A Piecewise Polynomial Lyapunov Function Approach to State Feedback and Estimation for a Class of Nonlinear Systems

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Abstract

The well-known Takagi-Sugeno (T-S) fuzzy systems has attracted many attentions for its capability to represent a large class of nonlinear systems. A wide range of applications, e.g. robotic systems, aircraft systems, etc, have also adopted the T-S fuzzy systems to approximate the nonlinear systems in terms of a set of fuzzy IF-THEN rules. Recently, a so-called polynomial fuzzy model has been proposed as an extension of the T-S fuzzy model. By allowing polynomial expression in the state or the input variables, polynomial fuzzy systems has tighter sectors compared to T-S fuzzy systems. Moreover, this reduces fuzzy rules number. These merits lead to the relaxation of polynomial fuzzy systems stability conditions.

Even though polynomial fuzzy model yields relaxation in stability analysis compared to that of the T-S fuzzy model, the conservativeness remains an issue. One of the important sources of the conservativeness is the selection of Lyapunov function candidate form, e.g. quadratic Lyapunov function, piecewise Lyapunov function, polynomial Lyapunov function (PLF), etc. In this thesis, a piecewise polynomial Lyapunov function-based (PPLF-based) approach is proposed in order to design the state feedback and the state estimation of polynomial fuzzy systems. In the PPLF-based approach, several PLFs are provided to analyze the stability of system. A switching index is then defined to simultaneously choose one Lyapunov function which is the minimum value among others. Based on this switching index, a switching controller is designed and selected in order to stabilize the polynomial fuzzy system. The effectiveness of the proposed design is demonstrated through simulation of major benchmark design examples. To show the possible utilization of the PPLF-based approach, the stabilization is expanded to robust stabilization by considering uncertainty parameters in the polynomial fuzzy systems. Finally, the PPLF-based approach is employed to design an observer.

This thesis is organized as follows.

Chapter 1 introduces the objectives and motivation of this study.

Chapter 2 presents a description to express nonlinear systems as fuzzy systems representation by utilizing a universal approximator. A sector nonlinearity concept which is used to convert nonlinear systems as fuzzy systems representation is also described in this chapter. The stability condition is analyzed based on the Lyapunov stability theory which is then represented as linear matrix inequalities (LMIs) problem. The conventional fuzzy systems representation is extended to polynomial vector field which is called polynomial fuzzy systems. The construction of polynomial fuzzy systems is also exploited by sector nonlinearity method. Since LMI design framework is restricted for a system with polynomial vector fields, another framework is required to perform the stability analysis and design for polynomial fuzzy systems. To date, one of the most powerful frameworks to prove nonnegativity of a polynomial is called sum of squares (SOS). The stability analysis by using this framework is described in this chapter. Furthermore, two relaxation techniques, i.e. copositive relaxation and positivstellensatz relaxation (P-satz), used in the subsequent chapters are also introduced. The copositive relaxation comes from the idea of Polya's theorem while P-satz was developed as the solution of Hilbert 17^{th} problem in 19^{th} century.

Chapter 3 provides a detailed derivation of polynomial fuzzy systems stabilization via a PPLF-based approach. Design of a switching controller based on parallel distributed compensation (PDC) is also presented. In the derivation process, two relaxations are considered, i.e. copositive and P-satz relaxation. Then, the stabilization conditions are formulated in terms of the SOS framework. Finally, sufficient conditions can be obtained to prove the nonnegativity of the polynomial fuzzy systems stabilization conditions. The effectiveness of the proposed design is demonstrated through the simulation results of two major benchmark design examples.

Chapter 4 introduces the proposed PPLF-based approach for robust control of polynomial fuzzy systems. A nonlinear system consisting of uncertainties are exactly converted to polynomial fuzzy systems with uncertainties by using sector nonlinearity concept. In the proposed robust control design, there are two schemes. In the first case, uncertainties appeared both in the system and in the input term. In the second case, the uncertainty appeared only in the system.

The robust stabilization conditions for these two schemes are then derived in terms of the SOS framework. To deal with the nonconvex term in the stabilization conditions, a path-following algorithm is applied. The validity of the proposed design was tested on two major benchmark design examples. The results are then compared to those of existing designs.

Chapter 5 provides a detailed description of an observer-controller design for the polynomial fuzzy systems. As a consequence of PPLF-based design, the augmented system contains both the switching polynomial fuzzy controller and the switching polynomial fuzzy observer. According to the switching information on the PPLF, the controller and observer can be switched simultaneously to stabilize the system and estimate the states. There are three schemes (Class I, Class II, Class III) that are introduced in the design of the polynomial fuzzy observer. The classification depends on the dependence of polynomial fuzzy matrices with respect to the state which is going to be estimated. The proposed design shows that separation principle design is not necessary for polynomial fuzzy observer. A significant improvement of the proposed design is demonstrated by using a design example for all the classes.

Chapter 6 summarizes this thesis and provides possible pathways for future development regarding in this topic.

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Chapter 1.

Introduction

The classical control theory began its history in 19th century. Dynamical systems were analytically modeled using linear differential equations. However, there are some real systems whose the mathematical model cannot be derived and represented as differential equation. They had also been frequently modeled using high-order differential equations which has large complexity for analysis and calculation. In addition, the physical system became more and more complex especially if it has high nonlinearities. These caused the classical approaches could not be relied to completely represent the nonlinear plant. Therefore, research interest on control design for nonlinear systems has been rapidly growing since the past decades.

By extending the concept of fuzzy set theory founded by Lotfi Zadeh, Takagi-Sugeno (T-S) fuzzy system was first proposed and published in [2]. The T-S fuzzy system have shown its great ability to describe any nonlinear systems in terms of a set of fuzzy IF-THEN rules representing the relation of local linear input-output of a nonlinear system. The local dynamic of each rule (fuzzy implication) is expressed by a linear system model [7]. Extensive research efforts have been done to show the utility of T-S fuzzy model including stability analysis of T-S fuzzy model.

In control design, stability is one of the most important issues. The so-called parallel distributed compensation (PDC) is one such control design framework that has been proposed and developed over the last two decades [9]. In the framework of T-S fuzzy model and PDC control design, stability analysis can be stated as a set of linear matrix inequalities (LMIs). By using the concept of Lyapunov stability theory, stability conditions of T-S fuzzy model can be reduce to the existence of positive definite function such that its time derivative is negative along the trajectories. Then the problem to find the positive definite function is formulated as LMI optimization problem which can be solved by LMI solver such as LMI toolbox in MATLAB.

Recently, a more general approximator called polynomial fuzzy systems has been proposed by Tanaka et al [24]. Polynomial fuzzy model is an extension of T-S fuzzy model that can represent any nonlinear systems in polynomial cases. Therefore the stability analysis for polynomial fuzzy system reduces to the existence of positive definite polynomials such that its partial derivative is negative along the trajectories. In this case, LMI optimization technique cannot be applied due to its restriction for a system with polynomial vector field. Moreover, finding such positive definite polynomials of the whole space is known as NP-hard problems [17]. Therefore, an approach called sum of squares (SOS) to deal with the problem has been proposed in [17, 18, 20, 21]. The idea is to prove the nonnegativity of a polynomial by the presence of an SOS decomposition. By utilizing SOS approach, stabilization conditions of polynomial fuzzy systems can be derived as SOS optimization problems which can be solved by MATLAB toolboxes such as SOSTOOLS [22] and SOSOPT [65].

Numerous investigations have been addressed to accomplish stability analysis and design for polynomial fuzzy systems based on SOS approach. The authors in [16, 24, 36, 43, 44, 63] have presented SOS-based design frameworks for polynomial fuzzy systems and the results shown that SOS-based approaches provide better results over LMI-based approaches. Those SOS-based design frameworks are applicable not only for polynomial fuzzy model but also for the T-S fuzzy model based control. Therefore, research on stability analysis of both T-S fuzzy systems and polynomial fuzzy systems has attracted many attentions. Some researches tried to bring more relaxations by considering the types of Lyapunov functions such as quadratic Lyapunov function, control Lyapunov function, piecewise Lyapunov function, polynomial Lyapunov function (PLF), etc.

Piecewise systems design and analysis widely proposed in [34, 66, 68, 69] performed attractive results. Other studies in [36, 64] compared piecewise polynomial Lyapunov function-based (PPLF-based) approach with other approaches i.e. PLF, multiple PLFs, and piecewise Lyapunov function-based approaches. According to the results, the authors concluded that PPLF-based approach brought more relaxation in comparison with the others. In the PPLF-based approach, Lyapunov function candidates are described as *piecewise* and *polynomial* functions. In addition, since the conservatism in SOS-based approach still exist, i.e. there is a gap between SOS forms and positive definite polynomial forms, considering a relaxation technique will contribute a better result. One of the techniques relies on positivstellensatz (P-satz) that has been used to tackle optimization problems related to analysis and design. Other relaxation that has been proposed in [22] is called copositive relaxation.

1.1 Objectives and Overview of Thesis

One of methods to provide more relaxed results in stability analysis is by selecting a good Lyapunov function candidate form. According to several literature, piecewise polynomial Lyapunov function (PPLF) brought promising results in stability analysis and design for fuzzy control systems. Motivating from the fact, this thesis proposes newly derived stabilization conditions of polynomial fuzzy systems based on PPLF approach. Some relaxations, i.e. copositive relaxation and $\mathcal{S}$-procedure, are also carried out in the derivation process in order to fully consider the PPLF properties. To prove the effectiveness of the proposed design, two benchmark design examples are demonstrated and the results are compared with other existing results.

In reality, an error of a plant model is possibly appear in the modeling process which means there is a differences between the actual plant and the used model in control design. Hence, robust control theory has became important feature in designing control systems. Robust control design of polynomial fuzzy systems, by considering uncertainties in the systems and input terms, has been performed in [44,61]. Seeing that PPLF-based stabilization conditions has successfully shown its effectiveness, this thesis also provides the robust stabilization of polynomial fuzzy systems via PPLF approach. In order to compare the proposed robust stabilization conditions with other existing approaches, i.e. [44, 61], two design examples are demonstrated and the results also showed the effectiveness of PPLF-based approach.

In designing the control systems, the states of a system are usually assumed to be available for feedback. However in practical applications, not all the states are available. This causes the necessity of unavailable states estimation. To fulfill such necessity, observer design becomes important feature in control systems. Hence, the works in this thesis also cover the observer design of polynomial fuzzy systems by taking the utility of PPLF-based approach, i.e. switching polynomial fuzzy observer and controller. In this case, the designed switching polynomial fuzzy controller depends on the state-estimation of the switching polynomial fuzzy observer. In addition, all the conditions are derived to guarantee the global stabilization and global state-estimation convergence of original nonlinear systems.

1.2 Chapters Outline

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Chapter 6 summarizes this thesis and provides possible pathways for future development regarding in this topic.

1.3 T-S Fuzzy Systems vs LPV Systems

This section provides an overview of T-S fuzzy systems and linear parameter varying (LPV) systems. The well known T-S fuzzy system, introduced by T. Takagi and M. Sugeno in 1985 [2], and linear parameter varying (LPV) systems, introduced by J. S. Shamma [45] and published in 1990 [51], received many attentions for their capability to deal with nonlinear systems control design. T-S fuzzy systems provides an effective way to approximate nonlinear systems by using the concept of fuzzy sets, fuzzy rules, and a set of linear models. By merging the linear models through fuzzy membership functions, the overall model of the nonlinear systems can be constructed. In practical applications, T-S fuzzy systems has successfully been applied for robotic systems [46], aircraft systems [47, 48], fault tolerant control [49, 50], power filter [52], etc.

Motivated from the methodology of gain scheduling control design , LPV systems was proposed in [51]. LPV systems approximate nonlinear plant as a linear system whose coefficients depend on some varying parameters. LPV systems has been also utilized in practical applications as T-S fuzzy systems: robotic systems, aircraft systems, fault tolerant control, and power filter.

In [53], the authors showed that the nonlinear embedding method, a method for the automated generation of LPV models, can be extended to construct T-S fuzzy model. On the other hands, the sector nonlinearity concept, a technique widely used for T-S fuzzy construction, can be utilized to construct a polytopic LPV model. The authors in [53] also conducted two measures for comparing between T-S fuzzy and LPV systems, i.e. overboundedness-based measure and region of attraction estimate-based measure. Through the result of a mathematical example, the authors stated that which model is the best depends on the context in which the model is used.

Stability analysis of both T-S fuzzy and LPV systems can be carried out as convex optimization problem and can be formulated as LMIs optimization problem. In order to reduce the conservativeness of T-S fuzzy model, polynomial fuzzy model which is a general form of T-S fuzzy model was proposed by K. Tanaka et al in [24]. In the same way, by considering polynomial term in the system and input, polynomial LPV has also been proposed by F. Wu et al in [55]. Stabilization conditions of both polynomial fuzzy model and polynomial LPV, based on Lyapunov stability theory, can be formulated in terms of SOS approach to provide sufficient conditions for the positive definiteness of the Lyapunov function and the negative definiteness of its partial derivative.

1.4 Related Researches in Other Classes

The advantage of PPLF-based approach was also utilized not only in fuzzy control area, but also other classes. For instance, in [19] a PPLF approach was used to conduct a stability analysis of switched and hybrid systems. The analysis showed significant improvements over the previous methods. Hence they extended the method for robust stability analysis of nonlinear hybrid systems with dynamic uncertainties in [25].

Another research in [58] showed an investigation of constructing piecewise polynomial Lyapunov function (PPLF) based approach was conducted for global stability analysis and global $\mathcal{L}_{2m}$ gain estimation of linear systems with deadzone/saturation with structured parametric uncertainties. The sufficient conditions of the system stability and system performance via PPLF are derived by using a candidate of Lyapunov function dependent on the uncertain parameter. A numerical example to show the maximum value of uncertain parameter was demonstrated. According to the results, PPLF provided contribution to reduce the conservativeness.

PPLF approach was also performed in [26] to conduct a stability analysis of nonlinear systems with polynomial vector fields. Instead of using SOS framework, the Handelman's theorem was used to provide a positive polynomial parametrization on the given polytope.

Besides PPLF, polynomial Lyapunov function is also considered as a good Lyapunov function form. In [57], research on stability region analysis, based on Lyapunov stability theory, of a nonlinear system was conducted. By utilizing polynomial Lyapunov function based approach, the author presented a method to enlarge the inner estimate of RoA (region of attraction) of nonlinear systems. The stability conditions are derived in terms of sum of squares (SOS) problems and solved by using SOS optimization technique. Through several examples, the works in [57] show the efficiency of the proposed method for finding RoA of nonlinear systems.

Other study in [60] conducted a finite time stability (FTS) analysis of nonlinear quadratic systems. It is stated that a system is said to be finite time stable if the state of the system is restricted within a given bounded region of the state-space. Sufficient conditions for FTS of nonlinear systems were derived by considering several types of Lyapunov functions, i.e. quadratic Lyapunov function, non-quadratic Lyapunov function, and polynomial Lyapunov function. The conditions were verified through a benchmark design example resulting that the derived conditions via polynomial Lyapunov function approach provided less conservative results compared with quadratic, non-quadratic Lyapunov functions.

Chapter 2.

Preliminaries

This chapter provides the necessary mathematical background on control theory for the later results.

2.1 Definitions and Notations

The following notations and definitions on polynomial are adopted through the whole of this paper [1, 5, 13, 20].

Definition 2.1.1 (Monomials)**.** Let α be an n-tuple of non-negative integers. A monomial in $\boldsymbol{x}(t) = [x_1(t) \cdots x_n(t)]$ is a function of the form $x_1(t)^{\alpha_1} \cdots x_n(t)^{\alpha n}$ where the degree is defined by $\sum_{i=1}^{n} \alpha_i$.

Definition 2.1.2 (Polynomials)**.** $\mathcal{P}$ is defined as the polynomials universe. A polynomial is a mathematical expression that consists of variables including any operations (e.g. addition, subtraction, multiplication, etc).

Definition 2.1.3 (Positive Semi Definite (PSD) Polynomials)**.** If $q(\boldsymbol{x}(t))$ is a polynomial of $\mathcal{P}$ and $q(\boldsymbol{x}(t)) \geq 0$ for all $\boldsymbol{x}(t) \in \mathbb{R}^n$, then $q(\boldsymbol{x}(t))$ is defined as a positive semi definite polynomial (PSD). The space of PSD polynomials is denoted as $\mathcal{P}^{0+}$.

Definition 2.1.4 (Sum of Squares (SOS) Polynomials)**.** $\mathcal{S}$ is defined as a set of sum of squares (SOS) polynomials. A polynomial is called as an SOS polynomial if it can be formulated as $q(\boldsymbol{x}(t)) = \sum_{i=1}^{n} f_i^2(\boldsymbol{x}(t))$ where $f_i(\boldsymbol{x}(t))_{i=1,\ldots,n} \in \mathcal{P}$. According to these definitions, we have $\mathcal{S} \subset \mathcal{P}^{0+} \subset \mathcal{P}$. By setting a polynomial such that $q(\boldsymbol{x}(t)) \in \mathcal{S}$, positive definiteness of the polynomial can be guaranteed. More explanation of SOS polynomial is given in Section 2.2.2.

2.2 Lyapunov Stability Theory

In this thesis, stabilization conditions of polynomial fuzzy systems were derived based on Lyapunov stability theory. Therefore, this chapter also provided description about Lyapunov theory adopted from [10].

Consider an autonomous system

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) \tag{2.1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the states and $f : \mathbb{R}^n \to \mathbb{R}^n$. Without loss of generality, it is assumed that $f(\boldsymbol{x}(t))$ satisfies $f(\mathbf{0}) = 0$. The stability in the origin, i.e., $\boldsymbol{x}(t) = \mathbf{0}$, is described as follows.

Definition 2.2.1 (Lyapunov Stability)**.** The equilibrium point $\boldsymbol{x}(t) = \mathbf{0}$ is

1. stable, if for each $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that

$$\|\boldsymbol{x}(0)\| < \delta \Rightarrow \|\boldsymbol{x}(t)\| < \epsilon, \forall t \geq 0,$$

2. unstable if it is not stable,

3. asymptotically stable if it is stable and δ can be chosen such that

$$\|\boldsymbol{x}(0)\| < \delta \Rightarrow \lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x}_e.$$

In 1892, instead of energy, Lyapunov determined the stability by employing a certain function instead of energy. Then the stability analysis was reduced to the existence of positive definite function such that its derivative is negative deninite along the trajectories. The statement of Lyapunov's stability theorem are described as follows.

Definition 2.2.2 (Stable and Asymptotically Stable)**.** Given a function $V(\boldsymbol{x}(t)) : \mathbb{D} \to \mathbb{R}$ such that

$$\begin{aligned} &V(0) \quad \text{and} \quad V(\boldsymbol{x}(t)) > 0 \quad \text{in} \quad \mathbb{D} - \{0\}, \\ &\dot{V}(\boldsymbol{x}(t)) \leq 0 \quad \text{in} \quad \mathbb{D}, \end{aligned} \tag{2.2}$$

then, $\boldsymbol{x}(t) = 0$ is stable. Moreover, if $\dot{V}(\boldsymbol{x}(t)) < 0$ in $\mathbb{D} - \{0\}$, then $\boldsymbol{x}(t) = 0$ is asymptotically stable.

For instance, we have a nonlinear system as follows.

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) \tag{2.3}$$

where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is a constant system matrix. Choose a Lyapunov function candidate as $V(\boldsymbol{x}(t)) = \boldsymbol{x}(t)^T\boldsymbol{P}\boldsymbol{x}(t)$ where $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ is called Gram matrix. According to Lyapunov's stability theorem, the system (2.3) is stable, i.e, all trajectories converge to the equilibrium point, if and only if there exist matrix $\boldsymbol{P}$ that is positive definite such that

$$\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} < 0. \tag{2.4}$$

Note that, positive-definiteness of matrix $\boldsymbol{P}$ has to be satisfied to guarantee $V(\boldsymbol{x}(t)) > 0$. Moreover, condition (2.4) is given to guarantee that the partial derivative of $V(\boldsymbol{x}(t))$ is negative definite, i.e., $\dot{V}(\boldsymbol{x}(t)) < 0$. The two conditions, i.e., $\boldsymbol{P} > 0$ and condition (2.4), are formulated as linear matrix inequality (LMI) problems, which can be solved by an LMI solver.

2.2.1 Linear Matrix Inequality

Linear matrix inequality (LMI) begins its history since Lyapunov theory was published in 1890. As presented in previous, the stability analysis can be reformulate as a problem to find a function that is positive definite such that its partial derivative is negative along the trajectories. The problem to find $\boldsymbol{P} > 0$ such that $\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} < 0$ is a special form of an LMI. An LMI in the variables $\boldsymbol{x}(t) \in \mathbb{R}^n$ has the form

$$\boldsymbol{Q}(\boldsymbol{x}(t)) = \boldsymbol{Q}_0 + \boldsymbol{x}_1(t)\boldsymbol{Q}_1 + \cdots + \boldsymbol{x}_n(t)\boldsymbol{Q}_n \geq 0, \tag{2.5}$$

where $\boldsymbol{Q}_0 \in \mathbb{R}^{m\times m}, \cdots, \boldsymbol{Q}_n \in \mathbb{R}^{m\times m}$ are symmetric matrices and $\boldsymbol{x}(t)$ is a vector of scalar.

2.2.2 Sum of Squares

In the previous section, according to Lyapunov theory, i.e. the so-called Lyapunov's direct method, a stability analysis of dynamical system can be reduced to the existence of a Lyapunov function V: a positive definite function V whose the derivative is negative semi-definite along the trajectories. Finding such a function can be expressed as LMI problems. However, the technique is restricted for a system with polynomial vector fields, i.e. the dynamical system that is described by polynomial equations. Therefore, for a system with polynomial terms Lyapunov theory reduces to the existence of nonnegative polynomials of the whole space which is known as NP-hard problems [17]. A sufficient condition to prove nonnegativity of a polynomial is the presence of a sum of squares (SOS) decomposition.

A multivariate polynomial $q(\boldsymbol{x}(t))$ is an SOS if it can be decomposed as

$$q(\boldsymbol{x}(t)) = \sum_{i=1}^{n} p_i^2(\boldsymbol{x}(t)). \tag{2.6}$$

for $p_1(\boldsymbol{x}(t)), p_2(\boldsymbol{x}(t)), \cdots, p_n(\boldsymbol{x}(t)) \in \mathbb{R}[\boldsymbol{x}(t)]$. Condition (2.6) is a sufficient condition to guarantee the nonnegativity of a polynomial. Remarkably, it can also be determined by the following lemma.

Lemma 2.2.1. *[20] Consider a polynomial $q(\boldsymbol{x}(t))$ in n variables of degree $2d$. The existence of an SOS decomposition of $q(\boldsymbol{x}(t))$ can be obtained by solving the following semi-definite programming (SDP) problem.*

$$q(\boldsymbol{x}(t)) = \boldsymbol{z}(t)^T \boldsymbol{P} \boldsymbol{z}(t), \quad \boldsymbol{z} = [1, x_1(t), x_2(t), \cdots, x_n(t), x_1(t)x_2(t), \cdots, x_n(t)^d] \tag{2.7}$$

If $\boldsymbol{P} \in \mathcal{P}^{0+}$, then $q(\boldsymbol{x}(t)) \in \mathcal{S}$

By applying an SOS approach, Lyapunov stability analysis reduces to the following lemma.

