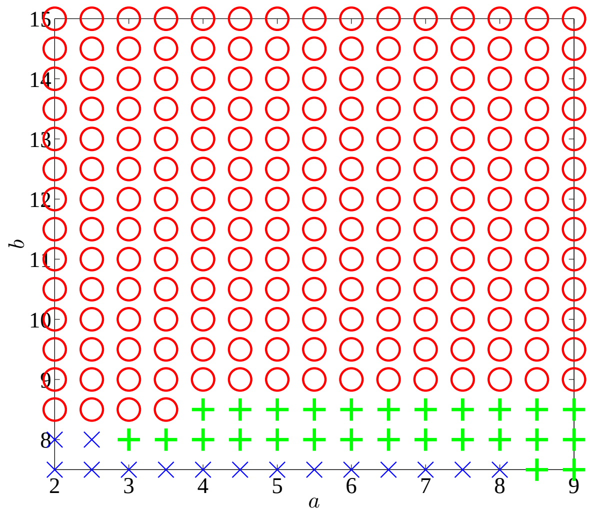

Fig. 7: Feasible region comparison, × for [22], + for [25], ○ for Theorem 1.

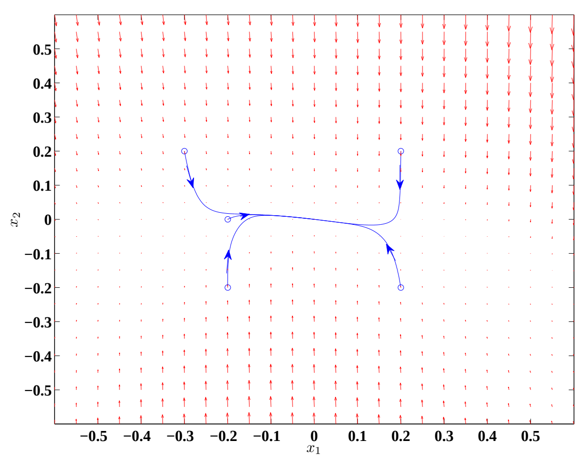

Fig. 8: Behavior of closed loop system ($a = 9$ and $b = 15$) .

Once again, as explained in Design Examples I and II, the region plot in Fig. 7 means that
× [22] ⊂ + [25] ⊂ ○ (Theorem 1).

As shown in Fig. 7, the proposed approach obtains more relaxed result than the existing SOS approaches even though in our approach we do not consider the shape of the membership function as in [25].

By applying Theorem 1 to the system with $a = 9$ and $b = 15$ that are an infeasible point for the existing SOS approaches [22] and [25], we still obtain feasible solutions. Fig. 8 shows the behavior of the closed loop system by the designed controller. As shown in Fig. 8, the system is stabilized by the designed controller.

Remark 2. *It has been reported in [25]–[30] that the polynomial SOS approaches provide more relaxed stability results than the existing LMI approaches. Furthermore, in the exact fuzzy model construction using the well-known sector nonlinearity [2], if a system with polynomial terms is represented by a T-S fuzzy model, the gradient (slope) of at least one of the sectors in the T-S fuzzy model generally becomes infinity. The infinity-gradient sector causes uncontrollability. To avoid it, in most of these cases, a T-S fuzzy model for a system with polynomial terms is constructed for a range of operation domains. The T-S fuzzy model can exactly represent the nonlinear system (with polynomial terms) only on the considered operation domain. On the other hand, the polynomial fuzzy model can exactly and globally represent a system with polynomial terms. This means that the polynomial fuzzy model approach guarantees the global stability even for a system with polynomial terms. This is an advantage of our approach. The polynomial fuzzy model (21) given in [25] can be regarded as being constructed from the following nonlinear system:*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ (-z_1(x_1) + 0.2172z_2(x_1))a & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ b \end{bmatrix} u, \tag{22}$$

where

$$z_1(x_1) = \frac{1}{1 + e^{(x_1+4)/2}}, \; z_2(x_1) = \frac{1}{1 + e^{-(x_1-4)/2}}.$$

Then, by applying the sector nonlinearity to the nonlinear system (22), we obtain the polynomial fuzzy model, where $h_1(x_1) = z_1(x_1)$, $h_3(x_1) = z_2(x_1)$, and $h_2(x_1) = 1 - z_1(x_1) - z_2(x_1)$. To construct the T-S fuzzy model for the nonlinear system (22), we need to define the operation domain by determining their domain ranges, i.e., $x_1 \in [-d_1 \; d_1]$ and $x_2 \in [-d_2 \; d_2]$ (d_1 and d_2 are positive values) like an example in [22], etc. In this case, the obtained T-S fuzzy model is a local model for the original nonlinear system. Thus, since it is difficult to exactly construct a global T-S fuzzy model for a nonlinear system with polynomial terms such as the nonlinear system (22), the global stability of a nonlinear system with polynomial terms cannot be generally guaranteed in T-S fuzzy model-based control.

E. Design Example IV

Consider the following polynomial fuzzy model with three state variables and two inputs.

Model Rule i:
IF x_1 is M_{i1}
THEN $\dot{\boldsymbol{x}} = \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{u}, \quad i = 1, 2, 3,$ (23)

where $\boldsymbol{x}^T = [x_1 \; x_2 \; x_3]$ and $\boldsymbol{u}^T = [u_1 \; u_2]$,

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 & 1 \\ -a & -6 & x_3 \\ 0 & 1 & -1 + x_3 + x_3^2 \end{bmatrix},$$

$$\boldsymbol{A}_2(\boldsymbol{x}) =$$

$$\begin{bmatrix} -1+x_1+x_1^2+x_1x_2-x_2^2 & 1 & 1 \\ 0 & -6 & x_3 \\ 0 & 1 & -1+x_3+x_3^2 \end{bmatrix},$$

$$\boldsymbol{A}_3(\boldsymbol{x}) =$$

$$\begin{bmatrix} -1+x_1+x_1^2+x_1x_2-x_2^2 & 1 & 1 \\ 0.2172a & -6 & x_3 \\ 0 & 1 & -1+x_3+x_3^2 \end{bmatrix},$$

$$\boldsymbol{B}_1(\boldsymbol{x}) = \begin{bmatrix} x_1 & 1 \\ b & 1 \\ 0 & x_3 \end{bmatrix},$$

$$\boldsymbol{B}_2(\boldsymbol{x}) = \begin{bmatrix} x_1 & 1 \\ b & 1 \\ 0 & x_3 \end{bmatrix},$$

$$\boldsymbol{B}_3(\boldsymbol{x}) = \begin{bmatrix} x_1 & 1 \\ b & 1 \\ 0 & x_3 \end{bmatrix},$$

with a and b are constant parameters, and the membership functions are

$$h_1(x_1) = \frac{1}{1+e^{(x_1+4)/2}},$$
$$h_3(x_1) = \frac{1}{1+e^{-(x_1-4)/2}},$$
$$h_2(x_1) = 1 - h_1(x_1) - h_3(x_1).$$

We set $a = 2$ and find the maximum value of b (b_{max}) in the discrete range $0 \le b \le 7$ with interval 0.5. The SOS stabilization conditions in [13], which is one of the best results excepting our approach (see Table I), find feasible solutions for $b_{max} = 3.5$ with 2nd order CLF when $a = 2$. Our approach obtains feasible solutions for $b_{max} = 6.5$ with 2nd order CLF. Thus, the proposed approach obtains more relaxed results than the existing SOS approach [13].

IV. Semi-Global Stabilization on Operation Domain

Section III has provided global stabilization for nonlinear systems represented by polynomial fuzzy models. In some cases, it is still a strict requirement to globally stabilize complicated nonlinear systems. Section IV presents semi-global stabilization on operation domain. It is useful for complicated nonlinear systems that are not globally stabilizable.

A. Semi-Global Stabilization Condition

Consider the operation domain

$$D = \{\boldsymbol{x} : x_\xi^{min} \le x_\xi \le x_\xi^{max}, \quad \xi = 1, 2, \cdots, \phi,\ \phi \le n\}, \tag{24}$$

that includes $\boldsymbol{x} = 0$ of system (3). Theorem 2 provides an SOS condition to obtain the CLF level set $\Omega_{V,\alpha} = \{\boldsymbol{x} \in \mathbb{R}^n : V(\boldsymbol{x}) \le \alpha\}$ in the operation domain D.