Lemma 2.2.2. *[18,21] Let consider a system with polynomial vector field $\dot{\boldsymbol{x}(t)} = f(\boldsymbol{x}(t))$ where $f(\mathbf{0}) = 0$. The equilibrium point $\boldsymbol{x}(t) = \mathbf{0}$ is asymptotically stable if the following conditions hold.*

$$V(\boldsymbol{x}(t)) - \epsilon(\boldsymbol{x}(t)) \in \mathcal{S} \tag{2.8}$$

$$-\frac{\partial V}{\partial \boldsymbol{x}(t)} f(\boldsymbol{x}(t)) \in \mathcal{S} \tag{2.9}$$

where $\epsilon(\boldsymbol{x}) \in \mathcal{P}^+$ is a slack variable to guarantee the positivity of Lyapunov function $V(\boldsymbol{x}(t))$.

In order to guarantee that $\dot{V}(\boldsymbol{x}(t)) < 0$ at $\boldsymbol{x} \neq \mathbf{0}$, Lemma 2.2.2 can be represented as follows.

Lemma 2.2.3. *The equilibrium point* $\boldsymbol{x}(t) = \mathbf{0}$ *is globally asymptotically stable if the following conditions hold.*

$$V(\boldsymbol{x}(t)) - \epsilon(\boldsymbol{x}(t)) \in \mathcal{S}, \tag{2.10}$$

$$-\frac{\partial V}{\partial \boldsymbol{x}(t)} f(\boldsymbol{x}(t)) + \alpha V(\boldsymbol{x}(t)) \in \mathcal{S} \tag{2.11}$$

where α *is a negative scalar.*

The above SOS problems are SOS optimization problems where α as the objective to be minimized, i.e., α is minimized until $\alpha < 0$ is obtained. To solve the SOS problems, two kinds of SOS solvers can be employed: SOSOPT [65] and SOSTOOLS [70]. These are free, third-party MATLAB toolboxes. Other tool is an SDP (semidefinite programming) solver like SeDuMi and SDPT3. The diagram depicting relation among sum of squares program (SOSP), SOSOPT/SOSTOOLS (SOS solver), SDP solver, and SOSP solution can be seen in the following Figure 2.1.

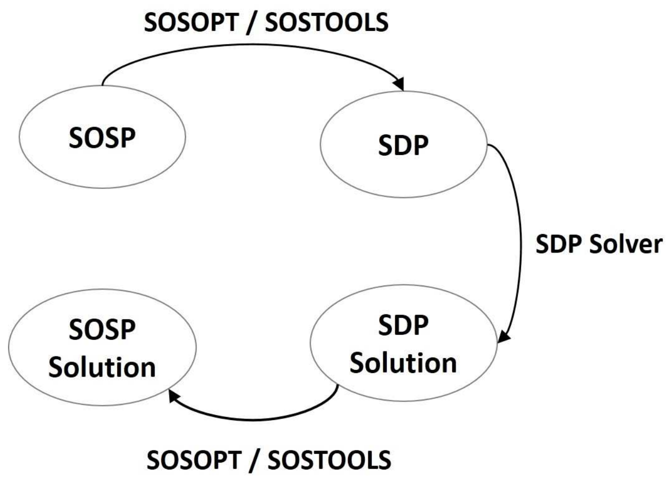

Fig. 2.1:Relation between SOSP, SOSOPT/SOSTOOLS, SDP, and SDP Solver [70]

First of all, I define SOS conditions (consisting of the symbolic forms) as SOSP. Then SOSOPT/SOSTOOLS will automatically convert SOSP to SDP (semidefinite programming). By calling the SDP solver, the SOSP can be numerically solved as SDP solution then convert the SDP solution backs to the solution of original. In this research, I used SOSOPT as an SOS solver because I can choose the reliability level. In all the simulations, I used the most reliable options to get the feasible solutions, i.e. “both” options. Moreover, all the feasible solutions have been verified by using "issos" command in SOSOPT with the most reliable options, i.e. "both" options. This double checking is important to obtain reliable solutions.

2.3 Relaxations Technique

2.3.1 Positivstellensatz Relaxation

The Positivstellensatz (P-satz) has been introduced as a solution of Hilberth 17^{th} problem. By employing P-satz, an infeasibility certificate or refutation, the emptyness of a set of polynomials can be determined [20]. In [13], the infeasibility certificate can be formulated in any commutative ring K via the real spectrum $Spec_r(K)$ as follows:

Lemma 2.3.1. *f_1, ..., f_r, g_1, ...g_t, h_1, ...$h_m \in K$, the empty set condition is described as follows:*

$$\begin{aligned}&\{\boldsymbol{x}(t) \in \mathbb{R}^n |\, f_1(\boldsymbol{x}(t)) \geq 0,\, ...,\, f_r(\boldsymbol{x}(t)) \geq 0,\\&g_1(\boldsymbol{x}(t)) \neq 0,\, ...,\, g_t(\boldsymbol{x}(t)) \neq 0,\\&h_1(\boldsymbol{x}(t)) = 0,\, ...,\, h_m(\boldsymbol{x}(t)) = 0\} \subset Spec_r(K) = \varnothing.\end{aligned}$$

The above condition can be converted as the presence of $f \in \mathcal{C}(f_1, ..., f_r)$, $g \in \mathcal{M}(g_1, ..., g_t)$, $h \in \mathcal{I}(h_1, ..., h_m)$ such that

$$f(\boldsymbol{x}(t)) + g^2(\boldsymbol{x}(t)) + h(\boldsymbol{x}(t)) = 0. \tag{2.12}$$

$\mathcal{M}$, $\mathcal{C}$, and $\mathcal{I}$ are defined in Definition 2.3.1, 2.3.2, and 2.3.3 as used in [13, 62].

Definition 2.3.1. Given $(g_i(\boldsymbol{x}(t)))_{i=1,\dots,n}$ is a set of polynomials, the multiplicative monoid $\mathcal{M}(g_i(\boldsymbol{x}(t)))$ is the set of all finite products of $g_i(\boldsymbol{x}(t))$ including I i.e. the empty product.

Definition 2.3.2. Given $(f_j(\boldsymbol{x}(t)))_{j=1,\dots,\xi}$ is a set of polynomials, the cone of $f_j(\boldsymbol{x}(t))$ is defined as follows.

$$\mathcal{C}(f_j(\boldsymbol{x}(t))) := s_0(\boldsymbol{x}(t)) + \sum_{j=1}^{\xi} s_j(\boldsymbol{x}(t)) e_j(\boldsymbol{x}(t)),$$

for $j = 1, ..., \xi$, a positive integer ξ, SOS polynomials $s_j(\boldsymbol{x}(t))$, and $e_j(\boldsymbol{x}(t)) \in \mathcal{M}(f_j(\boldsymbol{x}(t)))$.

Definition 2.3.3. Given $(h_\eta(\boldsymbol{x}(t)))_{\eta=1,...,\rho}$ is a set of polynomials. The Ideal of $h_\eta(\boldsymbol{x}(t))$ is defined as $\mathcal{I}(h_\eta(\boldsymbol{x}(t))) := \sum_{\eta=1}^{\rho} h_\eta(\boldsymbol{x}(t))p_\eta(\boldsymbol{x}(t))$ for ρ is a positive integer and $p_\eta(\boldsymbol{x}(t))$ are polynomials.

2.3.2 Copositive Relaxation

Lemma 2.3.2. *[17, 20] A symmetric matrix $\boldsymbol{J} \in \mathbb{R}^{n\times n}$ is copositive if*

$$\boldsymbol{\upsilon}^T \boldsymbol{J} \boldsymbol{\upsilon} = \sum_{i=1}^{n}\sum_{j=1}^{n} \upsilon_i \upsilon_j \boldsymbol{J}_{ij} \in \mathcal{P}^{0+} \tag{2.13}$$

for $\boldsymbol{\upsilon} = [\upsilon_1, \upsilon_2, \cdots, \upsilon_n]$ and $\upsilon_i \geq 0$.

Checking whether a matrix is copositive is a co-NP problem (see [4, 20, 22]). Thus, in [20] a sufficient condition to guarantee the copositivity has been proposed as described in Lemma 2.3.3.

Lemma 2.3.3. *[20, 22] Consider υ_i in Lemma 2.13 as $\upsilon_i = z_i^2$. A matrix $\boldsymbol{J}$ is copositive if the following condition is satisfied.*

$$\boldsymbol{Q}^\mu(\boldsymbol{z}) = \Big(\sum_{k=1}^{m} z_k^2\Big)^\mu \sum_{i=1}^{m}\sum_{j=1}^{m} z_i^2 z_j^2 J_{ij} \in \mathcal{S} \tag{2.14}$$

where $\boldsymbol{z} = [z_1, z_2, \cdots, z_m]^T$ and μ is a nonnegative integer.

The above relaxation technique can also be applied for a polynomial $J(\boldsymbol{x}(t))$.

2.4 Takagi-Sugeno Fuzzy Model

Takagi-Sugeno (T-S) fuzzy model was first proposed by Takagi and Sugeno in 1985 [2]. T-S fuzzy systems provides an effective way to approximate nonlinear systems by using the concept of fuzzy sets, fuzzy rules, and a set of linear models. By merging the linear models through fuzzy membership functions, the overall model of the nonlinear systems can be constructed.

Given a nonlinear system, the i-th rule of the T-S fuzzy model are expressed as followings:

$$\begin{aligned}
&\textit{Model Rule } i\textit{:}\\
&\text{IF } z_1(t) \text{ is } M_{i1} \text{ and } \cdots \text{ and } z_o(t) \text{ is } M_{io},\\
&\text{THEN } \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t),\\
&i = 1, 2, \cdots, r,
\end{aligned} \tag{2.15}$$

where, $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the system states, r is the number of linear models represented as number of fuzzy rules, M_{io} is the fuzzy set associated with i-th model rule, and $z_o(t)$ is known premise variables. $\boldsymbol{A}_i \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B}_i \in \mathbb{R}^{n\times q}$ are are matrices of the system and input respectively. By performing "fuzzy blending" of the local linear system models, the overall fuzzy system representation is constructed as follows:

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)\}, \tag{2.16}$$

where $\boldsymbol{z}(t) = [z_1(t) \quad z_2(t) \cdots \quad z_o(t)]^T \in \mathbb{R}^o$,

$$h_i(\boldsymbol{z}(t)) = \frac{\prod_{j=1}^{o} M_{ij}(z_j(t))}{\sum_{k=1}^{r}\prod_{j=1}^{o} M_{kj}(z_j(t))},$$

$$\sum_{i=1}^{r} h_i(\boldsymbol{z}(t)) = 1, \quad h_i(\boldsymbol{z}(t)) \geq 0, \quad \forall i.$$

A more detail explanation to construct a T-S fuzzy model is given in 2.4.1.

2.4.1 Construction of T-S Fuzzy Model

There are two methods to construct a Takagi-Sugeno (T-S) fuzzy model. One of the method is indentification using input-output data which has been proposed in [2]. The identification process of a fuzzy model is divided into two main parts: structures identification and parameters identification. However, this approach is more suitable for a plant that is hard to be represented by analytical or physical models. [14]. The other approach is by using derivation of the nonlinear systems which built upon sector nonlinearity concept. This approach has been employed in [9, 11, 12].

The main idea of sector nonlinearity is described in the Figure 2.2. By employing the sector

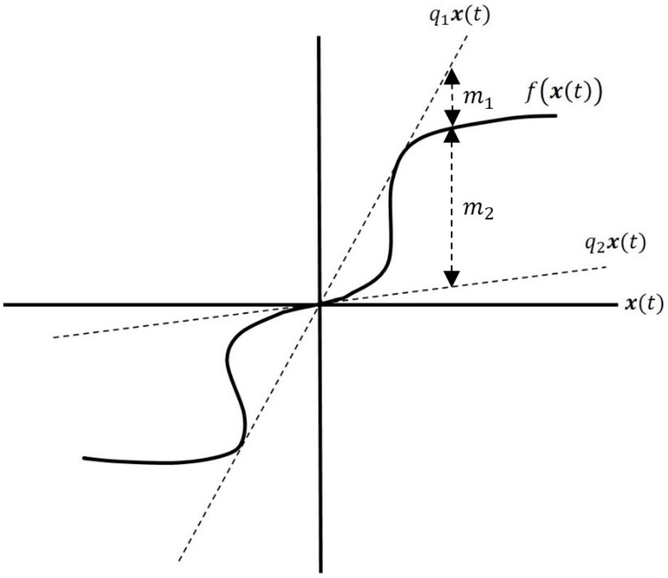

Fig. 2.2:Global sector nonlinearity [14]

nonlinearity concept, the nonlinear system $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t))$ can be exactly represented by the sectors $q_1(\boldsymbol{x}(t))$ and $q_2(\boldsymbol{x}(t))$ such that $\dot{\boldsymbol{x}}(t) \in \begin{bmatrix} q_1 & q_2 \end{bmatrix} \boldsymbol{x}(t)$.

The membership functions of T-S fuzzy model are

$$h_1(\boldsymbol{x}(t)) = \frac{m_2(\boldsymbol{x}(t))}{m_1(\boldsymbol{x}(t)) + m_2(\boldsymbol{x}(t))}, \quad h_2(\boldsymbol{x}(t)) = \frac{m_1(\boldsymbol{x}(t))}{m_1(\boldsymbol{x}(t)) + m_2(\boldsymbol{x}(t))}, \tag{2.17}$$

where $h_1(\boldsymbol{x}(t)) \geq 0$, $h_2(\boldsymbol{x}(t)) \geq 0$, and $h_1(\boldsymbol{x}(t)) + h_2(\boldsymbol{x}(t)) = 1$. Hence the T-S fuzzy representation is constructed as follows.

$$\dot{\boldsymbol{x}}(t) = \sum_i h_i(\boldsymbol{x}(t))q_i(\boldsymbol{x}(t)) = h_1(\boldsymbol{x}(t))q_1(\boldsymbol{x}(t)) + h_2(\boldsymbol{x}(t))q_2(\boldsymbol{x}(t)), \quad i = 1, 2, \cdots, r \tag{2.18}$$

where r is number of rules. For the case that the global sector for a nonlinear system cannot be found, we can consider a local sector nonlinearity to convert the nonlinear system as the T-S fuzzy model. Figure 2.3 describes the idea of local sector nonlinearity where the two dashed lines become the local sector under $-d < \boldsymbol{x} < d$. The local region of the nonlinear system $f(\boldsymbol{x}(t))$ can be exactly represented by applying the local sector nonlinearity concept.

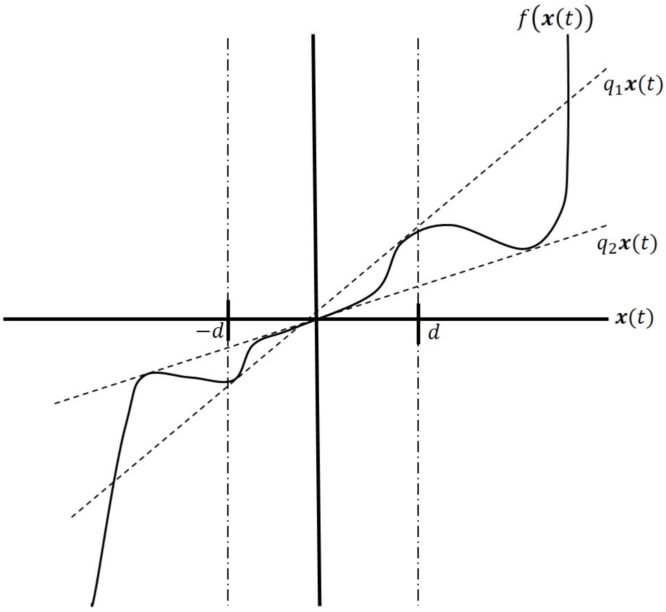

Fig. 2.3:Local sector nonlinearity [14]

2.4.2 Parallel Distributed Compensation

In this thesis, parallel distributed compensation, called PDC, was used to design a controller for stabilizing the T-S fuzzy model [14]. In PDC, the controller has the same membership functions as the T-S fuzzy model. Each controller is designed according to the corresponding rule of the T-S fuzzy model. For T-S fuzzy system (2.15), the controller is designed as follows:

Control Rule i:

IF z_1 is M_{i1} and $\cdots$ and z_o is M_{io},

$$\text{THEN } \boldsymbol{u}(t) = -\boldsymbol{F}_i\boldsymbol{x}(t), \quad i = 1, 2, \cdots, r, \tag{2.19}$$

where the defuzzified outputs of the fuzzy controller is represented as follows:

$$\boldsymbol{u}(t) = -\sum_{i=1}^{r} h_i(z)\boldsymbol{F}_i\boldsymbol{x}(t). \tag{2.20}$$

By substituting the above controller to the T-S fuzzy closed-loop system, the overall closed-loop system is represented as

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\left\{\boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{F}_j\right\}\boldsymbol{x}(t). \tag{2.21}$$

2.4.3 Stabilization of T-S Fuzzy Systems

This section provides LMI approach to perform T-S fuzzy model stability and stabilization analysis based on Lyapunov stability theorem. Let $V(\boldsymbol{x}(t)) = \boldsymbol{x}(t)^T \boldsymbol{P}\boldsymbol{x}(t)$ becomes a Lyapunov function candidate, and without loss of generality, assume that $\boldsymbol{x} = 0$ is the equilibrium point. A stability analysis, i.e., $\boldsymbol{u} = \boldsymbol{0}$ of T-S fuzzy model is described in Lemma 2.4.1.

Lemma 2.4.1. *[T-S Fuzzy Model Stability Analysis] The equilibrium $\boldsymbol{x}(t) = 0$ of (2.16) with zero control input is asymptotically stable if a matrix $\boldsymbol{P}$ is exist such that*

$$\boldsymbol{P} > 0, \tag{2.22}$$

$$\boldsymbol{A}_i^T \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_i < 0. \quad \forall i \tag{2.23}$$

LMI problems in Lemma 2.4.1 guarantee the positive-definitness of the Lyapunov function such that its partial derivative is negative definite along the trajectories.

Now, by considering the fuzzy controller $\boldsymbol{u}$, stabilization analysis of T-S fuzzy model can be described in the following Lemma.

Lemma 2.4.2. *[11][T-S Fuzzy Model Stabilization Analysis] The equilibrium $\boldsymbol{x} = 0$ of* (2.21) *is asymptotically stable if the exist matrix $\boldsymbol{X} = \boldsymbol{P}^{-1}$ and $\boldsymbol{F}_i$ such that*

$$\boldsymbol{X} > 0, \tag{2.24}$$

$$-\boldsymbol{X}\boldsymbol{A}_i^T - \boldsymbol{A}_i\boldsymbol{X} + \boldsymbol{M}_i^T\boldsymbol{B}_i^T + \boldsymbol{B}_i\boldsymbol{M}_i > 0, \tag{2.25}$$

$$-\boldsymbol{X}\boldsymbol{A}_i^T - \boldsymbol{A}_i\boldsymbol{X} - \boldsymbol{X}\boldsymbol{A}_j^T - \boldsymbol{A}_j\boldsymbol{X} + \boldsymbol{M}_j^T\boldsymbol{B}_i^T + \boldsymbol{B}_i\boldsymbol{M}_j + \boldsymbol{M}_i^T\boldsymbol{B}_j^T + \boldsymbol{B}_j\boldsymbol{M}_i \geq 0. \tag{2.26}$$

The local feedback gains are obtained as $\boldsymbol{F}_i = \boldsymbol{M}_i\boldsymbol{X}^{-1}$.

2.5 Polynomial Fuzzy Model

Polynomial fuzzy system was introduced [24] as a more general approximator than T-S fuzzy model. Polynomial fuzzy systems can represent any nonlinear system in polynomial cases, i.e. consequent parts are represented in polynomial vector field. Consider a nonlinear system represented in (2.27).

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \tag{2.27}$$

where $\boldsymbol{x}(t) = [x_1(t)\, x_2(t)\, \cdots\, x_n(t)]^T$ is the state vector, f is a nonlinear function, and $\boldsymbol{u}(t) = [u_1(t)\, u_2(t)\, \cdots\, u_m(t)]^T$ is the input vector. Here, similar to the construction of the T-S fuzzy model, by applying the sector nonlinearity concept [14], the nonlinear system can be exactly represented by the polynomial fuzzy model as proposed in [24]:

Model Rule i:

IF $z_1(t)$ is M_{i1} and $\cdots$ and $z_o(t)$ is M_{io},

THEN $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)$,

$i = 1, 2, \cdots, r,$ (2.28)

where r denotes the rules number, $z_o(t)$ and M_{io} are the premise variable and the fuzzy set respectively. $\boldsymbol{A}_i(\boldsymbol{x}(t)) \in \mathbb{R}^{n\times N}$ and $\boldsymbol{B}_i(\boldsymbol{x}(t)) \in \mathbb{R}^{n\times q}$ are polynomial matrices in the system and input respectively. $\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) \in \mathbb{R}^N$ is a monomial vector in $\boldsymbol{x}(t)$ under assumption that $\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) = 0 \iff \boldsymbol{x}(t) = 0$.

The overall closed-loop system is presented as follows [24]:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)\}, \tag{2.29}$$

where $\boldsymbol{z}(t) = [z_1(t)\, z_2(t)\, \cdots z_o(t)] \in \mathbb{R}^o$,

$$h_i(\boldsymbol{z}(t)) = \frac{\prod_{j=1}^{o} M_{ij}(\boldsymbol{z}_j(t))}{\sum_{k=1}^{r}\prod_{j=1}^{o} M_{kj}(\boldsymbol{z}_j(t))}.$$

We note from the property of the membership functions that

$$h_i(\boldsymbol{z}(t)) \geq 0, \quad \forall i,$$
$$\sum_{i=1}^{r} h_i(\boldsymbol{z}(t)) = 1.$$

2.5.1 Construction of Polynomial Fuzzy Model

Construction of polynomial fuzzy model also utilize the sector nonlinearity approach as described in Figure 2.4. By allowing the sectors in polynomial term, the nonlinear system which has polynomial term can be exactly represented as polynomial fuzzy model.

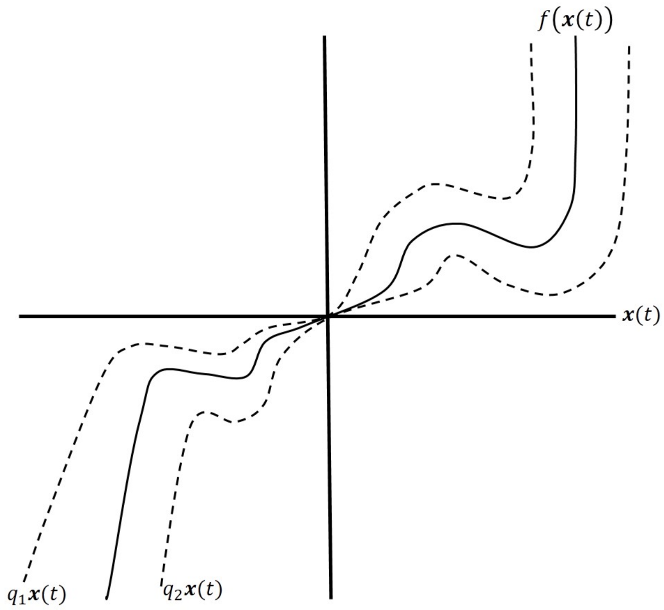

Fig. 2.4:Sector nonlinearity approach for polynomial fuzzy system construction

In order to bring more clarity, the construction of polynomial fuzzy model is provided as follows.