Theorem 2. *If there exist a smooth function $V(\boldsymbol{x})$, SOS polynomials $s_{10im}(\boldsymbol{x})$, $s_{11m}(\boldsymbol{x})$, $Q_\xi(\boldsymbol{x})$, $v_i(\boldsymbol{x})$, positive definite SOS polynomials $l_{5im}(\boldsymbol{x})$ and polynomials $p_{jm}(\boldsymbol{x})$ that satisfy (8) and (25) for a non-positive γ, the outmost CLF level set $\Omega_{V,\alpha}$ inside D is contractively invariant sets of (3).*

$$\sum_{m=1}^{r}\sum_{i=1}^{r}\hat{h}_i^2\hat{h}_m^2\Lambda_{im} + \sum_{m=1}^{r}\hat{h}_m^2 p_{2m}(\boldsymbol{x}) \in \mathcal{S}, \tag{25}$$

where $\hat{h}_i^2$ is a non-negative value as addressed in Remark 1.

$$\begin{aligned}\Lambda_{im} = -\Big\{&-s_{10im}(\boldsymbol{x})\sum_{\xi=1}^{\phi}Q_\xi(\boldsymbol{x})(x_\xi - x_\xi^{min})(x_\xi - x_\xi^{max}) \\ &+ s_{11m}(\boldsymbol{x})\Big[\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) - \gamma v_i(\boldsymbol{x})\Big] \\ &+ \sum_{j=1}^{q}p_{1jm}(\boldsymbol{x})\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{B}_{ij}(\boldsymbol{x}) + p_{2m}(\boldsymbol{x}) + l_{5im}(\boldsymbol{x})\Big\}.\end{aligned}$$

Moreover, the feedback controller can be constructed by applying (10).

Proof. For the operation domain (24), the following inequality holds:

$$\sum_{\xi=1}^{\phi}Q_\xi(\boldsymbol{x})(x_\xi - x_\xi^{min})(x_\xi - x_\xi^{max}) \le 0 \tag{26}$$

where $Q_\xi(\boldsymbol{x}) \ge 0$ that is achieved by restricting $Q_\xi(\boldsymbol{x})$ to be SOS polynomials. By introducing the non-negative variables (quadratic variables) $\hat{h}_i^2$ as in Theorem 1, we consider that when (26) holds and

$$\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}\hat{h}_i^2\boldsymbol{B}_{ij}(\boldsymbol{x}) = 0, \quad j = 1, \cdots, q,$$

we have

$$\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}\hat{h}_i^2\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) < 0,$$

where $\sum_{i=1}^{r}\hat{h}_i^2 = 1$. The above conditions can be expressed as

$$\begin{aligned}\Big\{&\boldsymbol{x} \in \mathbb{R}^n | \sum_{\xi=1}^{\phi}Q_\xi(\boldsymbol{x})(x_\xi - x_\xi^{min})(x_\xi - x_\xi^{max}) \le 0, \\ &\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}\hat{h}_i^2\boldsymbol{B}_{ij}(\boldsymbol{x}) = 0, j = 1, \cdots, q, \sum_{i=1}^{r}\hat{h}_i^2 = 1, \\ &\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}\hat{h}_i^2\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) < 0, \boldsymbol{x} \ne 0\Big\}\end{aligned} \tag{27}$$

Note that (27) is a subeset of the global one, that is,

$$\begin{aligned}\Big\{&\boldsymbol{x} \in \mathbb{R}^n | \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}\hat{h}_i^2\boldsymbol{B}_{ij}(\boldsymbol{x}) = 0, j = 1, \cdots, q, \\ &\sum_{i=1}^{r}\hat{h}_i^2 = 1, \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\sum_{i=1}^{r}\hat{h}_i^2\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) < 0, \boldsymbol{x} \ne 0\Big\}.\end{aligned}$$

Here, (27) can be rewritten as the following empty set condition:

$$\Big\{\boldsymbol{x} \in \mathbb{R}^n | \sum_{\xi=1}^{\phi}Q_\xi(\boldsymbol{x})(x_\xi - x_\xi^{min})(x_\xi - x_\xi^{max}) \le 0,$$

$$\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \sum_{i=1}^{r} \hat{h}_i^2 \boldsymbol{B}_{ij}(\boldsymbol{x}) = 0, j = 1, \cdots, q, \sum_{i=1}^{r} \hat{h}_i^2 - 1 = 0,$$
$$\left.\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \sum_{i=1}^{r} \hat{h}_i^2 \boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \geq 0, \boldsymbol{x} \neq 0\right\} = \varnothing. \quad (28)$$

In order to apply the SOS optimization technique, as in the derivation of the global conditions, we introduce $l_x(\boldsymbol{h}, \boldsymbol{x})$ to replace $\boldsymbol{x} \neq 0$ where $l_x(\boldsymbol{h}, \boldsymbol{x}) \neq 0$ iff $\boldsymbol{x} \neq 0$. In addition, we also introduce a non-positive real number γ and an SOS polynomial $v_i(\boldsymbol{h}, \boldsymbol{x})$ to the condition for the sake of applying the SOS optimization technique. The condition is now

$$\left\{\boldsymbol{x} \in \mathbb{R}^n | \sum_{\xi=1}^{\phi} Q_\xi(\boldsymbol{x})(x_\xi - x_\xi^{min})(x_\xi - x_\xi^{max}) \leq 0,\right.$$
$$\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \sum_{i=1}^{r} \hat{h}_i^2 \boldsymbol{B}_{ij}(\boldsymbol{x}) = 0, j = 1, \cdots, q, \sum_{i=1}^{r} \hat{h}_i^2 - 1 = 0$$
$$\sum_{i=1}^{r} \hat{h}_i^2 \Big[\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) - \gamma v_i(\boldsymbol{x})\Big] \geq 0,$$
$$\left. l_x(\boldsymbol{h}, \boldsymbol{x}) \neq 0\right\} = \varnothing. \quad (29)$$

Here, we apply the Positivstellensatz (Lemma 1) to (29) in the same fashion as in the proof of Theorem 1. Thus, we can construct a semi-global CLF by solving the sufficient conditions below.

$$\text{Find } V(\boldsymbol{x}) \text{ such that (8) and (25)}$$

for a non-positive γ, where $s_{10im}(\boldsymbol{x})$, $s_{11m}(\boldsymbol{x})$, $v_i(\boldsymbol{x})$, $Q_\xi(\boldsymbol{x}) \in \mathcal{S}$, $V(\boldsymbol{x})$, $l_1(\boldsymbol{x})$, $l_{5im}(\boldsymbol{x}) \in \mathcal{S}^+$ and $p_{1jm}(\boldsymbol{x})$, $p_{2m}(\boldsymbol{x}) \in \mathcal{P}$.

Therefore, if there exist $V(\boldsymbol{x})$, $s_{10im}(\boldsymbol{x})$, $s_{11m}(\boldsymbol{x})$, $v_i(\boldsymbol{x})$, $Q_\xi(\boldsymbol{x})$, $l_{5im}(\boldsymbol{x})$, $p_{1jm}(\boldsymbol{x})$, and $p_{2m}(\boldsymbol{x})$ that satisfy (8) and (25) for a non-positive γ, then (27) is satisfied, that is the outmost CLF $V(\boldsymbol{x})$ level set contained in D is contractively invariant sets of (3). □

Furthermore, the largest domain of attraction ($\Omega_{V,\alpha}$), inside the considered operation domain can be determined by solving the following optimization problem using the obtained semi-global CLF ($V(\boldsymbol{x})$)

$$\max_{\Phi} \alpha \quad \text{subject to}$$
$$\Phi(V(\boldsymbol{x}) - \alpha) - (x_\xi - x_\xi^{min})(x_\xi - x_\xi^{max}) \in \mathcal{S},$$
$$\xi = 1, 2, \cdots, \phi, \quad \phi \leq n. \quad (30)$$

where $\Phi > 0$. This condition can be obtained by applying the S-procedure (Lemma 2).

B. Design Example V

Consider the following operation domain for the fuzzy model of Section III-C:

$$D = \{x : -d \leq x_1 \leq d\}, \quad (31)$$

where $x_\xi^{max} = d$, $x_\xi^{min} = -d$ and $d = 0.2$.

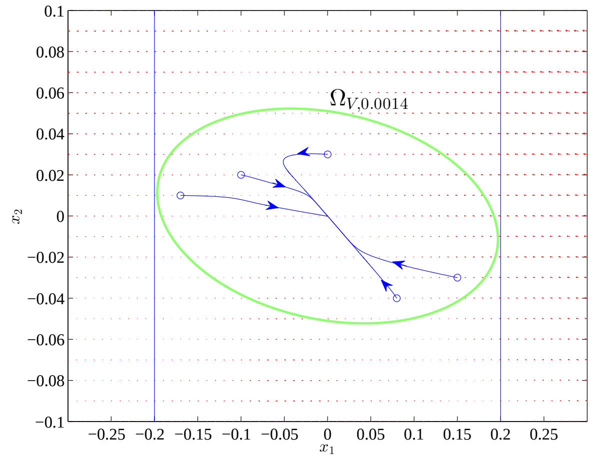

Fig. 9: States behavior ($a = 6$ and $b = 8$) within the domain of attraction inside D (Lyapunov function level set $\Omega_{V,0.0014}$).