Let consider the following nonlinear system.

$$\dot{x}_1 = -x_1 + x_1^2 + x_1^3 + x_1^2 x_2 - x_1 x_2^2 + x_2 + x_1 u \tag{2.30}$$

$$\dot{x}_2 = -\sin(x_1) - x_2 \tag{2.31}$$

The above nonlinear system can be rewritten as follows.

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1 x_2 - x_2^2 & 1 \\ -z_1 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} x_1 \\ 0 \end{bmatrix} u \tag{2.32}$$

where $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and $z_1 = \dfrac{\sin(x_1)}{x_1} \equiv \text{sinc}(x_1)$. Since

$$\max_{x_1} z_1 = 1 \equiv q_1, \qquad \min_{x_1} z_1 = -0.2172 \equiv q_2,$$

z_1 can be rewritten as

$$z_1 = \sum_{i}^{2} h_i(z_1) q_i \tag{2.33}$$

where

$$h_1(z_1) = \frac{z_1 - q_2}{q_1 - q_2}, \qquad h_2(z_1) = \frac{q_1 - z_1}{q_1 - q_2}. \tag{2.34}$$

By substituting z_1, q_1, q_2 into (2.34), the membership functions $h_i(z_1)$ can be represented as

$$h_1(z_1) = \frac{\sin(x_1) - 0.2172x_1}{1.2172x_1}, \qquad h_2(z_1) = \frac{x_1 - \sin(x_1)}{1.2172x_1}. \tag{2.35}$$

Hence, the nonlinear system can be exactly represented as polynomial fuzzy system for $x_1 \in (-\infty\ \infty)$ and $x_2 \in (-\infty\ \infty)$, i.e., globally:

$$\dot{x} = \sum_{i=1}^{2} h_i(z)\{\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}} + \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{u}\} \tag{2.36}$$

where $\boldsymbol{x} = \hat{\boldsymbol{x}} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$, $z = x_1$, and

$$\boldsymbol{A}_1 = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ -1 & -1 \end{bmatrix},$$

$$\boldsymbol{A}_2 = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ 0.2172 & -1 \end{bmatrix},$$

$$\boldsymbol{B}_1(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{B}_2(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

The membership functions are given as

$$h_1(z_1) = \frac{\sin(x_1) - 0.2172x_1}{1.2172x_1}, \qquad h_2(z_1) = \frac{x_1 - \sin(x_1)}{1.2172x_1}. \tag{2.37}$$

2.5.2 Parallel Distributed Compensation

By using the same concept as T-S fuzzy model, the following polynomial fuzzy controller (2.38) is typically utilized to stabilize polynomial fuzzy model in (2.29).

$$\boldsymbol{u}(t) = -\sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{F}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) \tag{2.38}$$

Then, by substituting (2.38) into (2.29), the overall closed-loop system becomes (2.39).

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z}(t))h_j(\boldsymbol{z}(t)) \times \\ &\quad \{\boldsymbol{A}_i(\boldsymbol{x}(t)) - \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{F}_j(\boldsymbol{x}(t))\}\hat{\boldsymbol{x}}(\boldsymbol{x}(t))\end{aligned} \tag{2.39}$$

2.5.3 $\mathcal{S}$-procedure

In [8], an $\mathcal{S}$-procedure to encounter a constraint in quadratic forms, i.e. some quadratic forms be negative implies other quadratic forms are negative, has been presented. Given $\vartheta_i(\boldsymbol{x}(t)) = \boldsymbol{x}(t)^T\boldsymbol{Q}_i\boldsymbol{x}(t) + \boldsymbol{L}_i\boldsymbol{x}(t) + c_i$ for $i = 0, ..., k$, and consider the condition (2.40).

$$\vartheta_0(\boldsymbol{x}(t)) \in \mathcal{P}^{0+}\ \forall\boldsymbol{x}(t) \text{ such that } \vartheta_i(\boldsymbol{x}(t)) \in \mathcal{P}^{0+} \tag{2.40}$$

for $i = 1, ..., k$. If $\tau_i \geq 0$ is exist satisfying

$$\vartheta_0(\boldsymbol{x}(t)) \geq \sum_{i=1}^{k}\tau_i\vartheta_i(\boldsymbol{x}(t))\ \forall\boldsymbol{x}(t), \tag{2.41}$$

then (2.40) holds. This technique cal also be applied in polynomial cases as described in Lemma 2.5.1.

Lemma 2.5.1. *[29] Given polynomials $g_1(\boldsymbol{x}(t))$ and $g_2(\boldsymbol{x}(t))$ define sets L_1 and L_2:*
$L_1 := \{\boldsymbol{x}(t) \in \mathbb{R}^n :\ g_1(\boldsymbol{x}(t)) \leq 0\}$,
$L_2 := \{\boldsymbol{x}(t) \in \mathbb{R}^n :\ g_2(\boldsymbol{x}(t)) \leq 0\}$,
if $\sigma(\boldsymbol{x}(t)) \in \mathcal{P}^{0+}$ is exist satisfying $-g_1(\boldsymbol{x}(t)) + \sigma(\boldsymbol{x}(t))g_2(\boldsymbol{x}(t)) \in \mathcal{P}^{0+}$ for all $\boldsymbol{x}(t)$ then $L_2 \subseteq L_1$.

Chapter 3.

Stability Analysis of Polynomial Fuzzy Systems

This section provides an analysis of polynomial fuzzy system stabilization.

3.1 PPLF and Switching Controller Design

Piecewise polynomial Lyapunov function can be categorized as an extension of polynomial Lyapunov function (PLF). One of several provided PLFs is chosen as Lyapunov function simultaneously. There are two types of PPLF-based approach: minimum-type and maximum-type. This thesis only focuses on the minimum-type PPLF. In minimum-type PPLF-based approach, the chosen Lyapunov function is selected when it becomes the minimum Lyapunov function among others. It is simply described as follows.

$$V(\boldsymbol{x}(t)) = \min_{1 \leq l \leq N} V_l(\boldsymbol{x}(t)), \tag{3.1}$$

where N described the PPLF number and $V_l(\boldsymbol{x}(t))$ is a PLF. It is worth mentioning that PPLF will be reduced to PLF when $N = 1$. In order to avoid ambiguity between $V_l(\boldsymbol{x}(t))$ and $V(\boldsymbol{x}(t))$, $V_l(\boldsymbol{x}(t))$ $(l = 1, 2, \cdots, N)$ are called as partial Lyapunov functions or partial Lyapunov function candidates. Moreover, switching controller is also employed so that the advantages of PPLF-based approach can be gained.

$$\boldsymbol{u}(t) = -\sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{F}_{il}(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) \tag{3.2}$$

when $V(\boldsymbol{x}(t)) = V_l(\boldsymbol{x}(t))$ for $l = 1, 2, \cdots, N$. If $N = 1$, (3.2) reduces to (2.38). By substituting (3.2) into (2.39), the closed-loop system is written as

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z}(t))h_j(\boldsymbol{z}(t)) \times \\ &\quad \{\boldsymbol{A}_i(\boldsymbol{x}(t)) - \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{F}_{jl}(\boldsymbol{x}(t))\}\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) \qquad (3.3)\end{aligned}$$

when $V(\boldsymbol{x}(t)) = V_l(\boldsymbol{x}(t))$. If $N = 1$, the closed-loop system (3.3) reduces to (2.39). From now, t notation will be dropped. Hence, in the later discussion, $\boldsymbol{x}(t)$ will be written as $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}(\boldsymbol{x}(t))$ will be written as $\hat{\boldsymbol{x}}(\boldsymbol{x})$.

3.2 Stabilization Conditions Based on Positivstellensatz Relaxation

Theorem 3.2.1. *The switching controller* (3.2) *stabilizes the closed-loop system* (3.3) *with the equilibrium* $\boldsymbol{x} = \boldsymbol{0}$ *if Lyapunov functions* $V_l(\boldsymbol{x})$ *(*$V_l(\boldsymbol{0}) = 0$*), polynomial matrices* $\boldsymbol{F}_{jl}(\boldsymbol{x})$*,* $\zeta_{ijml}(\boldsymbol{x}), s_{kl}(\boldsymbol{x}), q_{ikl}(\boldsymbol{x}) \in \mathcal{S}$*,* $p_{kl}(\boldsymbol{x}) \in \mathcal{P}$ *and a scalar* $\alpha < 0$ *are exist such that*

$$V_l(\boldsymbol{x}) - \gamma(\boldsymbol{x}) \in \mathcal{S} \qquad (3.4)$$

$$\begin{aligned}&-\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\hat{h}_i^2\hat{h}_j^2\hat{h}_k^2\Big\{s_{kl}(\boldsymbol{x})\Big[\frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}}\times \\ &\{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_{jl}(\boldsymbol{x})\}\hat{\boldsymbol{x}}(\boldsymbol{x}) - \alpha V_l(\boldsymbol{x}) \\ &+\sum_{m=1}^{N}\zeta_{ijml}(\boldsymbol{x})(V_m(\boldsymbol{x}) - V_l(\boldsymbol{x}))\Big] + q_{ikl}(\boldsymbol{x}) + p_{kl}(\boldsymbol{x})\Big\} \\ &+\sum_{k=1}^{r}\hat{h}_k^2 p_{kl}(\boldsymbol{x}) \in \mathcal{S} \qquad (3.5)\end{aligned}$$

$$\zeta_{ijml}(\boldsymbol{x}) \in \mathcal{S} \qquad (3.6)$$

for $i, j \in \{1, ..., r\}$*,* $m, l \in \{1, ..., N\}$*.* $N \geq 0$ *denotes the PPLF number, and* $\gamma(\boldsymbol{x}) \in \mathcal{P}^+$ *is a predefined radially unbounded polynomial.*

Proof. $V_l(\boldsymbol{x})$ for $l = 1, ..., N$ with $N \geq 0$ are chosen as the candidates of Lyapunov function. Since the Lyapunov functions have to be positive definite function, a predefined polynomial $\gamma(\boldsymbol{x}) \in \mathcal{P}^+$ is employed satisfying

$$V_l(\boldsymbol{x}) - \gamma(\boldsymbol{x}) \geq 0. \tag{3.7}$$

Moreover, partial derivatives of the Lyapunov functions have to be negative definite along the trajectories. The derivatives of $V_l(\boldsymbol{x})$ is presented in (3.8).

$$\dot{V}_l(\boldsymbol{x}) = \frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} \tag{3.8}$$

By substituting (3.3) into (3.8), the condition (3.8) can be rewritten as

$$\dot{V}_l(\boldsymbol{x}) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\boldsymbol{z}) h_j(\boldsymbol{z}) \frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}} \{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x}) \boldsymbol{F}_{jl}(\boldsymbol{x})\} \hat{\boldsymbol{x}}(\boldsymbol{x}). \tag{3.9}$$

Recall Lemma 2.5.1, hence condition (3.9) is represented as (4.21).

$$\begin{aligned} \dot{V}_l(\boldsymbol{x}) \quad = \quad & \sum_{j=1}^{r} \sum_{i=1}^{r} h_i(\boldsymbol{z}) h_j(\boldsymbol{z}) \Big\{ \frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}} \{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x}) \boldsymbol{F}_{jl}(\boldsymbol{x})\} \hat{\boldsymbol{x}}(\boldsymbol{x}) \\ & + \sum_{m=1}^{N} \zeta_{ijml}(\boldsymbol{x}) (V_m(\boldsymbol{x}) - V_l(\boldsymbol{x})) \Big\} < 0, \\ & \forall \boldsymbol{x} \neq 0. \end{aligned} \tag{3.10}$$

Now, introduce $\hat{h}_i^2$ ($\sum_{i=1}^{r} \hat{h}_i^2 = 1$) to replace the term $h_i(\boldsymbol{z})$ as performed in [62]. Moreover, in order to guarantee $\dot{V}_l(\boldsymbol{x}) < 0$ at $\boldsymbol{x} \neq 0$, a scalar $\alpha < 0$ is introduced satisfying $\dot{V}_l(\boldsymbol{x}) - \alpha V_l(\boldsymbol{x}) \leq 0$. Hence, (3.10) becomes (3.11).

$$\begin{aligned} & \sum_{j=1}^{r} \sum_{i=1}^{r} \hat{h}_i^2 \hat{h}_j^2 \Big\{ \frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}} \{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x}) \boldsymbol{F}_{jl}(\boldsymbol{x})\} \hat{\boldsymbol{x}}(\boldsymbol{x}) \\ & -\alpha V_l(\boldsymbol{x}) + \sum_{m=1}^{N} \zeta_{ijml}(\boldsymbol{x}) (V_m(\boldsymbol{x}) - V_l(\boldsymbol{x})) \Big\} \leq 0, \\ & \sum_{i=1}^{r} \sum_{j=1}^{r} \hat{h}_i^2 \hat{h}_j^2 = 1, \quad \boldsymbol{x} \neq 0 \end{aligned} \tag{3.11}$$

To apply the P-satz relaxation, the empty set condition of (3.11) is written as follows.

$$\big\{\boldsymbol{x} \in \mathbb{R}^N |\ \Lambda_l(\boldsymbol{x}) \geq 0,\ \boldsymbol{x} \neq 0,\ \sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i^2\hat{h}_j^2 - 1 = 0\big\} = \varnothing \tag{3.12}$$

where

$$\begin{aligned}\Lambda_{ijl}(\boldsymbol{x}) = &\sum_{j=1}^{r}\sum_{i=1}^{r}\hat{h}_i^2\hat{h}_j^2\Big\{\frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}}\{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_{jl}(\boldsymbol{x})\}\hat{\boldsymbol{x}}(\boldsymbol{x}) - \alpha V_l(\boldsymbol{x})\\ &+ \sum_{m=1}^{N}\zeta_{ijml}(\boldsymbol{x})(V_m(\boldsymbol{x}) - V_l(\boldsymbol{x}))\Big\}.\end{aligned} \tag{3.13}$$

Let $\boldsymbol{h} = [\hat{h}_i^2 \cdots\ \hat{h}_r^2]^T$. The constraint $\boldsymbol{x} \neq \boldsymbol{0}$ in (3.12) is replaced by $q_{xl}(\boldsymbol{h},\, \boldsymbol{x}) \in \mathcal{P}^+$ with under constraint $q_{xl}(\boldsymbol{h},\, \boldsymbol{x}) \neq 0 \iff \boldsymbol{x} \neq \boldsymbol{0}$. Then define $q_{xl}(\boldsymbol{h},\, \boldsymbol{x}) = \sum_{i=1}^{r}\hat{h}_i^2 q_{xil}(\boldsymbol{x})$ with under constraint $q_{xl}(\boldsymbol{h},\, \boldsymbol{x}) \neq 0 \iff \boldsymbol{x} \neq \boldsymbol{0}$. Hence, (3.12) can be expressed as

$$\begin{aligned}\big\{\boldsymbol{x} \in \mathbb{R}^N |\ \Lambda_{ijl}(\boldsymbol{x}) \geq 0,\ &\sum_{i=1}^{r}\hat{h}_i^2 q_{xil}(\boldsymbol{x}) \neq 0,\\ &\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i^2\hat{h}_j^2 - 1 = 0\big\} = \varnothing\end{aligned} \tag{3.14}$$

where $\Lambda_l(\boldsymbol{x})$ is defined in (3.13). Now, apply P-satz described in Lemma 2.3.1 to condition (3.14) where $f_1(\boldsymbol{x})$, $g_1(\boldsymbol{x})$, and $h_1(\boldsymbol{x})$ of (2.12) in P-satz correspond to $\Lambda_l(\boldsymbol{x})$, $\sum_{i=1}^{r}\hat{h}_i^2 q_{xil}(\boldsymbol{x})$, and $\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i^2\hat{h}_j^2 - 1$ in (3.14) respectively, the condition (3.14) can be rewritten as

$$\begin{aligned}&s_{1l}(\boldsymbol{h}, \boldsymbol{x}) + s_{2l}(\boldsymbol{h},\, \boldsymbol{x})\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i^2\hat{h}_j^2\Big[\frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}} \times\\ &\{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_{jl}(\boldsymbol{x})\}\hat{\boldsymbol{x}}(\boldsymbol{x}) - \alpha V_l(\boldsymbol{x})\\ &+ \sum_{m=1}^{N}\zeta_{ijml}(\boldsymbol{x})(V_m(\boldsymbol{x}) - V_l(\boldsymbol{x}))\Big]\\ &+ p_l(\boldsymbol{h},\, \boldsymbol{x})\Big[\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i^2\hat{h}_j^2 - 1\Big]\\ &+ \Big[\sum_{i=1}^{r} q_{xil}(\boldsymbol{x})\Big]^2 = 0\end{aligned} \tag{3.15}$$

where $s_{1l}(\boldsymbol{h},\boldsymbol{x}), s_{2l}(\boldsymbol{h},\boldsymbol{x}) \in \mathcal{S}$, and $p_l(\boldsymbol{h},\boldsymbol{x}) \in \mathcal{P}$. In order to simplify (3.15), $\Big[\sum_{i=1}^{r} q_{xil}(\boldsymbol{x})\Big]^2$ are written as $\sum_{i=1}^{r}\sum_{m=1}^{r} q_{xil}(\boldsymbol{x})q_{xml}(\boldsymbol{x})$, then we can represent $q_{xil}(\boldsymbol{x})q_{xml}(\boldsymbol{x}) = q_{ikl}(\boldsymbol{x})$. Moreover, we choose $p_l(\boldsymbol{h},\,\boldsymbol{x}) = \sum_{m=1}^{r}\hat{h}_k^2 p_{kl}(\boldsymbol{x})$ and $s_{2l}(\boldsymbol{h},\,\boldsymbol{x}) = \sum_{m=1}^{r}\hat{h}_k^2 s_{kl}(\boldsymbol{x})$ where $s_{kl}(\boldsymbol{x}) \in \mathcal{S}$ and $p_{kl}(\boldsymbol{x}) \in \mathcal{P}$. Therefore, the condition (3.15) becomes (3.16).

$$
\begin{aligned}
&-\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\hat{h}_i^2\hat{h}_j^2\hat{h}_k^2\Big\{s_{kl}(\boldsymbol{x})\Big[\frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}}\times\\
&\{\boldsymbol{A}_i(\boldsymbol{x})-\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_{jl}(\boldsymbol{x})\}\hat{\boldsymbol{x}}(\boldsymbol{x})-\alpha V_l(\boldsymbol{x})\\
&+\sum_{m=1}^{N}\zeta_{ijml}(\boldsymbol{x})(V_m(\boldsymbol{x})-V_l(\boldsymbol{x}))\Big]+q_{ikl}(\boldsymbol{x})+p_{kl}(\boldsymbol{x})\Big\}\\
&+\sum_{m=1}^{r}\hat{h}_k^2 p_{kl}(\boldsymbol{x}) \in \mathcal{S}
\end{aligned}
\tag{3.16}
$$

According to the relation between a space of nonnegative polynomials and a set of SOS polynomials, the conditions (3.4) and (3.5) are sufficient conditions of (3.7) and (3.16) respectively. Therefore, the SOS problems presented in Theorem 3.2.1 are to find polynomial Lyapunov functions $V_l(\boldsymbol{x})$, polynomials $s_{kl}(\boldsymbol{x}), \zeta_{ijml}(\boldsymbol{x}), q_{ikl}(\boldsymbol{x}) \in \mathcal{S}$, $p_{kl}(\boldsymbol{x}) \in \mathcal{P}$, polynomial matrices $\boldsymbol{F}_{jl}(\boldsymbol{x})$, and a scalar $\alpha < 0$ such that conditions (3.4) - (3.6) are satisfied. □

To solve the SOS conditions in Theorem 3.2.1, an SOS solver called SOSOPT is used. However, since nonconvex term appears in (3.5), a path following algorithm (see [36,61]) is required to solve the problem which is explained in the next section.

3.2.1 Path Following Algorithm

In order to solve the nonconvex condition, path-following algorithm is presented as follows.

Step 1: Set $\eta = 0$, randomly choose $V_l(\boldsymbol{x})$ and $s_m(\boldsymbol{x})$, $l = \{1, ..., N\}$, $m = \{1, ..., r\}$

Step 2: Set $s_m(\boldsymbol{x}) = s_m^{\eta}(\boldsymbol{x})$, and $V_l(\boldsymbol{x}) = V_l^{\eta}(\boldsymbol{x})$ and solve the following optimization problem

$$\min_{\boldsymbol{F}_{jl}(\boldsymbol{x}),\zeta_{ijsl}(\boldsymbol{x}),q_{ikl}(\boldsymbol{x}),p_{kl}(\boldsymbol{x})} \alpha \textit{ subject to } (3.4),\ (3.5),\ \textit{and } (3.6).$$

If $\alpha < 0$ is obtained in Step 2, it is a strict solution and the iteration will be stopped otherwise go to Step 3.

Step 3: For the $\boldsymbol{F}_{jl}(\boldsymbol{x})$, $\zeta_{ijml}(\boldsymbol{x})$, $q_{ikl}(\boldsymbol{x})$, $p_{kl}(\boldsymbol{x})$ obtained from step 2, solve the following optimization problem

$$
\begin{aligned}
&\min_{\delta V_l(\boldsymbol{x}),\delta \boldsymbol{F}_{jl}(\boldsymbol{x}),\delta\zeta_{ijml}(\boldsymbol{x}),\delta s_{kl}(\boldsymbol{x}),\delta q_{ikl}(\boldsymbol{x}),\delta p_{kl}(\boldsymbol{x})} \alpha\ subject\ to \\
&V_l(\boldsymbol{x}) + \delta V_l(\boldsymbol{x}) - \epsilon(\boldsymbol{x}) \,\text{is SOS}, l \in \{1, ..., N\} \qquad (3.17) \\
&-\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\hat{h}_i^2\hat{h}_j^2\hat{h}_k^2\Big\{(s_{kl}(\boldsymbol{x}) + \delta s_{kl}(\boldsymbol{x}))\Big[\frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}} \times \{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_{jl}(\boldsymbol{x})\}\hat{(}\boldsymbol{x})(\boldsymbol{x}) \\
&-\alpha V_l(\boldsymbol{x}) + \sum_{m=1}^{N}\zeta_{ijml}(\boldsymbol{x})(V_m(\boldsymbol{x}) - V_l(\boldsymbol{x}))\Big] \\
&+s_{kl}(\boldsymbol{x})\Big[\frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}} \times \{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\delta\boldsymbol{F}_{jl}(\boldsymbol{x})\}\hat{\boldsymbol{x}}(\boldsymbol{x}) - \alpha\delta V_l(\boldsymbol{x}) \\
&+\{\sum_{m=1}^{N}\lambda_{ijsl}(\boldsymbol{x})(\delta V_m(\boldsymbol{x}) - \delta V_l(\boldsymbol{x})) + \delta\zeta_{ijml}(\boldsymbol{x})(V_m(\boldsymbol{x}) - V_l(\boldsymbol{x}))\}\Big] \\
&+s_{kl}(\boldsymbol{x})\Big[\frac{\partial \delta V_l(\boldsymbol{x})}{\partial \boldsymbol{x}} \times \{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_{jl}(\boldsymbol{x})\}\hat{\boldsymbol{x}}(\boldsymbol{x})\Big] \\
&+(q_{ikl}(\boldsymbol{x}) + \delta q_{ikl}(\boldsymbol{x})) + \sum_{m=1}^{r}\hat{h}_k^2(p_{kl}(\boldsymbol{x}) + \delta p_{kl}(\boldsymbol{x}))\Big\} \in \mathcal{S} \qquad (3.18) \\
&\zeta_{ijml}(\boldsymbol{x}) + \delta\zeta_{ijml}(\boldsymbol{x})\text{is SOS}, \\
&i \in \{1, ..., r\},\ \ s,\, l \in \{1, ..., N\} \qquad (3.19)
\end{aligned}
$$

$$\boldsymbol{v}_1^T \begin{bmatrix} \epsilon_v V_l^2(\boldsymbol{x}) & \delta V_l(\boldsymbol{x}) \\ \delta V_l(\boldsymbol{x}) & 1 \end{bmatrix} \boldsymbol{v}_1 \in \mathcal{S} \tag{3.20}$$

$$\boldsymbol{v}_2^T \begin{bmatrix} \epsilon_\zeta \zeta_{ijml}^2(\boldsymbol{x}) & \delta\zeta_{ijml}(\boldsymbol{x}) \\ \delta\zeta_{ijml}(\boldsymbol{x}) & 1 \end{bmatrix} \boldsymbol{v}_2 \in \mathcal{S} \tag{3.21}$$

$$\boldsymbol{v}_3^T \begin{bmatrix} \epsilon_f \boldsymbol{F}_{jl}^T(\boldsymbol{x})\boldsymbol{F}_{jl}(\boldsymbol{x}) & \delta\boldsymbol{F}_{jl}^T(\boldsymbol{x}) \\ \delta\boldsymbol{F}_{jl}(\boldsymbol{x}) & I \end{bmatrix} \boldsymbol{v}_3 \in \mathcal{S} \tag{3.22}$$

$$\boldsymbol{v}_4^T \begin{bmatrix} \epsilon_s s_{kl}^2(\boldsymbol{x}) & \delta s_{kl}(\boldsymbol{x}) \\ \delta s_{kl}(\boldsymbol{x}) & 1 \end{bmatrix} \boldsymbol{v}_4 \in \mathcal{S} \tag{3.23}$$

$$\boldsymbol{v}_5^T \begin{bmatrix} \epsilon_p p_{kl}^2(\boldsymbol{x}) & \delta p_{kl}(\boldsymbol{x}) \\ \delta p_{kl}(\boldsymbol{x}) & 1 \end{bmatrix} \boldsymbol{v}_5 \in \mathcal{S} \tag{3.24}$$

$$\boldsymbol{v}_6^T \begin{bmatrix} \epsilon_q q_{ikl}^2(\boldsymbol{x}) & \delta q_{ikl}(\boldsymbol{x}) \\ \delta q_{ikl}(\boldsymbol{x}) & 1 \end{bmatrix} \boldsymbol{v}_6 \in \mathcal{S} \tag{3.25}$$

$$i,\, j \in \{1,\, ...,\, r\},\;\; m,\, l \in \{1,\, ...,\, N\}$$

where ϵ_v, ϵ_ζ, ϵ_f, ϵ_s, ϵ_p, ϵ_q are small positive scalars for small perturbation.

Step 4:

For $\delta s_{kl}(\boldsymbol{x})$, $\delta V_l(\boldsymbol{x})$ obtained from step 3, update $s_{kl}^\eta(\boldsymbol{x})$ and $V_l^\eta(\boldsymbol{x})$ such that $s_{kl}^{(\eta+1)}(\boldsymbol{x}) = s_{kl}^\eta(\boldsymbol{x}) + \delta s_{kl}(\boldsymbol{x})$ and $V_l^{(\eta+1)}(\boldsymbol{x}) = V_l^\eta(\boldsymbol{x}) + \delta V_l(\boldsymbol{x})$; then set $\eta = \eta + 1$ and go back to step 2.

Remark 1. The path-following algorithm in this thesis is performed by utilizing SOSOPT with the most reliable options 'both'. Please note that if any feasible solutions in Step 2 are found, the solutions will be substituted into the original SOS conditions in theorem 3.2.1 and checked whether they satisfy the SOS conditions (i.e. by using "issos" command with the checking options 'both'). This double checking is important to obtain reliable solutions.

3.2.2 Design Example

In order to show the effectiveness of the proposed design described in Theorem 3.2.1, a benchmark design example employed in many literature is demonstrated in this section.

Consider the following continuous T-S fuzzy model.

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{3} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}} + \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{u}\}$$

where $\boldsymbol{z} = x_1$, $\hat{\boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{x} = [x_1 \ x_2]^T$, and

$$\boldsymbol{A}_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}, \qquad \boldsymbol{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\boldsymbol{A}_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix}, \qquad \boldsymbol{B}_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix},$$
$$\boldsymbol{A}_3 = \begin{bmatrix} -a & -4.33 \\ 0 & 0.05 \end{bmatrix}, \qquad \boldsymbol{B}_3 = \begin{bmatrix} -b+6 \\ -1 \end{bmatrix}.$$

The membership functions are given as

$$h_1(x_1) = \frac{\cos 10x_1 + 1}{4}, \ h_2(x_1) = \frac{\sin 10x_1 + 1}{4},$$
$$h_3(x_1) = \frac{-\cos 10x_1 - \sin 10x_1 + 2}{4}.$$

In this design example, set $a = 2$ and find the maximum value of b that can be achieved by solving the SOS conditions in Theorem 3.2.1. Obtaining a larger b means a more relaxed result can be achieved. The feasible region of b by applying the proposed design will then be compared with the existing results from the existing approaches. To solve the SOS conditions in Theorem 3.2.1, the following setting are used (see Table 3.1).

Table 3.1:Order setting of decision variables in Theorem 3.2.1 for design example 3.2.2.

Decision variable	Degree
$V_l(\boldsymbol{x})$	2^{nd}
$\zeta_{ijml}(\boldsymbol{x}) \in \mathcal{S}$	0
$s_{kl}(\boldsymbol{x}) \in \mathcal{S}$	0
$q_{ikl}(\boldsymbol{x}) \in \mathcal{S}$	2^{nd}
$p_{kl}(\boldsymbol{x}) \in \mathcal{P}$	2^{nd}

By using the proposed design in Theorem 3.2.1, the feasible solution is obtained for $b_{\max} = 6.5$

for second order PPLF when $N = 1$. The feasible solutions are as follows:

$$
\begin{aligned}
V(\boldsymbol{x}) &= 0.866346x_1^2 + 1.0400x_1x_2 + 10.93201x_2^2, \\
\boldsymbol{F}_{11} &= \begin{bmatrix} 31.340180 & 13.45745 \end{bmatrix}, \\
\boldsymbol{F}_{21} &= \begin{bmatrix} 173.50990 & 104.23202 \end{bmatrix}, \\
\boldsymbol{F}_{31} &= \begin{bmatrix} -33.88348 & -292.6525 \end{bmatrix}, \\
s_{11} &= 1.0471144,\ s_{21} = 5.0495888,\ s_{31} = 1.2063872, \\
p_{11}(\boldsymbol{x}) &= 0.015393x_1^2 + 0.020307x_1x_2 + 0.011605x_2^2, \\
p_{21}(\boldsymbol{x}) &= 1.780719x_1^2 + 2.087569x_1x_2 + 0.612413x_2^2, \\
p_{31}(\boldsymbol{x}) &= 0.033510x_1^2 + 0.036831x_1x_2 + 0.103569x_2^2, \\
q_{111}(\boldsymbol{x}) &= 0.016624x_1^2 + 0.021287x_1x_2 + 0.009619x_2^2, \\
q_{121}(\boldsymbol{x}) &= 2.310126x_1^2 + 2.675464x_1x_2 + 0.792256x_2^2, \\
q_{131}(\boldsymbol{x}) &= 0.064400x_1^2 - 1.334756x_1x_2 + 12.49034x_2^2, \\
q_{211}(\boldsymbol{x}) &= 2.310126x_1^2 + 2.675642x_1x_2 + 0.792256x_2^2, \\
q_{221}(\boldsymbol{x}) &= 2.635839x_1^2 + 3.092061x_1x_2 + 0.907256x_2^2, \\
q_{231}(\boldsymbol{x}) &= 1.648303x_1^2 + 1.897572x_1x_2 + 0.586931x_2^2, \\
q_{311}(\boldsymbol{x}) &= 0.064400x_1^2 - 1.334756x_1x_2 + 12.49034x_2^2, \\
q_{321}(\boldsymbol{x}) &= 1.648303x_1^2 + 1.897572x_1x_2 + 0.586931x_2^2, \\
q_{331}(\boldsymbol{x}) &= 0.044975x_1^2 - 0.187010x_1x_2 + 2.407984x_2^2, \\
\zeta_{ijml} &= 71.2881515.
\end{aligned}
$$

The comparison of the result with other existing results is shown in Table 3.2.

Table 3.2:Comparison of b_{max}.

Approach	b_{max}
Theorem 3.2.1 (2^{nd} order)	6.5
M. C. M. Teixeira et al [40]	6.0
Y. -J. Chen et al. [33]	6.0
F. Delmotte et al [38]	6.0
C. -H. fang et al [41]	6.0
M. C. M. Teixeira et al. [39]	5.0
X. Liu et al [42]	2.5

The controlled behavior of the design example can be seen in Figure 3.1. From the figure, we can see that the switching controller stabilizes the system, i.e., all the initial conditions converge to the equilibrium point.

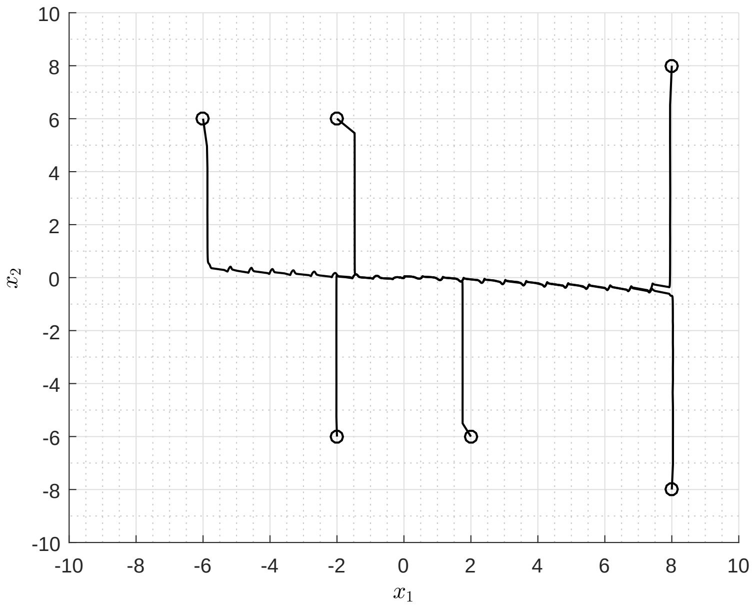

Fig. 3.1:Control results of design example 3.2.2 by solving Theorem 3.2.1

3.3 Stabilization Conditions based on Copositive Relaxation

In this section, the other relaxation technique, i.e. copositive relaxation, is applied to obtain more relaxed SOS PPLF-based stabilization conditions. The derived SOS conditions are given in Theorem 3.3.1.

Theorem 3.3.1. *The equilibrium point* $\boldsymbol{x} = \boldsymbol{0}$ *of the polynomial fuzzy systems* (2.29) *can be stabilized by the switching controller*(3.2) *if there exist polynomial matrices* $\boldsymbol{F}_{jl}(\boldsymbol{x})$*, radially unbounded polynomials (polynomial partial Lyapunov functions)* $V_l(\boldsymbol{x})$ *with* $V_l(\boldsymbol{0}) = 0$*, SOS polynomials* $\xi_{ijml}(\boldsymbol{x}) \in \mathcal{S}$*, a nonnegative integer* μ*, and a negative scalar* τ *such that*

$$V_l(\boldsymbol{x}) - \epsilon(\boldsymbol{x}) \in \mathcal{S}, \tag{3.26}$$

$$-\Big(\sum_{k=1}^{r} \hat{h}_k^2\Big)^{\mu} \sum_{i=1}^{r}\sum_{j=1}^{r} \hat{h}_i^2 \hat{h}_j^2 \Big\{\Theta_{ijl}(\boldsymbol{x}) - \tau V_l(\boldsymbol{x}) + \sum_{m=1}^{N} \xi_{ijml}(\boldsymbol{x})\{V_m(\boldsymbol{x}) - V_l(\boldsymbol{x})\}\Big\} \in \mathcal{S} \tag{3.27}$$

$$\xi_{ijml}(\boldsymbol{x}) \in \mathcal{S} \tag{3.28}$$

for $i, j \in \{1, ..., r\}$*,* $l \in \{1, ..., N\}$ *and* N *denotes the PPLF number. A polynomial* $\epsilon(\boldsymbol{x}) \in \mathcal{P}^+$ *is a predefined radially unbounded polynomial to guarantee the positivity of* $V_l(\boldsymbol{x})$*.* $\Theta_{ijl}(\boldsymbol{x})$ *are defined as*

$$\Theta_{ijl}(\boldsymbol{x}) = \frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}}\Big\{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_{jl}(\boldsymbol{x})\Big\}\hat{\boldsymbol{x}}(\boldsymbol{x}). \tag{3.29}$$

Proof. $V_l(\boldsymbol{x})$ ($\exists l \in \{1, 2, \cdots, N\}$) are partial Lyapunov function candidates selected as minimum polynomial Lyapunov functions (PLFs). If $V_l(\boldsymbol{x})$ are radially unbounded polynomials $\forall l$, then $V(\boldsymbol{x})$ is a radially unbounded polynomial. $\epsilon(\boldsymbol{x}) \in \mathcal{P}^+$ is introduced as predefined radially unbounded polynomial satisfying $V_l(\boldsymbol{x}) - \epsilon(\boldsymbol{x}) \geq 0$. This condition is used to guarantee that $V_l(\boldsymbol{x})$ are radially unbounded polynomials $\forall l$. It implies $V_l(\boldsymbol{x}) \geq \epsilon(\boldsymbol{x}) > 0$ at $\boldsymbol{x} \neq \boldsymbol{0}$. According to the definitions in 2.1, the condition can be satisfied if (3.30) holds.

$$V_l(\boldsymbol{x}) - \epsilon(\boldsymbol{x}) \in \mathcal{S} \tag{3.30}$$

Time derivatives of $V_l(\boldsymbol{x})$ are described in (3.31).

$$\dot{V}_l(\boldsymbol{x}) = \frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}}. \tag{3.31}$$

By substituting (3.3) into (3.31), (3.31) is rewritten as (3.32).

$$\dot{V}_l(\boldsymbol{x}) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\Theta_{ijl}(\boldsymbol{x}) \tag{3.32}$$

To guarantee $\dot{V}_l(\boldsymbol{x}) < 0$ at $\boldsymbol{x} \neq 0$, a scalar $\tau < 0$ is considered satisfying $\dot{V}_l(\boldsymbol{x}) \leq \tau V_l(\boldsymbol{x})$, that is,

$$\dot{V}_l(\boldsymbol{x}) - \tau V_l(\boldsymbol{x}) \leq 0. \tag{3.33}$$

The condition (3.33) is equivalent to

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\big(\Theta_{ijl}(\boldsymbol{x}) - \tau V_l(\boldsymbol{x})\big) \leq 0. \tag{3.34}$$

As a consequence of minimum-type PPLF-based approach (5.5), i.e. $V_l(\boldsymbol{x}(t))$ are chosen in the partial Lyapunov functions, (3.35) is necessary for $m = 1, 2, \cdots, N$.

$$V_l(\boldsymbol{x}) - V_m(\boldsymbol{x}) \leq 0 \tag{3.35}$$

Now, apply the $\mathcal{S}$-procedure in Lemma 2.5.1 by defining sets L_1 and L_2 as

$$\begin{aligned} L_1 &:= \{\boldsymbol{x} \in \mathbb{R}^n : \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\big(\Theta_{ijl}(\boldsymbol{x}) - \tau V_l(\boldsymbol{x})\big) \leq 0\}, \\ L_2 &:= \{\boldsymbol{x} \in \mathbb{R}^n : V_l(\boldsymbol{x}) - V_m(\boldsymbol{x}) \leq 0\}. \end{aligned}$$

According to Lemma 2.5.1, if $\xi_{ijml}(\boldsymbol{x}) \in \mathcal{P}^{0+}$ are exist such that

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\big(\Theta_{ijl}(\boldsymbol{x}) - \tau V_l(\boldsymbol{x}) + \sum_{m=1}^{N} \xi_{ijml}(\boldsymbol{x})\{V_m(\boldsymbol{x}) - V_l(\boldsymbol{x})\}\big) \leq 0 \tag{3.36}$$

holds $\forall\boldsymbol{x}$ then $L_2 \subseteq L_1$. In other words, the condition (3.34) is satisfied only if (3.37) is satisfied.

$$\sum_{m=1}^{N} \xi_{ijml}(\boldsymbol{x})\{V_m(\boldsymbol{x}) - V_l(\boldsymbol{x})\} \in \mathcal{P}^{0+} \tag{3.37}$$

According to the relation between a space of nonnegative polynomials and a set of SOS polynomials (see Section 2.1), (3.38) is a sufficient condition of (3.36).

$$-\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\big(\Theta_{ijl}(\boldsymbol{x}) - \tau V_l(\boldsymbol{x}) + \sum_{m=1}^{N} \xi_{ijml}(\boldsymbol{x})\{V_m(\boldsymbol{x}) - V_l(\boldsymbol{x})\}\big) \in \mathcal{S} \tag{3.38}$$

Apply copositive relaxation in Lemma 2.3.3, (3.38) can be rewritten as

$$-\Big(\sum_{k=1}^{r} \hat{h}_k^2\Big)^{\mu} \sum_{i=1}^{r}\sum_{j=1}^{r} \hat{h}_i^2\hat{h}_j^2\Big\{\Theta_{ijl}(\boldsymbol{x}) - \tau V_l(\boldsymbol{x}) + \sum_{m=1}^{N} \xi_{ijml}(\boldsymbol{x})\{V_m(\boldsymbol{x}) - V_l(\boldsymbol{x})\}\Big\} \in \mathcal{S}. \tag{3.39}$$

□

In order to use SOS design frameworks, $\xi_{ijml}(\boldsymbol{x})$ in (3.39) are set as SOS polynomials even though $\xi_{ijml}(\boldsymbol{x}) \in \mathcal{P}^{0+}$ are sufficient for (3.39).

Recent frameworks to efficiently solve the SOS conditions in Theorem 3.3.1 have widely discussed in many literature, e.g. [29–31, 33, 36, 54, 61, 62, 64], and so on. The SOS stabilization conditions fully takes the advantage of the PPLF properties and provides several new ideas in the derivation process of both the stabilization and robust stabilization conditions over the existing approaches. The utility of the approach can be seen though two design examples. The SOS solver that is used to solve the SOS conditions is SOSOPT [65] with the most reliable computational accuracy option 'both'.

Remark 2. Note that $\epsilon(\boldsymbol{x})$ is a slack radially unbounded polynomial to keep the positivity of $V_l(\boldsymbol{x})$. In most of cases, an extremely small positive definite polynomial is set to $\epsilon(\boldsymbol{x})$. For example, we can set $\epsilon(\boldsymbol{x}) = 10^{-6}(x_1^d + x_2^d + \cdots + x_n^d)$ where $d \in [2, 4, 6, \cdots]$ means d-th Lyapunov function.

Remark 3. Polynomial Lyapunov functions $V_l(\boldsymbol{x})$ are even degree polynomials. For fourth degree $V_l(\boldsymbol{x})$, the form is defined as

$$V_l(\boldsymbol{x}) = \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}^T \boldsymbol{P}_l^{3\times 3} \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$$

where $\boldsymbol{P}_l$ are called the Gram matrices whose elements are decision variables. SOS solver can numerically find the Gram matrices $\boldsymbol{P}_l$ such that $V_l(\boldsymbol{x})$ are positive definite polynomials. By employing an SOS solver, the SOS problems in Theorem 3.3.1 will be determined as an SOSP (sum of squares program). SOSOPT [65] or SOSTOOLS [70], SOS solvers, call an SDP solver (SDPT3, SeDuMi, etc) after the SOSPs have been automatically converted as SDPs (semidefinite programs). Then, SDP solutions can be obtained after calling the SDP solver. Finally, a conversion from SDP solutions to SOSP solutions is the last step in SOS solver. The SOSP solutions are called as feasible solutions in this thesis.

3.3.1 Path Following Algorithm

This section provides the algorithm, path following algorithm, to solve the SOS stabilization Conditions in Theorem 3.3.1.

Step 1: Set $\eta = 0$, choose Δp_{il} for $i = 1, 2, ..., \rho$ and $l = 1, ..., N$ where N is positive integer indicating the number of PPLF. For all the combinations $(p_{1l}, p_{2l}, ..., p_{\rho l})$ on all grid points $\begin{bmatrix} p_{1l}^{min} & p_{1l}^{max} \end{bmatrix} \times \cdots \times \begin{bmatrix} p_{\rho l}^{min} & p_{\rho l}^{max} \end{bmatrix}$ with the intervals $\Delta p_{1l}, \Delta p_{2l}, \ldots, \Delta p_{\rho l}$, set $V_l = \begin{bmatrix} \rho_{1l} & 0 \\ 0 & \rho_{2l} \end{bmatrix}$ where $l = 1, 2, ..., N$.

Step 2: Set $V_l(\boldsymbol{x}) = V_l^{\eta}(\boldsymbol{x})$ and solve the following optimization problem

$$\min_{\boldsymbol{F}_{jl}(\boldsymbol{x}), \xi_{ijml}(\boldsymbol{x})} \tau \text{ subject to (3.26) - (3.28) }.$$

Choose the appropriate $V_l(\boldsymbol{x})$ for $l = 1, 2, ..., N$ such that the minimum τ is obtained in step 2. If $\tau < 0$ is obtained (the feasible solution is obtained), then stop the iteration otherwise go to step 3.