1) Second order CLF: First, we consider 2nd order CLF for Theorem 2. We focus on an infeasible point ($a = 6$ and $b = 8$) for the global stabilization condition (Theorem 1).

By applying Theorem 2 to the fuzzy model with $a = 6$ and $b = 8$ that are an infeasible point for the global stabilization condition (Theorem 1), we get

$$V(\boldsymbol{x}) = 0.039x_1^2 + 0.063x_1x_2 + 0.55x_2^2. \quad (32)$$

Thus, Theorem 2 provides more relaxed results compared to Theorem 1.

Furthermore, to obtain the largest domain of attraction inside D, we solve (30) and obtain

$$\alpha = 0.0015,$$
$$\Phi = 27.$$

Fig. 9 shows that all points within the level set $\Omega_{V,\alpha}$ is stabilized to the equilibrium point.

2) High order CLF: Next, we consider 4th, 6th, and 8th order CLF for $a = 5$ and $b = 9$ that are an infeasible point even for the 2nd order semi-global CLF (32). For 4th, 6th and 8th order CLFs, we serach the maximum values of d satisfying Theorem 2 by solving (30). The maximum d is obtained as $d_4 = 0.3$ $d_6 = d_8 = 0.5$ for 4th, 6th and 8th order CLFs, respectively.

Fig. 10 shows the largest domain of attractions (inside D) of 4th, 6th 8th order CLFs for $a = 5$ and $b = 9$ that are an infeasible point even for the 2nd order semi-global CLF (32), where the following CLFs are obtained.

[4th order CLF ($d_4 = 0.3$)]

$$V(\boldsymbol{x}) = 0.00088x_1^4 - 0.0047x_1^3x_2 + 0.027x_1^2x_2^2 + 0.039x_1x_2^3 + 0.24x_2^4$$

[6th order CLF ($d_6 = 0.5$)]

$$V(\boldsymbol{x}) = 0.00071x_1^6 - 0.0057x_1^5x_2 + 0.037x_1^4x_2^2 - 0.12x_1^3x_2^3 + 0.46x_1^2x_2^4 + 0.76x_1x_2^5 + 1.9x_2^6$$

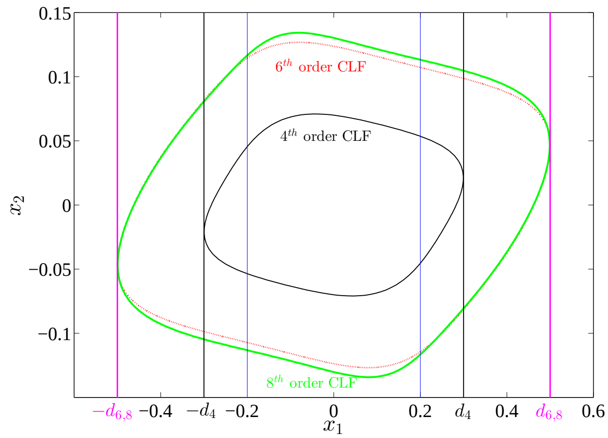

Fig. 10: CLF level set comparison at $a = 5$ and $b = 9$.

[8th order CLF ($d_8 = 0.5$)]

$$\begin{aligned} V(\boldsymbol{x}) &= 0.0023x_1^8 - 0.025x_1^7x_2 + 0.20x_1^6x_2^2 - 0.83x_1^5x_2^3 \\ &\quad + 3.3x_1^4x_2^4 - 6.7x_1^3x_2^5 + 22x_1^2x_2^6 + 37x_1x_2^7 \\ &\quad + 57x_2^8 \end{aligned}$$

It can be seen from Fig. 10 that increasing the order of CLF enlarges the domain of attraction.

V. Conclusion

This paper has dealt with an SOS-based CLF design for polynomial fuzzy control of nonlinear systems. Next, global stabilization conditions represented in terms of SOS have been provided in the framework of the CLF design. Furthermore, semi-global stabilization conditions on operation domains have been derived in the same fashion as in the global stabilization conditions. Both global and semi-global stabilization problems have been formulated as SOS optimization problems which reduce to numerical feasibility problems. Five design examples have been given to show the effectiveness of our proposed approach over the existing linear matrix inequality (LMI) and SOS approaches.

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