Step 3: For $\boldsymbol{F}_{jl}(\boldsymbol{x})$ and $\xi_{ijml}(\boldsymbol{x})$ obtained from step 2, solve the following optimization problem, which is the linearized version of (3.26) and (3.27).

$$\begin{aligned}
&\min_{\delta V_l(\boldsymbol{x}), \delta \boldsymbol{F}_{jl}(\boldsymbol{x}), \delta \xi_{ijml}(\boldsymbol{x})} \tau \text{ subject to} \\
&-\Big(\sum_{k=1}^{r} \hat{h}_k^2\Big)^{\mu} \sum_{i=1}^{r} \sum_{j=1}^{r} \hat{h}_i^2 \hat{h}_j^2 \Big\{ \Theta_{ijl}(\boldsymbol{x}) + \delta\Theta_{ijl}(\boldsymbol{x}) - \tau(V_l(\boldsymbol{x}) + \delta V_l(\boldsymbol{x})) \\
&- \frac{\partial V_l(\boldsymbol{x})}{\partial x} (\boldsymbol{B}_i(\boldsymbol{x}) \delta \boldsymbol{F}_{jl}(\boldsymbol{x})) \hat{\boldsymbol{x}}(\boldsymbol{x}) \\
&+ \sum_{m=1}^{N} \big\{ (\xi_{ijml}(\boldsymbol{x}) + \delta\xi_{ijml}(\boldsymbol{x}))(V_m(\boldsymbol{x}) - V_l(\boldsymbol{x})) + \xi_{ijml}(\boldsymbol{x})(\delta V_m(\boldsymbol{x}) - \delta V_l(\boldsymbol{x})) \big\} \Big\} \in \mathcal{S} \quad (3.40) \\
&\xi_{ijml}(\boldsymbol{x}) + \delta\xi_{ijml}(\boldsymbol{x}) \in \mathcal{S} \quad (3.41) \\
&\boldsymbol{v}_1^T \begin{bmatrix} \epsilon_v V_l^2(\boldsymbol{x}) & \delta V_l(\boldsymbol{x}) \\ \delta V_l(\boldsymbol{x}) & 1 \end{bmatrix} \boldsymbol{v}_1 \in \mathcal{S} \quad (3.42) \\
&\boldsymbol{v}_2^T \begin{bmatrix} \epsilon_\xi \xi_{ijml}^2(\boldsymbol{x}) & \delta\xi_{ijml}(\boldsymbol{x}) \\ \delta\xi_{ijml}(\boldsymbol{x}) & 1 \end{bmatrix} \boldsymbol{v}_2 \in \mathcal{S} \quad (3.43) \\
&\boldsymbol{v}_3^T \begin{bmatrix} \epsilon_f \boldsymbol{F}_{jl}^T(\boldsymbol{x}) \boldsymbol{F}_{jl}(\boldsymbol{x}) & \delta \boldsymbol{F}_{jl}^T(\boldsymbol{x}) \\ \delta \boldsymbol{F}_{jl}(\boldsymbol{x}) & I \end{bmatrix} \boldsymbol{v}_3 \in \mathcal{S} \quad (3.44)
\end{aligned}$$

for $i, j \in \{1, ..., r\}$, $m, l \in \{1, ..., N\}$, and N is a positive integer indicating the number of PPLF. $\Theta_{ijl}(\boldsymbol{x})$ is defined as in (16) while $\delta\Theta_{ijl}(\boldsymbol{x})$ is defined as follows.

$$\delta\Theta_{ijl}(\boldsymbol{x}) = \frac{\partial \delta V_l(\boldsymbol{x})}{\partial \boldsymbol{x}} \times \boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_{jl}(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \tag{3.45}$$

ϵ_v, ϵ_ξ, ϵ_f are small positive scalars for small perturbations. In this simulation, we use $\epsilon_v = \epsilon_\xi = \epsilon_f = 0.001$. In step 3, solve SOS conditions in (4.35)-(4.40) such that minimum τ can be obtained.

Step 4:

For $\delta V_l(\boldsymbol{x})$ obtained from Step 3, update $V_l^\eta(\boldsymbol{x})$ such that $V_l^{(\eta+1)}(\boldsymbol{x}) = V_l^\eta(\boldsymbol{x}) + \delta V_l(\boldsymbol{x})$; then set $\eta = \eta + 1$ and solve the following optimization problem.

$\min_{\boldsymbol{F}_{jl}(\boldsymbol{x}),\xi_{ijml}(\boldsymbol{x})} \tau$ subject to (3.26) - (3.28).

If $\tau < 0$ is obtained, then stop the iteration otherwise go back to Step 3.

Remark 4. Please note that if $\tau < 0$ is obtained, double checking is performed to obtain reliable solutions. This is performed by substituting the solutions into the original SOS conditions and check whether they satisfy as SOS or not, i.e. using "issos" command with the checking options 'both'. If

3.3.2 Design Examples

As mentioned before, two benchmark design examples are presented to prove the effectiveness of the proposed design in Theorem 3.3.1. Example I is the well-known benchmark design example for T-S fuzzy systems. This example are used in several literature e.g. [24, 33, 35–37] . By using this benchmark design example, the result can be fairly compared between the proposed approach and other existing approaches. Example II is a benchmark design example for polynomial fuzzy systems. This example is used not only to show the effectiveness of the proposed SOS conditions but also to show that the proposed design is applicable for both T-S fuzzy systems and polynomial fuzzy systems.

Example I

Consider the following T-S fuzzy systems:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{3} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}} + \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{u}\}, \tag{3.46}$$

where $\hat{\boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{x} = [x_1 \ x_2]^T$ and $\boldsymbol{z} = x_1$,

$$\boldsymbol{A}_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}, \qquad \boldsymbol{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\boldsymbol{A}_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix}, \qquad \boldsymbol{B}_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix},$$
$$\boldsymbol{A}_3 = \begin{bmatrix} -a & -4.33 \\ 0 & 0.05 \end{bmatrix}, \qquad \boldsymbol{B}_3 = \begin{bmatrix} -b+6 \\ -1 \end{bmatrix}.$$

The membership functions are given as

$$h_1(x_1) = \frac{\cos 10x_1 + 1}{4}, \tag{3.47}$$
$$h_2(x_1) = \frac{\sin 10x_1 + 1}{4}, \tag{3.48}$$
$$h_3(x_1) = \frac{-\cos 10x_1 - \sin 10x_1 + 2}{4}. \tag{3.49}$$

In this design example, we set $a = 2.0$, the same setting as used in [24, 33, 35–37] . Then, the maximum value of b, $0 \leq b \leq 8.5$ with interval 0.5, is calculated by applying the proposed stabilization conditions in Theorem 3.3.1. The results of the proposed design can be seen in Table 3.3 which also provides comparison of $b_{\max}$ with other reported results.

From Table 3.3, it can be concluded that the proposed stabilization conditions in Theorem 3.3.1 provides the most relaxed results compared with other existing approaches. Moreover, the effect of increasing the PPLF number has also been investigated by setting $N = 1, 2, 3$ which indicates PPLF_1(PLF), PPLF_2, and PPLF_3, respectively. The Lyapunov functions are set to as second and fourth degree polynomials. The results of PLF approach ($N = 1$) and PPLF approach ($N > 1$) are summarized in Table 3.4.

Table 3.3:Comparison of b_{max}.

Method	b_{max}
Theorem 3.3.1 (4^{th} order)	8.0
Theorem 3.3.1 (2^{nd} order)	7.0
Y. -J. Chen et al. [36]	6.5
V. F. Montagner et al. [37]	6.5
A. Sala [35]	6.5
Y. -J. Chen et al. [33]	6.0
F. Delmotte et al [38]	6.0
M. C. M. Teixeira et al. [39]	5.0
X. Liu et al [42]	2.5

Table 3.4:Comparison of b_{max} for PLF and PPLF approach with $a = 2$.

	2^{nd} order	4^{th} order
PLF	$b_{max} = 6.5$	$b_{max} = 7.0$
PPLF_2	$b_{max} = 7.0$	$b_{max} = 8.0$
PPLF_3	$b_{max} = 7.0$	$b_{max} = 8.0$

For $b = 8.0$, feasible solutions (4^{th} order PPLF_2) are obtained as follows:

$$\begin{aligned}
V_1(\boldsymbol{x}) &= 0.000585x_1^4 - 0.001959x_1^3x_2 + 0.01213x_1^2x_2^2 \\
&\quad + 0.021719x_1x_2^3 + 0.0116808x_2^4, \\
V_2(\boldsymbol{x}) &= 0.023715x_1^4 - 0.026073x_1^3x_2 + 0.478558x_1^2x_2^2 \\
&\quad + 0.503281x_1x_2^3 + 3.666267x_2^4, \\
\boldsymbol{F}_{11} &= \begin{bmatrix} 5.403357 & 0.063883 \end{bmatrix}, \\
\boldsymbol{F}_{12} &= \begin{bmatrix} 4.91963 & -3.5975 \end{bmatrix}, \\
\boldsymbol{F}_{21} &= \begin{bmatrix} 12.6513 & 10.95309 \end{bmatrix}, \\
\boldsymbol{F}_{22} &= \begin{bmatrix} 13.2867 & 12.06171 \end{bmatrix}, \\
\boldsymbol{F}_{31} &= \begin{bmatrix} -5.4542 & -40.982 \end{bmatrix}, \\
\boldsymbol{F}_{32} &= \begin{bmatrix} -4.6248 & -45.402 \end{bmatrix}.
\end{aligned}$$

For 4^{th} order PPLF_2, the control result of an initial state $\boldsymbol{x}(0) = \begin{bmatrix} 5 & -10 \end{bmatrix}^T$ is presented in

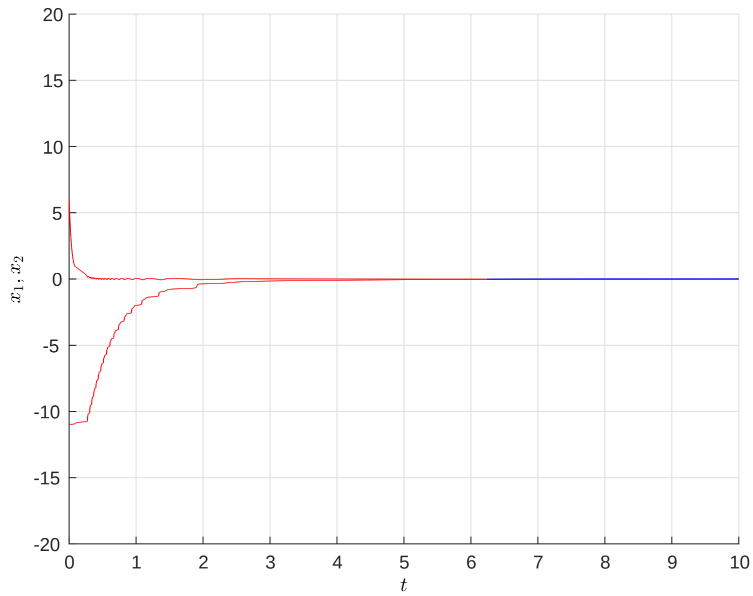

Fig. 3.2:Control result for an initial state $\boldsymbol{x}(0) = [5 \quad -10]^T$ in fourth order PPLF_2 approach.

Figure 3.2. The blue line indicates $V(\boldsymbol{x}) = V_1(\boldsymbol{x})$ while the red line indicates $V(\boldsymbol{x}) = V_2(\boldsymbol{x})$. The use of switching controller can also be seen in Figure 3.2, i.e. $\boldsymbol{F}_{i2}$ is switched to $\boldsymbol{F}_{i1}$ around 6 [sec.] in order to stabilize the system.

Controlled behavior of the system with six initial states is given in Figure 3.3 for 4^{th} order PPLF_2 approach ($a = 2.0$ and $b = 8.0$). By employing the designed switching controller, it shows that all the initial states converge to zero. The region of $V_1(\boldsymbol{x})$ ("$\times$")and $V_2(\boldsymbol{x})$ ("$+$") for 4^{th} order PPLF_2 is shown in Figure 3.4 for $a = 2.0$ and $b = 8.0$.

Example II

Consider the following polynomial fuzzy system:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{3} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}} + \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{u}\}, \tag{3.50}$$

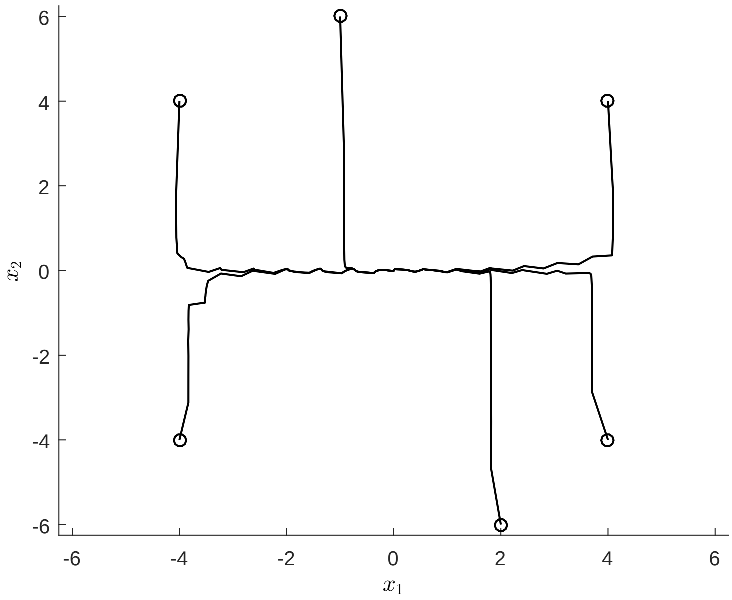

Fig. 3.3:Control trajectories of six initial states: $\boldsymbol{x}(0) = [2 \quad -6]^T$, $\boldsymbol{x}(0) = [4 \quad -4]^T$, $\boldsymbol{x}(0) = [4 \quad 4]^T$, $\boldsymbol{x}(0) = [1 \quad 6]^T$, $\boldsymbol{x}(0) = [-4 \quad 4]^T$, $\boldsymbol{x}(0) = [-4 \quad -4]^T$ for fourth order PPLF$_2$ approach ($a = 2.0$ and $b = 8.0$).

where $\hat{\boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{x} = [x_1 \ x_2]^T$ and $\boldsymbol{z} = x_1$,

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} 1.59 + x_1^2 - 2x_2^2 - x_1x_2 & -7.29 + 2x_1x_2 \\ 0.01 & -x_1^2 - x_2^2 \end{bmatrix},$$

$$\boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} 0.02 + x_1^2 - 2x_2^2 - x_1x_2 & -4.64 + 2x_1x_2 \\ 0.35 & 0.21 - x_1^2 - x_2^2 \end{bmatrix},$$

$$\boldsymbol{A}_3(\boldsymbol{x}) = \begin{bmatrix} -a + x_1^2 - 2x_2^2 - x_1x_2 & -4.33 + 2x_1x_2 \\ 0 & 0.05 - x_1^2 - x_2^2 \end{bmatrix},$$

$$\boldsymbol{B}_1(\boldsymbol{x}) = \begin{bmatrix} 1 + x_1 + x_1^2 \\ 0 \end{bmatrix},$$

$$\boldsymbol{B}_2(\boldsymbol{x}) = \begin{bmatrix} 8 + x_1 + x_1^2 \\ 0 \end{bmatrix},$$

$$\boldsymbol{B}_3(\boldsymbol{x}) = \begin{bmatrix} -b + 6 + x_1 + x_1^2 \\ -1 \end{bmatrix}.$$

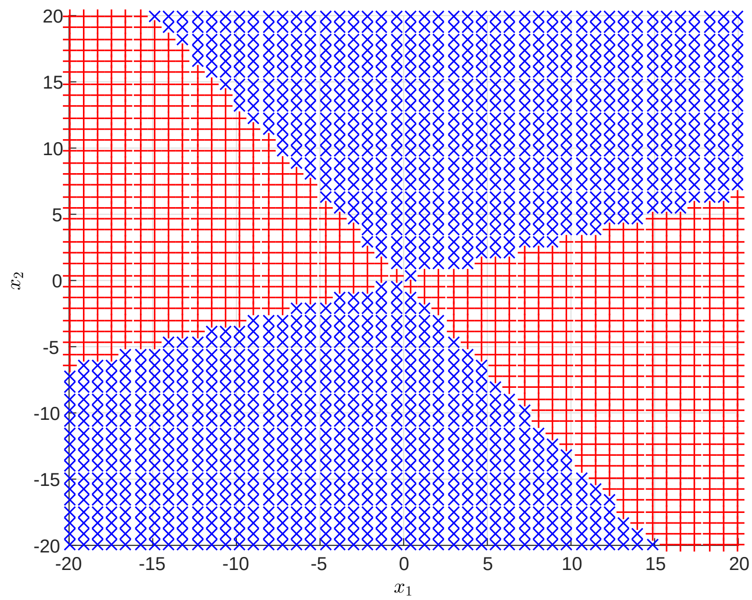

Fig. 3.4:Switching boundaries of fourth order of $V_1(\boldsymbol{x})$ and $V_2(\boldsymbol{x})$ for $a = 2.0$ and $b = 8.0$. $\times$ for $V_1(\boldsymbol{x})$ and $+$ for $V_2(\boldsymbol{x})$.

In the polynomial fuzzy system, a and b are constant parameters, and the membership functions are given as

$$h_1(x_1) = \frac{1}{1 + e^{(125x_1+12)/2}},$$
$$h_2(x_1) = \frac{1}{1 + e^{-(125x_1-12)/2}},$$
$$h_3(x_1) = 1 - h_1(x_1) - h_2(x_1).$$

Since other existing approaches are mostly in LMI-based frameworks and they cannot be applied for this design example, we only compare our proposed design with other SOS-based approach, i.e. PLF approach. The global stabilization conditions in Theorem 3.3.1 are solved for all combinations of $2 \leq a \leq 5.5$ for 2^{nd} and 4^{th} order of PLFs and PPLFs. By setting $2 \leq a \leq 5.5$, we try to obtain the maximum value of b that can be achieved for both PLFs and PPLFs. The feasible solutions can be seen in Figure 3.5. Note thet the plotted mark in Figure 3.5 is accumulative. In other words, plotted region in Figure 3.5 indicates that o (2^{nd} order PPLF$_2$) $\subset$ $+$ (4^{th} order PLF) $\subset$ $\times$ (4^{th} order PPLF$_2$).

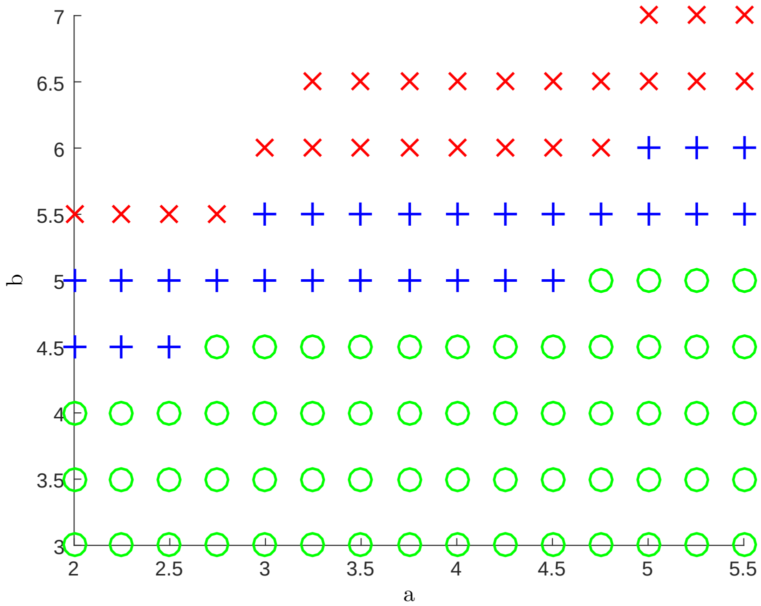

Fig. 3.5:Feasible solutions area of Example II (o for 2^{nd} order $PPLF_2$, + for 4^{th} order PLF, and × for 4^{th} order $PPLF_2$).

For 2^{nd} order Lyapunov function, PLF approach (the existing approach) fails to obtain any feasible solutions, while in the $PPLF_2$, $b_{max} = 5.0$ can be achieved for $a = 5.5$. From the results, the PPLF approach ($N > 1$) provides better results compared with the existing PLF approach.

For 4^{th} order $PPLF_2$ with $a = 2$ and $b = 5.5$ which is an infeasible point of both 4^{th} order

PLF and 2^{nd} order PPLF$_2$), the obtained feasible solutions are as follows:

$$\begin{aligned}
V_1(\boldsymbol{x}) &= 0.334122x_1^4 + 0.657829x_1^3x_2 + 1.069435x_1^2x_2^2 \\
&\quad + 0.900811x_1x_2^3 + 3.974909x_2^4, \\
V_2(\boldsymbol{x}) &= 0.3663079x_1^4 + 0.7750424x_1^3x_2 + 1.036988x_1^2x_2^2 \\
&\quad + 0.789109x_1x_2^3 + 3.942412x_2^4, \\
\boldsymbol{F}_{11} &= \begin{bmatrix} 12.52659 & 4.115568 \end{bmatrix}, \\
\boldsymbol{F}_{12} &= \begin{bmatrix} 14.13437 & 3.423371 \end{bmatrix}, \\
\boldsymbol{F}_{21} &= \begin{bmatrix} 10.98544 & 7.711715 \end{bmatrix}, \\
\boldsymbol{F}_{22} &= \begin{bmatrix} 8.382026 & 6.896101 \end{bmatrix}, \\
\boldsymbol{F}_{31} &= \begin{bmatrix} 6.852751 & -12.30076 \end{bmatrix}, \\
\boldsymbol{F}_{32} &= \begin{bmatrix} 13.43515 & -19.20387 \end{bmatrix}.
\end{aligned}$$

Control result of an initial state $\boldsymbol{x}(0) = \begin{bmatrix} -6 & -4 \end{bmatrix}^T$ is shown in Figure 3.6 for 4^{th} order PPLF$_2$ ($a = 2$ and $b = 5.5$). The blue line implies $V(\boldsymbol{x}) = V_1(\boldsymbol{x})$ which indicates $\boldsymbol{u} = \boldsymbol{F}_{i1}(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x})$ is used as control input. The red line implies $V(\boldsymbol{x}) = V_2(\boldsymbol{x})$ which indicates $\boldsymbol{u} = \boldsymbol{F}_{i2}(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x})$ is used as control input.

From the figure, it can be seen that the controller switches for four times which indicates by color changing (from blue to red and vice versa). For instance, at $t = 0$, the chosen Lyapunov function is $V_1(\boldsymbol{x})$ so that controller $\boldsymbol{F}_{i1}(\boldsymbol{x})$ is employed to stabilize the system. After that, the controller switches from $\boldsymbol{F}_{i1}(\boldsymbol{x})$ to $\boldsymbol{F}_{i2}(\boldsymbol{x})$. At around 1.5 [sec.], the controller switches from $\boldsymbol{F}_{i2}(\boldsymbol{x})$ to $\boldsymbol{F}_{i1}(\boldsymbol{x})$. The controlled behavior of the system with six initial states is presented in Figure 3.7. From the figure, it can be seen that all the initial states converge to the equilibrium point.

The region of $V_1(\boldsymbol{x})$ and $V_2(\boldsymbol{x})$ is plotted in Figure 3.8 which are marked by "$\times$" and "$+$", respectively.

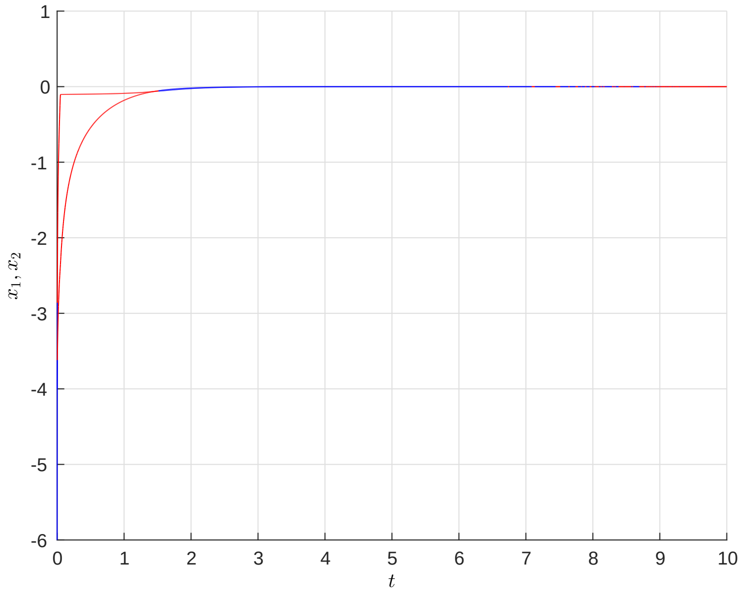

Fig. 3.6:Control result for an initial state $\boldsymbol{x}(0) = [-6 \quad -4]^T$ in fourth order PPLF$_2$ approach.

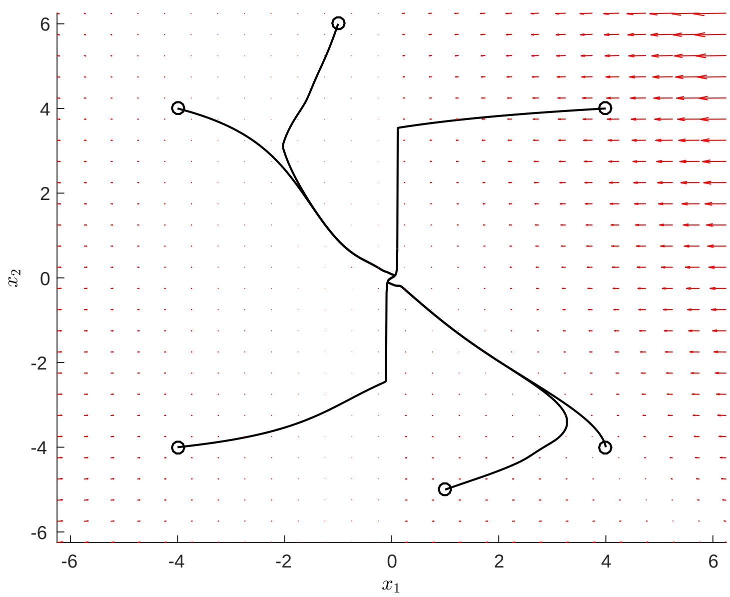

Fig. 3.7:Trajectories of control results ($a = 2.0$, and $b = 5.5$) for six initial states: $\boldsymbol{x}(0) = [2 \quad -6]^T$, $\boldsymbol{x}(0) = [4 \quad -4]^T$, $\boldsymbol{x}(0) = [4 \quad 4]^T$, $\boldsymbol{x}(0) = [1 \quad 6]^T$, $\boldsymbol{x}(0) = [-4 \quad 4]^T$, $\boldsymbol{x}(0) = [-4 \quad -4]^T$ in fourth order PPLF$_2$ approach.

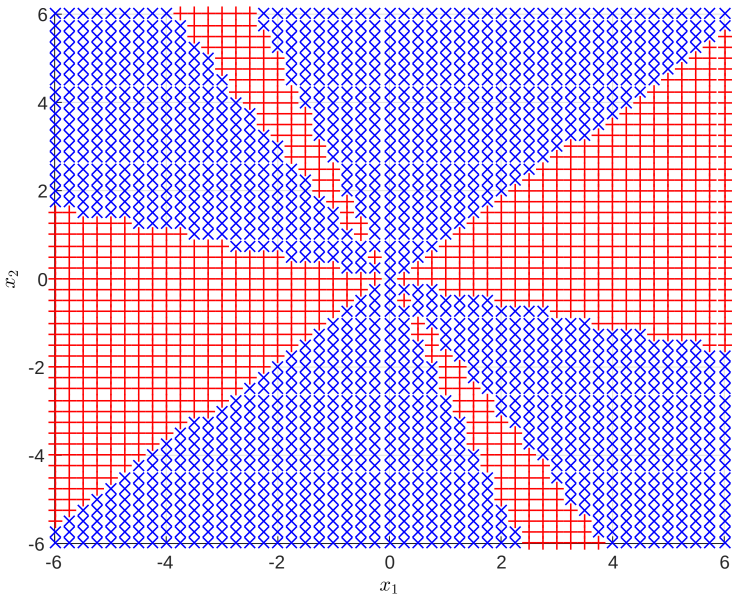

Fig. 3.8:Switching boundaries of fourth order Lyapunov functions $V_1(\boldsymbol{x})$ and $V_2(\boldsymbol{x})$. $\times$ for $V_1(\boldsymbol{x})$ and $+$ for $V_2(\boldsymbol{x})$.

Chapter 4.

Robust Control of Polynomial Fuzzy Systems

In reality, an error of a plant model is possibly appear in the modeling process which means there is a differences between the actual plant and the used model in control design. Hence, robust control theory has became important feature in designing control systems. This section provides robust control of such systems represented as polynomial fuzzy model by assuming there are uncertain elements in plant dynamics.

In [44,61], robust control design of polynomial fuzzy systems, by considering uncertainties in the systems and input terms, has been performed by utilizing a polynomial Lyapunov function (PLF) approach. In order to compare the proposed robust stabilization conditions (PPLF-based approach) with other existing approaches, i.e. [44,61], two design examples are also demonstrated in this section.

4.1 Polynomial Fuzzy Systems with Uncertainties

Global robust stabilization of polynomial fuzzy system first has been performed by Cao et al [44] and Tanaka et al [61] by employing PLF approach. Motivating from the results showed in the previous section, i.e. global stabilization of polynomial fuzzy systems, PPLF approach is also considered to be used in robust control of polynomial fuzzy systems. Newly derived robust stabilization conditions of polynomial fuzzy systems with uncertainties are presented in this section. The definition of polynomial fuzzy system with uncertainties is described as follows.

Model Rule i:

IF $z_1(t)$ is M_{i1} and $\cdots$ and $z_o(t)$ is M_{io},

$$\text{THEN } \dot{\boldsymbol{x}}(t) = \{\boldsymbol{A}_i(\boldsymbol{x}(t)) + \boldsymbol{D}_{ai}(\boldsymbol{x}(t))\boldsymbol{\Delta}_{ai}(\boldsymbol{x}(t))\boldsymbol{E}_{ai}(\boldsymbol{x}(t))\}\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) + \{\boldsymbol{B}_i(\boldsymbol{x}(t)) + \boldsymbol{D}_{bi}(\boldsymbol{x}(t))\boldsymbol{\Delta}_{bi}(\boldsymbol{x}(t))\boldsymbol{E}_{bi}(\boldsymbol{x}(t))\}\boldsymbol{u}(t), \quad i = 1, 2, \cdots, r, \tag{4.1}$$

The defuzzification process of the polynomial fuzzy system (4.1) is represented as

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\big\{\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) + \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{D}_{ai}(\boldsymbol{x})\boldsymbol{\Delta}_{ai}(\boldsymbol{x})\boldsymbol{E}_{ai}(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) + \boldsymbol{D}_{bi}(\boldsymbol{x})\boldsymbol{\Delta}_{bi}(\boldsymbol{x})\boldsymbol{E}_{bi}(\boldsymbol{x})\boldsymbol{u}\big\}. \tag{4.2}$$

$\boldsymbol{D}_{ai}(\boldsymbol{x}), \boldsymbol{D}_{bi}(\boldsymbol{x}), \boldsymbol{E}_{ai}(\boldsymbol{x})$, and $\boldsymbol{E}_{bi}(\boldsymbol{x})$ are polynomial matrices in $\boldsymbol{x}(\boldsymbol{x})$. $\boldsymbol{\Delta}_{ai}(\boldsymbol{x})$ and $\boldsymbol{\Delta}_{bi}(\boldsymbol{x})$ are the uncertain matrices that satisfy $||\boldsymbol{\Delta}_{ai}(\boldsymbol{x})|| \leq \beta_{ai}(\boldsymbol{x})$ and $||\boldsymbol{\Delta}_{bi}(\boldsymbol{x})|| \leq \beta_{bi}(\boldsymbol{x})$ respectively where $\beta_{ai}(\boldsymbol{x})$ and $\beta_{bi}(\boldsymbol{x})$ denotes the upper bound of the uncertainty norm.

The global robust stabilization conditions of polynomial fuzzy systems with uncertainties (4.2) are presented in Theorem 4.2.1.

4.2 PPLF-based Robust Stabilization Design of Polynomial Fuzzy Systems

Theorem 4.2.1. *The polynomial fuzzy system with uncertainties* (4.2) *is stabilized by the switching controller* (3.2) *if radially unbounded polynomials* $V_l(\boldsymbol{x})$ *(*$V_l(\boldsymbol{0}) = 0$*), polynomial matrices* $\boldsymbol{F}_{jl}(\boldsymbol{x})$, $\xi_{ijml}(\boldsymbol{x}) \in \mathcal{S}$, $\Pi_{ijl}(\boldsymbol{x}) \in \mathcal{P}$, $\mu \geq 0$, *a scalar* $\tau < 0$ *and a scalar* $\phi_l > 0$ *are exist and* *satisfythe following conditions.*

$$V_l(\boldsymbol{x}) - \epsilon(\boldsymbol{x}) \in \mathcal{S}, \tag{4.3}$$

$$-\Big(\sum_{k=1}^{r}\hat{h}_k^2\Big)^{\mu}\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i^2\hat{h}_j^2\Big(\Theta_{ijl}(\boldsymbol{x}) + \Pi_{ijl}(\boldsymbol{x}) - \tau V_l(\boldsymbol{x}) + \sum_{m=1}^{N}\xi_{ijml}(\boldsymbol{x})\{V_m(\boldsymbol{x}) - V_l(\boldsymbol{x})\}\Big) \in \mathcal{S}, \tag{4.4}$$

$$\boldsymbol{v}_1^T\boldsymbol{G}_{ijl}(\phi_l, \boldsymbol{x})\boldsymbol{v}_1 \in \mathcal{S} \tag{4.5}$$

for i, $j \in \{1, ..., r\}$, $l \in \{1, ..., N\}$, *and* $N \geq 1$ *denotes PPLF number.* $\boldsymbol{v}_1$ *is an independent vector, and* $\epsilon(\boldsymbol{x}) \in \mathcal{P}^+$ *is a predefined radially unbounded polynomial to guarantee positive definiteness of* $V_l(\boldsymbol{x})$*.* $\Theta_{ijl}(\boldsymbol{x})$ *have the same definition as in* (3.29)*, and* $\boldsymbol{G}_{ijl}(\phi_l, \boldsymbol{x})$ *are defined as*

$$\boldsymbol{G}_{ijl}(\phi_l, \boldsymbol{x}) = \begin{bmatrix} \phi_l\Pi_{ijl}(\boldsymbol{x}) - \Xi_{ijml}(\boldsymbol{x}) & \boldsymbol{\mathcal{M}}_{ijl}^T(\phi_l, \boldsymbol{x}) \\ \boldsymbol{\mathcal{M}}_{ijl}(\phi_l, \boldsymbol{x}) & \boldsymbol{\mathcal{T}} \end{bmatrix},$$

where

$$\Xi_{ijml}(\boldsymbol{x}) = \sum_{m=1}^{N}\zeta_{ijml}(\boldsymbol{x})(V_m(\boldsymbol{x}) - V_l(\boldsymbol{x})), \tag{4.6}$$

$$\boldsymbol{\mathcal{M}}_{ijl}(\phi_l, \boldsymbol{x}) = \begin{bmatrix} \phi_l\beta_{ai}(\boldsymbol{x})\boldsymbol{D}_{ai}^T(\frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}})^T \\ \phi_l\beta_{bi}(\boldsymbol{x})\boldsymbol{D}_{bi}^T(\frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}})^T \\ \boldsymbol{E}_{ai}(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \\ \boldsymbol{E}_{bi}(\boldsymbol{x})\boldsymbol{F}_{jl}(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \end{bmatrix}, \tag{4.7}$$

$$\boldsymbol{\mathcal{T}} = 2\boldsymbol{I}_4. \tag{4.8}$$

Proof. Define $V_l(\boldsymbol{x})$ ($\exists l \in \{1, 2, \cdots, N\}$) as Lyapunov function candidates chosen as the minimum PLFs. $V(\boldsymbol{x}) \to \infty$ at $||\boldsymbol{x}|| \to \infty$ (radially unbounded) if $V_l(\boldsymbol{x}) \to \infty$ at $||\boldsymbol{x}|| \to \infty$ for all l.

In order to guarantee radially unboundedness of $V_l(\boldsymbol{x})$, $\epsilon(\boldsymbol{x}) \in \mathcal{P}^+$ is introduced satisfying $V_l(\boldsymbol{x}) - \epsilon(\boldsymbol{x}) \geq 0$. That condition implies $V_l(\boldsymbol{x}) \geq \epsilon(\boldsymbol{x}) > 0$ at $\boldsymbol{x} \neq \boldsymbol{0}$. According to the relation of a soace of nonnegative polynomials and a set of SOS polynomials (see Section 2.1), the following condition is a sufficient condition of $V_l(\boldsymbol{x}) - \epsilon(\boldsymbol{x}) \geq 0$.

$$V_l(\boldsymbol{x}) - \epsilon(\boldsymbol{x}) \in \mathcal{S} \tag{4.9}$$

Partial derivatives of $V_l(\boldsymbol{x})$ are described as

$$\dot{V}_l(\boldsymbol{x}) = \frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}}. \tag{4.10}$$

Substitute the closed-loop system (4.2) and the control input (3.2) into (4.10), condition (4.10) becomes

$$\dot{V}_l(\boldsymbol{x}) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\boldsymbol{z}) h_j(\boldsymbol{z}) \Big(\Theta_{ijl}(\boldsymbol{x}) + \Gamma_{ijl}(\boldsymbol{x}) \Big) \tag{4.11}$$

where $\Gamma_{ijl}(\boldsymbol{x}) = \boldsymbol{\varphi}_{il}(\boldsymbol{x}) \boldsymbol{\vartheta}_{ijl}(\boldsymbol{x})$, $\boldsymbol{\varphi}_{il}(\boldsymbol{x})$ and $\boldsymbol{\vartheta}_{ijl}(\boldsymbol{x})$ are as follows:

$$\boldsymbol{\varphi}_{il}(\boldsymbol{x}) = \left[\frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{\Omega}_{ai}(\boldsymbol{x}) \quad -\frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{\Omega}_{bi}(\boldsymbol{x}) \right], \tag{4.12}$$

$$\boldsymbol{\vartheta}_{ijl}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{E}_{ai}(\boldsymbol{x}) \hat{\boldsymbol{x}}(\boldsymbol{x}) \\ \boldsymbol{E}_{bi}(\boldsymbol{x}) \boldsymbol{F}_{jl}(\boldsymbol{x}) \hat{\boldsymbol{x}}(\boldsymbol{x}) \end{bmatrix} \tag{4.13}$$

for $\boldsymbol{\Omega}_{ai}(\boldsymbol{x}) = \boldsymbol{D}_{ai}(\boldsymbol{x}) \boldsymbol{\Delta}_{ai}(\boldsymbol{x})$ and $\boldsymbol{\Omega}_{bi}(\boldsymbol{x}) = \boldsymbol{D}_{bi}(\boldsymbol{x}) \boldsymbol{\Delta}_{bi}(\boldsymbol{x})$. Relation between $\boldsymbol{\varphi}_{il}(\boldsymbol{x})$ and $\boldsymbol{\vartheta}_{ijl}(\boldsymbol{x})$ can be expressed as

$$\begin{aligned} \phi_l \boldsymbol{\varphi}_{il}(\boldsymbol{x}) \boldsymbol{\varphi}_{il}^T(\boldsymbol{x}) + & \frac{1}{\phi_l} \boldsymbol{\vartheta}_{ijl}^T(\boldsymbol{x}) \boldsymbol{\vartheta}_{ijl}(\boldsymbol{x}) \\ & \geq \boldsymbol{\varphi}_{il}(\boldsymbol{x}) \boldsymbol{\vartheta}_{ijl}(\boldsymbol{x}) + \boldsymbol{\varphi}_{il}^T(\boldsymbol{x}) \boldsymbol{\vartheta}_{ijl}^T(\boldsymbol{x}) \end{aligned} \tag{4.14}$$

for any $\phi_l > 0$ and positive integer l. Since $\boldsymbol{\varphi}_{il}(\boldsymbol{x}) \boldsymbol{\vartheta}_{ijl}(\boldsymbol{x}) = \boldsymbol{\varphi}_{il}^T(\boldsymbol{x}) \boldsymbol{\vartheta}_{ijl}^T(\boldsymbol{x})$, we have the following relation.

$$\phi_l \boldsymbol{\varphi}_{il}(\boldsymbol{x}) \boldsymbol{\varphi}_{il}^T(\boldsymbol{x}) + \frac{1}{\phi_l} \boldsymbol{\vartheta}_{ijl}^T(\boldsymbol{x}) \boldsymbol{\vartheta}_{ijl}(\boldsymbol{x}) \geq 2 \boldsymbol{\varphi}_{il}(\boldsymbol{x}) \boldsymbol{\vartheta}_{ijl}(\boldsymbol{x}) \tag{4.15}$$

Hence, by substituting (4.12) and (4.13) into (4.15), the condition becomes (4.16).

$$
\begin{aligned}
\boldsymbol{\varphi}_{il}(\boldsymbol{x})\boldsymbol{\vartheta}_{ijl}(\boldsymbol{x}) &\leq \frac{\phi_l}{2}\boldsymbol{\varphi}_{il}(\boldsymbol{x})\boldsymbol{\varphi}_{il}^T(\boldsymbol{x}) + \frac{1}{2\phi_l}\boldsymbol{\vartheta}_{ijl}^T(\boldsymbol{x})\boldsymbol{\vartheta}_{ijl}(\boldsymbol{x}) \\
&= \frac{\phi_l}{2}\frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{\Omega}_{ai}(\boldsymbol{x})\boldsymbol{\Omega}_{ai}^T(\boldsymbol{x})\Big(\frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}}\Big)^T \\
&\quad + \frac{\phi_l}{2}\frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{\Omega}_{bi}(\boldsymbol{x})\boldsymbol{\Omega}_{bi}^T(\boldsymbol{x})\Big(\frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}}\Big)^T \\
&\quad + \frac{1}{2\phi_l}\hat{\boldsymbol{x}}^T(\boldsymbol{x})\boldsymbol{E}_{ai}^T(\boldsymbol{x})\boldsymbol{E}_{ai}(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \\
&\quad + \frac{1}{2\phi_l}\hat{\boldsymbol{x}}^{\boldsymbol{T}}(\boldsymbol{x})\boldsymbol{F}_{jl}^T(\boldsymbol{x})\boldsymbol{E}_{bi}^T(\boldsymbol{x}) \\
&\quad \times \boldsymbol{E}_{bi}(\boldsymbol{x})\boldsymbol{F}_{jl}(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \\
&\leq \boldsymbol{\Lambda}_{ijl}(\phi_l, \boldsymbol{x})
\end{aligned} \tag{4.16}
$$

$\boldsymbol{\Lambda}_{ijl}(\phi_l, \boldsymbol{x})$ in (4.16) are given as follows.

$$
\boldsymbol{\Lambda}_{ijl}(\phi_l, \boldsymbol{x}) = \frac{1}{2\phi_l}\boldsymbol{\mathcal{M}}_{ijl}(\phi_l, \boldsymbol{x})^T\boldsymbol{\mathcal{M}}_{ijl}(\phi_l, \boldsymbol{x}) \tag{4.17}
$$

From inequality (4.17), (4.11) becomes

$$
\dot{V}_l(\boldsymbol{x}) \leq \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\Big(\Theta_{ijl}(\boldsymbol{x}) + \boldsymbol{\Lambda}_{ijl}(\boldsymbol{x})\Big). \tag{4.18}
$$

Now, introduce polynomials $\Pi_{ijl}(\boldsymbol{x})$ satisfying (4.19).

$$
\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\Big(-\boldsymbol{\Lambda}_{ijl}(\phi_l, \boldsymbol{x}) + \Pi_{ijl}(\boldsymbol{x})\Big) \in \mathcal{P}^{0+} \tag{4.19}
$$

By introducing polynomials $\Pi_{ijl}(\boldsymbol{x})$ satisfying (4.19), (4.18) is converted to (4.20).

$$
\dot{V}_l(\boldsymbol{x}) \leq \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\Big(\Theta_{ijl}(\boldsymbol{x}) + \Pi_{ijl}(\boldsymbol{x})\Big) \tag{4.20}
$$

To utilize the properties of PPLF approach (minimum type), Lemma 2.5.1 is applied to the condition (4.20) as applied in global stabilization analysis.

Moreover, the negative definiteness of $\dot{V}_l(\boldsymbol{x}) < 0$ at $\boldsymbol{x} \neq 0$ can be guaranteed by considering a scalar $\tau < 0$ such that $\dot{V}_l(\boldsymbol{x}) \leq \Theta_{ijl}(\boldsymbol{x}) + \Pi_{ijl}(\boldsymbol{x}) \leq \tau V_l(\boldsymbol{x})$. Those conditions are represented in the following condition.

$$
\begin{aligned}
\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\Big(&\Theta_{ijl}(\boldsymbol{x}) + \Pi_{ijl}(\boldsymbol{x}) - \tau V_l(\boldsymbol{x}) \\
&+ \sum_{m=1}^{N} \xi_{ijml}(\boldsymbol{x})\{V_m(\boldsymbol{x}) - V_l(\boldsymbol{x})\}\Big) \leq 0.
\end{aligned} \tag{4.21}
$$

The following SOS condition is a sufficient condition of (4.21).

$$\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\boldsymbol{z})h_j(\boldsymbol{z})\Big(-\Theta_{ijl}(\boldsymbol{x})-\Pi_{ijl}(\boldsymbol{x})+\tau V_l(\boldsymbol{x}) - \sum_{m=1}^{N}\xi_{ijml}(\boldsymbol{x})\{V_m(\boldsymbol{x})-V_l(\boldsymbol{x})\}\Big)\in\mathcal{S}. \tag{4.22}$$

By utilizing the same technique as used in the global stabilization analysis based on copositive relaxation, we arrive at the following SOS condition.

$$-\Big(\sum_{k=1}^{r}\hat{h}_k^2\Big)^{\mu}\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i^2\hat{h}_j^2\Big(\Theta_{ijl}(\boldsymbol{x})+\Pi_{ijl}(\boldsymbol{x})-\tau V_l(\boldsymbol{x}) + \sum_{m=1}^{N}\xi_{ijml}(\boldsymbol{x})\{V_m(\boldsymbol{x})-V_l(\boldsymbol{x})\}\Big)\in\mathcal{S} \tag{4.23}$$

Note that $\xi_{ijml}(\boldsymbol{x})\in\mathcal{P}^{0+}$ are sufficient in condition (4.23). However, since all conditions should be reformulated in SOS frameworks,hence $\xi_{ijml}(\boldsymbol{x})$ are defined as SOS polynomials.

From (4.19), condition (4.24) is satisfied for $\phi_l>0$.

$$\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\boldsymbol{z})h_j(\boldsymbol{z})\Big(\phi_l\Pi_{ijl}(\boldsymbol{x})-\phi_l\Lambda_{ijl}(\phi_l,\boldsymbol{x})\Big)\in\mathcal{P}^{0+} \tag{4.24}$$

Again, by applying Lemma 2.5.1 to the condition (4.24), it can be rewritten as (4.25).

$$\phi_l\Pi_{ijl}(\boldsymbol{x})-\sum_{m=1}^{N}\zeta_{ijml}(\boldsymbol{x})\{V_m(\boldsymbol{x})-V_l(\boldsymbol{x})\} - \phi_l\boldsymbol{\Lambda}_{ijl}(\phi_l,\boldsymbol{x})\in\mathcal{P}^{0+} \tag{4.25}$$

According to the Schur complement, (4.25) can be transformed to (4.26)

$$\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{G}_{ijl}(\phi_l,\boldsymbol{x})\in\mathcal{P}^{0+} \tag{4.26}$$

where

$$\boldsymbol{G}_{ijl}(\phi_l,\boldsymbol{x})=\begin{bmatrix}\phi_l\Pi_{ijl}(\boldsymbol{x})-\Xi_{ijml}(\boldsymbol{x}) & \boldsymbol{\mathcal{M}}_{ijl}^T(\phi_l,\boldsymbol{x})\\ \boldsymbol{\mathcal{M}}_{ijl}(\phi_l,\boldsymbol{x}) & \boldsymbol{\mathcal{T}}\end{bmatrix}. \tag{4.27}$$

$\Xi_{ijml}(\boldsymbol{x})$, $\boldsymbol{\mathcal{M}}_{ijl}(\phi_l,\boldsymbol{x})$ and $\boldsymbol{\mathcal{T}}$ are as given in (4.6), (4.7) and (4.8), respectively. □

Remark 5. Conditions (4.5) always hold even if $\beta_{ai}(\boldsymbol{x})=0$ and $\beta_{bi}(\boldsymbol{x})=0$, i.e., no uncertainties case, $\boldsymbol{D}_{ai}(\boldsymbol{x})=\boldsymbol{D}_{bi}(\boldsymbol{x})=\boldsymbol{E}_{ai}(\boldsymbol{x})=\boldsymbol{E}_{bi}(\boldsymbol{x})=\boldsymbol{0}$. In other words, Theorem 4.2.1 is equivalent with 3.3.1 if the uncertainties are not exist.

Corollary 4.2.2. *Assume there is no uncertainties with respect to the input, i.e.* $\boldsymbol{\Delta}_{bi}(\boldsymbol{x}) = \mathbf{0} \forall i$. *The closed-loop system* (4.2) *is stabilized by the switching controller* (3.2) *if Lyapunov functions* $V_l(\boldsymbol{x})$ ($V_l(\mathbf{0}) = 0$), *feedback gains* $\boldsymbol{F}_{jl}(\boldsymbol{x})$, $\xi_{ijml}(\boldsymbol{x}) \in \mathcal{S}$, $\zeta_{iml}(\boldsymbol{x}) \in \mathcal{S}$, $\Pi_{il}(\boldsymbol{x}) \in \mathcal{P}$, *scalars* $\mu \geq 0$, $\tau < 0$ *and* $\phi_l > 0$ *are exist satisfying*

$$V_l(\boldsymbol{x}) - \epsilon(\boldsymbol{x}) \in \mathcal{S} \tag{4.28}$$

$$-\Big(\sum_{k=1}^{r} \hat{h}_k^2\Big)^{\mu} \sum_{i=1}^{2}\sum_{j=1}^{r} \hat{h}_i^2 \hat{h}_j^2 \Big(\Theta_{ijl}(\boldsymbol{x}) + \Pi_{ijl}(\boldsymbol{x}) - \tau V_l(\boldsymbol{x}) + \sum_{m=1}^{N} \xi_{ijml}(\boldsymbol{x})\{V_m(\boldsymbol{x}) - V_l(\boldsymbol{x})\}\Big) \in \mathcal{S} \tag{4.29}$$

$$\boldsymbol{v}_1^T \boldsymbol{G}_{il}(\phi_l, \boldsymbol{x}) \boldsymbol{v}_1 \in \mathcal{S} \tag{4.30}$$

where i, $j \in \{1, ..., r\}$, $l \in \{1, ..., N\}$, N is a positive integer, and $\boldsymbol{v}_1$ denotes a vector that is independent of $\boldsymbol{x}$. A small radially unbounded polynomial $\epsilon(\boldsymbol{x}) \in \mathcal{P}^+$ is a given slack variable to keep the positivity of $V_l(\boldsymbol{x})$. $\Theta_{ijl}(\boldsymbol{x})$ in $\Theta_{ijl}(\boldsymbol{x})$ have the same definition as in (3.29) while $\boldsymbol{G}_{il}(\phi_l, \boldsymbol{x})$ are defined as

$$\boldsymbol{G}_{il}(\phi_l, x) = \begin{bmatrix} \phi_l \Pi_{il}(\boldsymbol{x}) - \Xi_{iml}(\boldsymbol{x}) & \boldsymbol{\mathcal{M}}_{il}^T(\phi_l, \boldsymbol{x}) \\ \boldsymbol{\mathcal{M}}_{il}(\phi_l, \boldsymbol{x}) & \boldsymbol{\mathcal{T}} \end{bmatrix} \tag{4.31}$$

where

$$\Xi_{iml}(\boldsymbol{x}) = \sum_{m=1}^{N} \zeta_{iml}(\boldsymbol{x})(V_m(\boldsymbol{x}) - V_l(\boldsymbol{x})), \tag{4.32}$$

$$\boldsymbol{\mathcal{M}}_{il}(\phi_l, \boldsymbol{x}) = \begin{bmatrix} \phi_l \beta_{ai}(\boldsymbol{x}) \boldsymbol{D}_{ai}^T \big(\frac{\partial V_l(\boldsymbol{x})}{\partial \boldsymbol{x}}\big)^T \\ \boldsymbol{E}_{ai}(\boldsymbol{x}) \hat{\boldsymbol{x}}(\boldsymbol{x}) \end{bmatrix}, \tag{4.33}$$

$$\boldsymbol{\mathcal{T}} = 2\boldsymbol{I}_2. \tag{4.34}$$

4.3 Path Following Algorithm

This section provides the algorithm, path following algorithm, to solve the SOS robust stabilization conditions in Theorem 4.2.1.

Step 1: Set $\eta = 0$, choose Δp_{il} for $i = 1, 2, ..., \rho$ and $l = 1, ..., N$ where N is positive integer indicating the number of PPLF. For all the combinations $(p_{1l}, p_{2l}, ..., p_{\rho l})$ on all grid points $\begin{bmatrix} p_{1l}^{min} & p_{1l}^{max} \end{bmatrix} \times \cdots \times \begin{bmatrix} p_{\rho l}^{min} & p_{\rho l}^{max} \end{bmatrix}$ with the intervals $\Delta p_{1l}, \Delta p_{2l}, \ldots, \Delta p_{\rho l}$, set $V_l = \begin{bmatrix} \rho_{1l} & 0 \\ 0 & \rho_{2l} \end{bmatrix}$ where $l = 1, 2, ..., N$.

Step 2: Set $V_l(\boldsymbol{x}) = V_l^{\eta}(\boldsymbol{x})$ and solve the following optimization problem

$$\min_{\boldsymbol{F}_{jl}(\boldsymbol{x}), \xi_{ijml}(\boldsymbol{x}), \zeta_{ijml}(\boldsymbol{x}), \Pi_{ijl}(\boldsymbol{x})} \tau \text{ subject to (4.3), (4.4), and (4.5).}$$

Choose the appropriate $V_l(\boldsymbol{x})$ for $l = 1, 2, ..., N$ such that the minimum τ is obtained in step 2. If $\tau < 0$ is obtained (the feasible solution is obtained), then stop the iteration otherwise go to step 3.

Step 3: For $\boldsymbol{F}_{jl}(\boldsymbol{x}), \zeta_{ijml}(\boldsymbol{x}), \Pi_{ijl}(\boldsymbol{x})$, and $\xi_{ijml}(\boldsymbol{x})$ obtained from step 2, solve the following optimization problem, which is the linearized version of (4.3), (4.4), and (4.5).

$$\min_{\delta V_l(\boldsymbol{x}),\delta \boldsymbol{F}_{jl}(\boldsymbol{x}),\delta\xi_{ijml}(\boldsymbol{x}),\delta\Xi_{ijml}(\boldsymbol{x}),\delta\Pi_{ijl}(\boldsymbol{x})} \tau \text{ subject to}$$

$$-\Big(\sum_{k=1}^{r}\hat{h}_k^2\Big)^{\mu}\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i^2\hat{h}_j^2\Big\{\Theta_{ijl}(\boldsymbol{x})+\delta\Theta_{ijl}(\boldsymbol{x})-\tau(V_l(\boldsymbol{x})+\delta V_l(\boldsymbol{x}))$$
$$+\Pi_{ijl}(\boldsymbol{x})+\delta\Pi_{ijl}(\boldsymbol{x})-\frac{\partial V_l(\boldsymbol{x})}{\partial x}(\boldsymbol{B}_i(\boldsymbol{x})\delta\boldsymbol{F}_{jl}(\boldsymbol{x}))\hat{\boldsymbol{x}}(\boldsymbol{x})$$
$$+\sum_{m=1}^{N}\{(\xi_{ijml}(\boldsymbol{x})+\delta\xi_{ijml}(\boldsymbol{x}))(V_m(\boldsymbol{x})-V_l(\boldsymbol{x}))+\xi_{ijml}(\boldsymbol{x})(\delta V_m(\boldsymbol{x})-\delta V_l(\boldsymbol{x}))\Big\}\in\mathcal{S} \quad (4.35)$$

$$-\Big(\sum_{k=1}^{r}\hat{h}_k^2\Big)^{\mu}\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i^2\hat{h}_j^2\boldsymbol{v}_G^T\big(\boldsymbol{G}_{ijl}(\phi_l,\boldsymbol{x})+\delta\boldsymbol{G}_{ijl}(\phi,\boldsymbol{x})\big)\boldsymbol{v}_G\in\mathcal{S} \quad (4.36)$$

$$\xi_{ijml}(\boldsymbol{x})+\delta\xi_{ijml}(\boldsymbol{x})\in\mathcal{S} \quad (4.37)$$

$$\boldsymbol{v}_1^T\begin{bmatrix}\epsilon_v V_l^2(\boldsymbol{x}) & \delta V_l(\boldsymbol{x})\\ \delta V_l(\boldsymbol{x}) & 1\end{bmatrix}\boldsymbol{v}_1\in\mathcal{S} \quad (4.38)$$

$$\boldsymbol{v}_2^T\begin{bmatrix}\epsilon_\xi\xi_{ijml}^2(\boldsymbol{x}) & \delta\xi_{ijml}(\boldsymbol{x})\\ \delta\xi_{ijml}(\boldsymbol{x}) & 1\end{bmatrix}\boldsymbol{v}_2\ \in\mathcal{S} \quad (4.39)$$

$$\boldsymbol{v}_3^T\begin{bmatrix}\epsilon_f\boldsymbol{F}_{jl}^T(\boldsymbol{x})\boldsymbol{F}_{jl}(\boldsymbol{x}) & \delta\boldsymbol{F}_{jl}^T(\boldsymbol{x})\\ \delta\boldsymbol{F}_{jl}(\boldsymbol{x}) & I\end{bmatrix}\boldsymbol{v}_3\in\mathcal{S} \quad (4.40)$$

$$\boldsymbol{v}_4^T\begin{bmatrix}\epsilon_\phi\phi_l^2\delta\phi_l\\ \delta\phi_l\ \ 1\end{bmatrix}\boldsymbol{v}_4\in\mathcal{S} \quad (4.41)$$

$$\boldsymbol{v}_5^T\begin{bmatrix}\epsilon_\Pi\Pi_{ijl}(\boldsymbol{x})^2\delta\Pi_{ijl}(\boldsymbol{x})\\ \delta\Pi_{ijl}(\boldsymbol{x})\quad 1\end{bmatrix}\boldsymbol{v}_5 \quad (4.42)$$

$$\boldsymbol{v}_6^T\begin{bmatrix}\epsilon_\zeta\zeta_{ijml}(\boldsymbol{x})^2\delta\zeta_{ijml}(\boldsymbol{x})^2\\ \delta\zeta_{ijml}(\boldsymbol{x})^2\quad 1\end{bmatrix}\boldsymbol{v}_6 \quad (4.43)$$

for $i,\ j\in\{1,\ ...,\ r\}$, $m,\ l\in\{1,\ ...,\ N\}$, and N is a positive integer indicating the number of PPLF. $\Theta_{ijl}(\boldsymbol{x})$ is defined as in (16) while $\delta\Theta_{ijl}(\boldsymbol{x})$ is defined as follows.

$$\delta\Theta_{ijl}(\boldsymbol{x})=\frac{\partial\delta V_l(\boldsymbol{x})}{\partial\boldsymbol{x}}\times\boldsymbol{A}_i(\boldsymbol{x})-\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_{jl}(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) \quad (4.44)$$

ϵ_v, ϵ_ξ, ϵ_f are small positive scalars for small perturbations. In this simulation, we use ϵ_v =

$\epsilon_\xi = \epsilon_f = 0.001$. In step 3, solve SOS conditions in (4.35)-(4.40) such that minimum τ can be obtained.

Step 4:

For $\delta V_l(\boldsymbol{x})$ obtained from step 3, update $V_l^\eta(\boldsymbol{x})$ such that $V_l^{(\eta+1)}(\boldsymbol{x}) = V_l^\eta(\boldsymbol{x}) + \delta V_l(\boldsymbol{x})$; then set $\eta = \eta + 1$ and go back to step 2.

4.4 Design Examples

4.4.1 Example III

Consider the following polynomial fuzzy system with uncertainties:

$$\begin{aligned}\dot{\boldsymbol{x}} =& \sum_{i=1}^{r} h_i(\boldsymbol{z}) \Big\{ \big(\boldsymbol{A}_i(\boldsymbol{x}) + \boldsymbol{D}_{ai}(\boldsymbol{x}) \boldsymbol{\Delta}_{ai}(\boldsymbol{x}) \boldsymbol{E}_{ai}(\boldsymbol{x}) \big) \boldsymbol{x} \\ & + \boldsymbol{B}_i(\boldsymbol{x}) \boldsymbol{u} \Big\}\end{aligned}$$

where $r = 3$, $\hat{\boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, $\boldsymbol{z} = x_1$, and

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} 1.59 + x_1^2 - 2x_2^2 - x_1x_2 & -7.29 + 2x_1x_2 \\ 0.01 & -x_1^2 - x_2^2 \end{bmatrix},$$

$$\boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} 0.02 + x_1^2 - 2x_2^2 - x_1x_2 & -4.64 + 2x_1x_2 \\ 0.35 & 0.21 - x_1^2 - x_2^2 \end{bmatrix},$$

$$\boldsymbol{A}_3(\boldsymbol{x}) = \begin{bmatrix} -a + x_1^2 - 2x_2^2 - x_1x_2 & -4.33 + 2x_1x_2 \\ 0 & 0.05 - x_1^2 - x_2^2 \end{bmatrix},$$

$$\boldsymbol{B}_1(\boldsymbol{x}) = \begin{bmatrix} 1 + x_1 + x_1^2 \\ 0 \end{bmatrix},$$

$$\boldsymbol{B}_2(\boldsymbol{x}) = \begin{bmatrix} 8 + x_1 + x_1^2 \\ 0 \end{bmatrix},$$

$$\boldsymbol{B}_3(\boldsymbol{x}) = \begin{bmatrix} -b + 6 + x_1 + x_1^2 \\ -1 \end{bmatrix},$$

$$\boldsymbol{D}_{a1}(\boldsymbol{x}) = \boldsymbol{D}_{a2}(\boldsymbol{x}) = \boldsymbol{D}_{a3}(\boldsymbol{x}) = \begin{bmatrix} c \\ 0 \end{bmatrix},$$

$$\boldsymbol{\Delta}_{a1}(\boldsymbol{x}) = \boldsymbol{\Delta}_{a2}(\boldsymbol{x}) = \boldsymbol{\Delta}_{a3}(\boldsymbol{x}) = \Delta(t)/c,$$

$$\boldsymbol{E}_{a1}(\boldsymbol{x}) = \boldsymbol{E}_{a2}(\boldsymbol{x}) = \boldsymbol{E}_{a3}(\boldsymbol{x}) = \begin{bmatrix} x_1 & 0 \end{bmatrix},$$

with a, b, and c are constant parameters. The membership functions are given as

$$h_1(x_1) = \frac{1}{1 + e^{(125x_1+12)/2}},$$
$$h_2(x_1) = \frac{1}{1 + e^{-(125x_1-12)/2}},$$
$$h_3(x_1) = 1 - h_1(x_1) - h_2(x_1).$$

$\Delta(t)$ is the uncertainty satisfying $|\Delta(t)| \leq c$ and $\|\boldsymbol{\Delta}_{a1}(\boldsymbol{x})\| = \|\boldsymbol{\Delta}_{a2}(\boldsymbol{x})\| = \|\Delta(t)/c\| \leq 1$ from $|\Delta(t)| \leq c$, $\beta_{a1}(\boldsymbol{x}) = \beta_{a2}(\boldsymbol{x}) = 1$. The effectiveness of the proposed design (Corollary 4.2.2) can be demonstrated by finding maximum c, a parameter for the norm of the uncertainty. The feasible region of the proposed design (PPLF approach) can be compared with the feasible region of the existing robust stabilization designs [44, 61].

By setting $a = 3.0$ and $4.5 \leq b \leq 5.5$ with the interval 0.5, the uncertainty value c_{max} of

Table 4.1: c_{max} of PPLF-based robust stabilization (Corollary 4.2.2) and PLF-based robust stabilization [44, 61] for $a = 3$.

Approach	$b = 4.5$	$b = 5.0$	$b = 5.5$
Cao et al [44]	infeasble	infeasible	infeasible
Tanaka et al [61]	$c \leq 0.79$	$c \leq 0.64$	$c \leq 0.42$
Corollary 4.2.2 $PPLF_2$	$c \leq 1.88$	$c \leq 1.71$	$c \leq 0.68$

the proposed approach (Corollary 4.2.2) and the existing approaches [44, 61] are calculated. In this design example, we consider fourth order Lyapunov functions since PLF-based approaches proposed in [61] fails to find any feasible solutions for second order PLF. Therefore, to fairly compare our proposed robust stabilization design with the existing PLF approaches ([44, 61]), fourth order Lyapunov function are investigated.

Table 4.1 shows c_{max} of Corollary 4.2.2, Theorem 1 in [44] and Theorem 2 in [61] for $a = 3$ and $4.5 \leq b \leq 5.5$. The calculation results $c_{max} = 0.79$ by performing PLF approach in [61] while PPLF approach results $c_{max} = 1.88$ for $b = 4.5$. Robust stabilization design proposed in [44] fails to find any feasible solutions by using the same setting. It can be seen that our proposed design produces the most relaxed results compared to [44, 61]. For $a = 3$, $b = 5.0$ and $c = 1.71$, a point where the existing approaches [44, 61] fail to find any feasible solutions, we find feedback gains and partial Lyapunof functions as follows:

$$V_1(\boldsymbol{x}) = 0.092513x_1^4 + 0.187352x_1^3x_2 + 0.430232x_1^2x_2^2 + 0.575256x_1x_2^3 + 3.149431x_2^4,$$
$$V_2(\boldsymbol{x}) = 0.089803x_1^4 + 0.064798x_1^3x_2 + 0.324644x_1^2x_2^2 + 0.474259x_1x_2^3 + 3.114964x_2^4,$$
$$\boldsymbol{F}_{11} = \begin{bmatrix} 20.049363 & 10.029594 \end{bmatrix},$$
$$\boldsymbol{F}_{12} = \begin{bmatrix} 4.7361688 & -1.169653 \end{bmatrix},$$
$$\boldsymbol{F}_{21} = \begin{bmatrix} 29.851701 & 30.126719 \end{bmatrix},$$
$$\boldsymbol{F}_{22} = \begin{bmatrix} 5.3369071 & 4.3670417 \end{bmatrix},$$
$$\boldsymbol{F}_{31} = \begin{bmatrix} 71.144731 & -57.69251 \end{bmatrix},$$
$$\boldsymbol{F}_{32} = \begin{bmatrix} 6.7869621 & -12.81281 \end{bmatrix}.$$

The regions of $V_1(\boldsymbol{x})$ and $V_2(\boldsymbol{x})$ are shown in Figure 4.1 indicated by "×" and "+", respectively. Figure 4.2 shows controlled behavior ($a = 3.0$, $b = 5.0$ and $c = 1.71$) of six initial states. From the control results, it can be seen that all the initial states go to the equilibrium point. In the simulations, $\Delta(t)$ satisfying $|\Delta(t)| \leq c$ is set as $\Delta(t) = c\sin(200\pi t)$. The polynomial fuzzy system with uncertainty can be stabilized by the designed switching controller for all the initial states.

4.4.2 Example IV

Consider the following polynomial fuzzy systems with uncertainties:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\big\{\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) + \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{D}_{ai}(\boldsymbol{x})\boldsymbol{\Delta}_{ai}(\boldsymbol{x})\boldsymbol{E}_{ai}(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) + \boldsymbol{D}_{bi}(\boldsymbol{x})\boldsymbol{\Delta}_{bi}(\boldsymbol{x})\boldsymbol{E}_{bi}(\boldsymbol{x})\boldsymbol{u}\big\}.$$

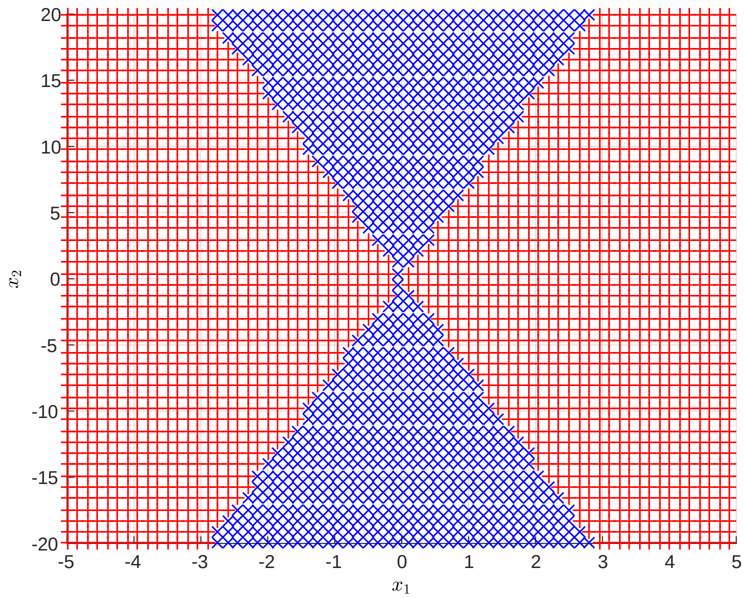

Fig. 4.1:Switching boundaries of fourth order of $V_1(\boldsymbol{x})$ and $V_2(\boldsymbol{x})$ for $a = 3.0$, $b = 5.0$, and $c = 1.71$. $\times$ for $V_1(\boldsymbol{x})$ and $+$ for $V_2(\boldsymbol{x})$.

with $r = 2$, $\hat{\boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$, $\boldsymbol{z} = x_1$, and

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ -2 & -6 \end{bmatrix},$$

$$\boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ 2 & -6 \end{bmatrix},$$

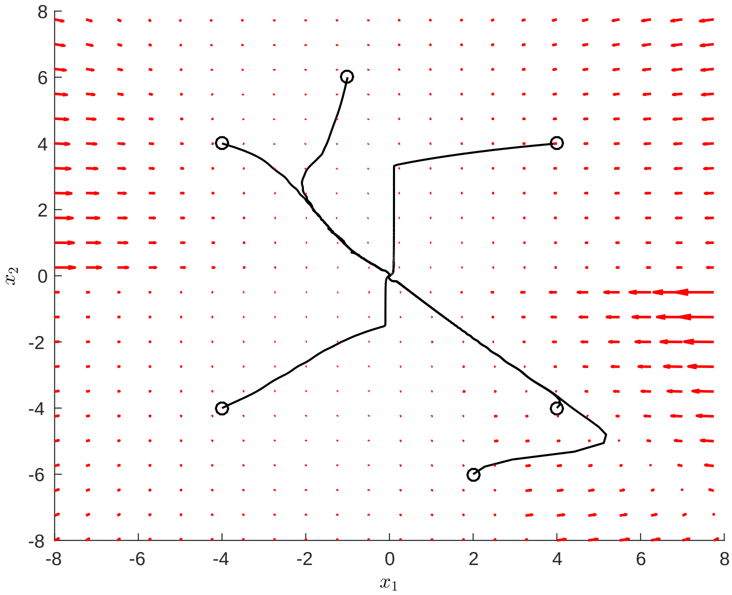

Fig. 4.2:Controlled behavior of six initial states: $\boldsymbol{x}(0) = [2 \quad -6]^T$, $\boldsymbol{x}(0) = [4 \quad -4]^T$, $\boldsymbol{x}(0) = [4 \quad 4]^T$, $\boldsymbol{x}(0) = [1 \quad 6]^T$, $\boldsymbol{x}(0) = [-4 \quad 4]^T$, $\boldsymbol{x}(0) = [-4 \quad -4]^T$ for fourth order PPLF$_2$ approach ($a = 3.0$, $b = 5.0$, and $c = 1.71$).

$$\boldsymbol{B}_1(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ -4 \end{bmatrix}, \quad \boldsymbol{B}_2(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ 4 \end{bmatrix},$$

$$\boldsymbol{D}_{a1}(\boldsymbol{x}) = \boldsymbol{D}_{a2}(\boldsymbol{x}) = \begin{bmatrix} q_a \\ 0 \end{bmatrix},$$

$$\boldsymbol{D}_{b1}(\boldsymbol{x}) = \begin{bmatrix} 0 \\ -4q_b \end{bmatrix}, \quad \boldsymbol{D}_{b2}(\boldsymbol{x}) = \begin{bmatrix} 0 \\ 4q_b \end{bmatrix},$$

$$\boldsymbol{\Delta}_{a1}(\boldsymbol{x}) = \boldsymbol{\Delta}_{a2}(\boldsymbol{x}) = \Delta_a(t)/q_a,$$

$$\boldsymbol{\Delta}_{b1}(\boldsymbol{x}) = \boldsymbol{\Delta}_{b2}(\boldsymbol{x}) = \Delta_b(t)/q_b,$$

$$\boldsymbol{E}_{a1}(\boldsymbol{x}) = \boldsymbol{E}_{a2}(\boldsymbol{x}) = [1 \ 0],$$

$$\boldsymbol{E}_{b1}(\boldsymbol{x}) = \boldsymbol{E}_{b2}(\boldsymbol{x}) = 1,$$

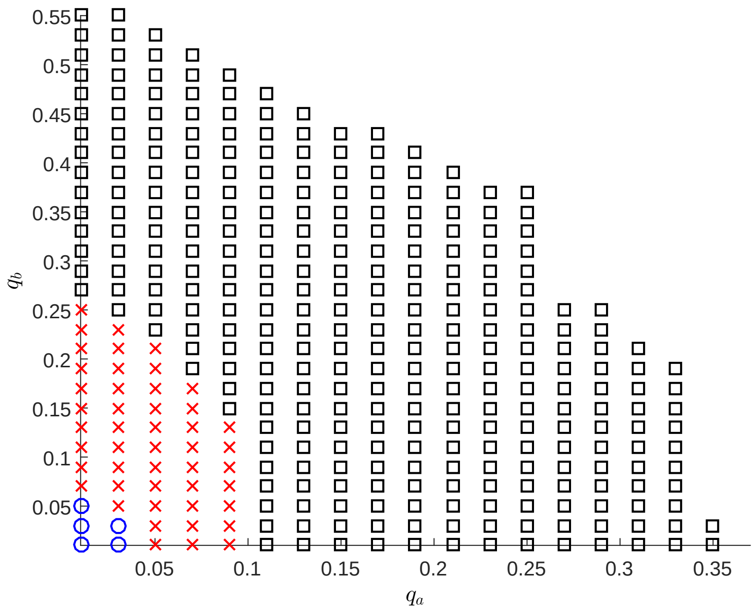

Fig. 4.3:Feasible regions of PPLF approach in Theorem 4.2.1 (□), and the existing SOS framework designs in [61] (×) and [44] (∘).

where q_a and q_b are constant parameters. Description of the membership functions are

$$h_1(\boldsymbol{z}) = \frac{\sin(x_1) + 1}{2}, \quad h_2(\boldsymbol{z}) = \frac{1 - \sin(x_1)}{2}.$$

$\Delta_a(t)$ and $\Delta_b(t)$ are the uncertainties satisfying $|\Delta_a(t)| \leq q_a$ and $|\Delta_b(t)| \leq q_b$. As performed in the previous design example, in this design example we also compare the results of our proposed robust control design with those of [44,61]. The comparison can be conducted by comparing the feasible regions of q_a and q_b parameters that can be achieved by the three robust control designs. Since $\|\boldsymbol{\Delta}_{a1}(\boldsymbol{x})\| = \|\boldsymbol{\Delta}_{a2}(\boldsymbol{x})\| = \|\Delta_a(t)/q_a\| \leq 1$ and $\|\boldsymbol{\Delta}_{b1}(\boldsymbol{x})\| = \|\boldsymbol{\Delta}_{b2}(\boldsymbol{x})\| = \|\Delta_b(t)/q_b\| \leq 1$, $\beta_{a1}(\boldsymbol{x}) = \beta_{a2}(\boldsymbol{x}) = \beta_{b1}(\boldsymbol{x}) = \beta_{b2}(\boldsymbol{x}) = 1$.

Figure 4.3 provides the feasible regions obtained by our PPLF based robust control design (Theorem 4.2.1) and PLF approaches performed in [44, 61]. In this design example (a more complex design example compared with the previous design example), our PPLF-based robust control design also produces the most relaxed results. Note that, the plotted mark in Figure 4.3 is accumulative. Hence, Figure 4.3 indicates that ∘ (PLF-based robust stabilization in [44]) $\subset$ × (PLF-based robust stabilization in [61]) $\subset$ □ (PPLF-based robust stabilization in Theorem 4.2.1). For $q_a = 0.17$ and $q_b = 0.30$, a point where the existing SOS based robust control

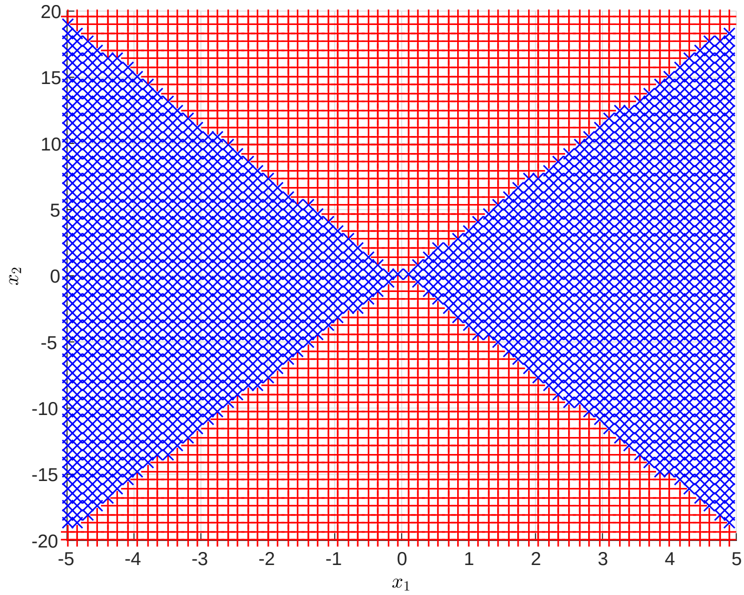

Fig. 4.4:The regions of second order $V_1(\boldsymbol{x})$ ($\times$) and $V_2(\boldsymbol{x})$ ($+$) for $q_a = 0.17$, and $q_b = 0.3$.

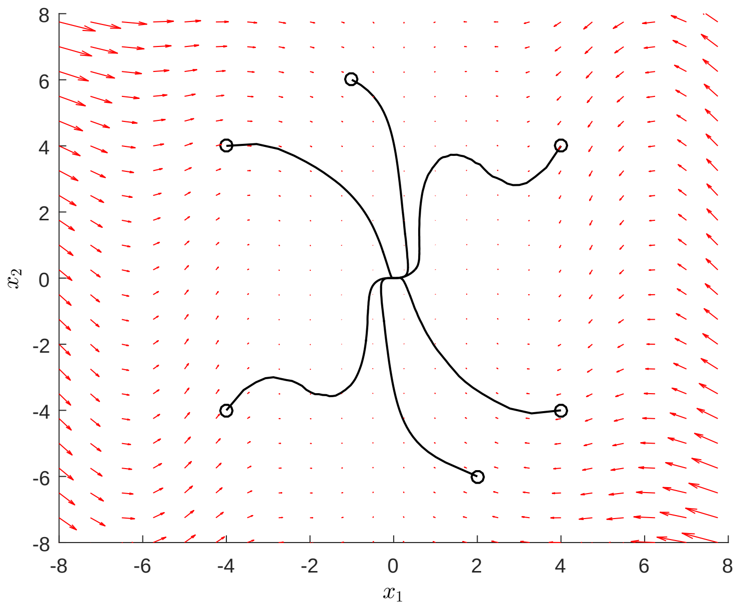

Fig. 4.5:Controlled behavior of six initial conditions: $\boldsymbol{x}(0) = [2 \quad -6]^T$, $\boldsymbol{x}(0) = [4 \quad -4]^T$, $\boldsymbol{x}(0) = [4 \quad 4]^T$, $\boldsymbol{x}(0) = [1 \quad 6]^T$, $\boldsymbol{x}(0) = [-4 \quad 4]^T$, $\boldsymbol{x}(0) = [-4 \quad -4]^T$ ($q_a = 0.17$, $q_b = 0.3$).

designs ([44, 61]) fail to find any feasible solutions, feedback gains and the Lyapunov functions are obtained as follows:

$$\phi_1 = 7.0813189,$$

$$\phi_2 = 1.5071593,$$

$$V_1(\boldsymbol{x}) = 1.101004x_1^2 + 0.270211x_2^2,$$

$$V_2(\boldsymbol{x}) = 1.776559x_1^2 + 0.224898x_2^2,$$

$$F_{11}(\boldsymbol{x}) = \begin{bmatrix} 1.871427x_1 + 0.248278x_2 + 0.571009 \\ 0.248278x_1 - 0.2821642 \end{bmatrix}^T,$$

$$F_{21}(\boldsymbol{x}) = \begin{bmatrix} 1.773991x_1 + 0.232586x_2 + 1.019811 \\ 0.232586x_1 + 0.166421 \end{bmatrix}^T,$$

$$F_{12}(\boldsymbol{x}) = \begin{bmatrix} 1.900791x_1 + 0.313101x_2 + 0.755888 \\ 0.313101x_1 - 0.113636 \end{bmatrix}^T,$$

$$F_{22}(\boldsymbol{x}) = \begin{bmatrix} 1.881896x_1 + 0.283963x_2 + 0.849027 \\ 0.283963x_1 + 0.087415 \end{bmatrix}^T.$$

Figure 4.4 shows the regions of $V_1(\boldsymbol{x})$ ("$\times$") and $V_2(\boldsymbol{x})$ ("$+$") when $q_a = 0.1$, and $q_b = 0.3$. The control trajectory results of the stabilized polynomial fuzzy system with uncertainties can be seen in Figure 4.5 with six initial conditions. All the initial states converge to the equilibrium point which means the designed switching controller successfully stabilizes the system.

The calculation time of the three robust control designs to find a feasible solution for this complex design example is given in Table 4.2. Although the complexity and the required calculation time (see Table 4.2) are slightly inferior as compared to other approaches, i.e. [44, 61], these demerits are well compensated by the advantage of the proposed design, i.e. wider robust stabilization region. The computer environment is an Intel(R) core(TM)i7-6700K CPU (4.00 GHz) with 16.0-GB RAM.

Table 4.2:Calculation time for design example 4.4.2

q_a, q_b	K. Cao et al. [21]	K. Tanaka et al.	Theorem 4.2.1
$q_a = 0.01$, $q_b = 0.05$	3 s	6 s	14 s
$q_a = 0.03$, $q_b = 0.03$	3 s	6 s	14 s
$q_a = 0.01$, $q_b = 0.25$	infeasible	13 s	15 s
$q_a = 0.03$, $q_b = 0.23$	infeasible	13 s	15 s
$q_a = 0.05$, $q_b = 0.21$	infeasible	13 s	16 s
$q_a = 0.07$, $q_b = 0.17$	infeasible	13 s	16 s
$q_a = 0.09$, $q_b = 0.13$	infeasible	13 s	16 s
$q_a = 0.01$, $q_b = 0.55$	infeasible	infeasible	18 s
$q_a = 0.03$, $q_b = 0.55$	infeasible	infeasible	18 s
$q_a = 0.05$, $q_b = 0.53$	infeasible	infeasible	18 s
$q_a = 0.07$, $q_b = 0.51$	infeasible	infeasible	18 s
$q_a = 0.09$, $q_b = 0.49$	infeasible	infeasible	18 s
$q_a = 0.11$, $q_b = 0.47$	infeasible	infeasible	17 s
$q_a = 0.13$, $q_b = 0.45$	infeasible	infeasible	17 s
$q_a = 0.15$, $q_b = 0.43$	infeasible	infeasible	17 s
$q_a = 0.17$, $q_b = 0.43$	infeasible	infeasible	17 s
$q_a = 0.19$, $q_b = 0.41$	infeasible	infeasible	19 s
$q_a = 0.21$, $q_b = 0.39$	infeasible	infeasible	18 s
$q_a = 0.23$, $q_b = 0.37$	infeasible	infeasible	19 s
$q_a = 0.25$, $q_b = 0.37$	infeasible	infeasible	17 s
$q_a = 0.27$, $q_b = 0.25$	infeasible	infeasible	18 s
$q_a = 0.29$, $q_b = 0.25$	infeasible	infeasible	18 s
$q_a = 0.31$, $q_b = 0.21$	infeasible	infeasible	19 s
$q_a = 0.33$, $q_b = 0.19$	infeasible	infeasible	18 s
$q_a = 0.35$, $q_b = 0.03$	infeasible	infeasible	19 s

Chapter 5.

PPLF-based Controller-Observer Design

In designing the control systems, the states of a system are usually assumed to be available for feedback. However in practical applications, not all the states are available. This causes the necessity of unavailable states estimation. To fulfill such necessity, observer design becomes important feature in control systems. Hence, the works in this thesis also cover the observer design of polynomial fuzzy systems by taking the utility of PPLF-based approach, i.e. switching polynomial fuzzy observer and controller. In this case, the designed switching polynomial fuzzy controller depends on the state-estimation of the switching polynomial fuzzy observer. In addition, all the conditions are derived to guarantee the global stabilization and global state-estimation convergence of original nonlinear systems.

The polynomial fuzzy observer has fisrt been proposed by Tanaka et al in [16]. The polynomial fuzzy observer designs are divided into three classes, i.e. class I, class II, and class III.

Through this thesis, the polynomial fuzzy observer based on PPLF approach is proposed in a more general design. The proposed design is applicable for the all three classes which brings more efficiency design compared with the work in [16]. In this thesis, switching polynomial fuzzy observer and controller are designed without considering the separation principles used in [16]. The difficulty part on applying PPLF-based approach is on the switching information according to the value of Lyapunov function. Since the chosen Lyapunov function also depends on the unknown state $\boldsymbol{x}$, a technique of Lyapunov function structure is proposed to deal with the problem.

5.1 Switching Polynomial Fuzzy Observer

This section provides SOS conditions for switching polynomial fuzzy observer and controller design. Consider the following polynomial fuzzy system representation:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{\rho}_a)\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{\rho}_b)\boldsymbol{u}\}, \tag{5.1}$$

where $\boldsymbol{z}(t) = [z_1(t)\ z_2(t)\ \cdots z_o(t)] \in \mathbb{R}^o$,

$$h_i(\boldsymbol{z}(t)) = \frac{\prod_{j=1}^{o} M_{ij}(\boldsymbol{z}_j(t))}{\sum_{k=1}^{r}\prod_{j=1}^{o} M_{kj}(\boldsymbol{z}_j(t))}, \tag{5.2}$$

$$h_i(\boldsymbol{z}(t)) \geq 0, \quad \forall i, \tag{5.3}$$

$$\sum_{i=1}^{r} h_i(\boldsymbol{z}(t)) = 1. \tag{5.4}$$

Even though the proposed polynomial fuzzy observer design can be applied for all classes as categorized in [16], this Thesis also consider the same classification in order to show the merits of the proposed design compared with [16]. For the polynomial fuzzy system (5.1), the classification is categorized as follows:

1. Class I: $\boldsymbol{\rho}_a = \boldsymbol{\eta}$ and $\boldsymbol{\rho}_b = \boldsymbol{\eta}$
2. Class II: Class I: $\boldsymbol{\rho}_a = \boldsymbol{x}$ and $\boldsymbol{\rho}_b = \boldsymbol{\eta}$
3. Class III: $\boldsymbol{\rho}_a = \boldsymbol{x}$ and $\boldsymbol{\rho}_b = \boldsymbol{x}$

$\boldsymbol{\eta}$ is an independent state of $\boldsymbol{x}$. From the above classification, it can be seen that Class III is the most complicated class since the polynomial matrices $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ depend on the state $\boldsymbol{x}$. SOS conditions for Class III design are derived in Section 5.1.1.

5.1.1 SOS conditions

This section provides SOS conditions for polynomial fuzzy observer and controller based on PPLF approach. A description of PPLF:

$$V(\boldsymbol{x}_v) = \min_{1\leq l\leq K} V_l(\boldsymbol{x}_v), \tag{5.5}$$

where l is the switching information and K is the PPLF number. Obviously, if $K = 1$ then PPLF will reduce to PLF. By applying PPLF-based approach, a switching observer estimating the state is designed as follows.

$$\dot{\tilde{\boldsymbol{x}}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}} + \boldsymbol{B}_i(\tilde{\boldsymbol{x}})\boldsymbol{u} + \boldsymbol{L}_{il}(\tilde{\boldsymbol{x}})(\boldsymbol{y} - \hat{\boldsymbol{y}})\} \tag{5.6}$$

$$\hat{\boldsymbol{y}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\tilde{\boldsymbol{x}} \tag{5.7}$$

where $\tilde{\boldsymbol{x}} \in \mathbb{R}^p$ is a vector of state estimated by the observer, $\tilde{\boldsymbol{y}} \in \mathbb{R}^q$ is an estimated output, $\boldsymbol{u}_l$ is the switching control input, and $\boldsymbol{L}_{il}(\tilde{\boldsymbol{x}})$ is the observer gain according to the switching index l. By using the estimated state feedback (5.6), a switching controller to stabilize the system is designed as

$$\boldsymbol{u} = -\sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{F}_{il}(\tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}} \tag{5.8}$$

Remark 6. In this paper, Lyapunov functions $V_l(\boldsymbol{x}_v)$ are defined as $V_l(\boldsymbol{x}_v) = \boldsymbol{x}_v^T\boldsymbol{Y}_l(\tilde{\boldsymbol{x}})\boldsymbol{x}_v$. Gram matrices $\boldsymbol{Y}_l \in \mathbb{R}^{n\times n}$ of the selected Lyapunov functions $V_l(\tilde{\boldsymbol{x}})$ are defined as

$$\boldsymbol{Y}_l(\tilde{\boldsymbol{x}}) = \begin{bmatrix} Y_l^{11}(\tilde{\boldsymbol{x}}) & Y_l^{12}(\tilde{\boldsymbol{x}}) & \cdots & Y_l^{1n}(\tilde{\boldsymbol{x}}) \\ Y_l^{21}(\tilde{\boldsymbol{x}}) & Y_l^{22}(\tilde{\boldsymbol{x}}) & \cdots & Y_l^{2n}(\tilde{\boldsymbol{x}}) \\ \vdots & \vdots & \ddots & \vdots \\ Y_l^{n1}(\tilde{\boldsymbol{x}}) & Y_l^{n2}(\tilde{\boldsymbol{x}}) & \cdots & Y_l^{nn}(\tilde{\boldsymbol{x}}) \end{bmatrix}. \tag{5.9}$$

The Lyapunov function is switching according to the value of $Y_K^{11}(\tilde{\boldsymbol{x}}), Y_K^{12}(\tilde{\boldsymbol{x}})$, and $Y_K^{22}(\tilde{\boldsymbol{x}})$ for K is the number of PPLF. For instance, if the number of PPLF is 2 ($K = 2$) then $Y_1^{11}(\tilde{\boldsymbol{x}}) \neq Y_2^{11}(\tilde{\boldsymbol{x}})$, $Y_1^{12}(\tilde{\boldsymbol{x}}) \neq Y_2^{12}(\tilde{\boldsymbol{x}})$, and $Y_1^{22}(\tilde{\boldsymbol{x}}) \neq Y_2^{22}(\tilde{\boldsymbol{x}})$ while other values of the Gram matrices are the same for $l = 1$ and $l = 2$.

The SOS conditions for switching observer and controller are given in Theorem 5.1.1.

Theorem 5.1.1. *The polynomial fuzzy system is stabilized by the polynomial fuzzy controller if there exist positive definite polynomial matrices* $\boldsymbol{Y}_l(\tilde{\boldsymbol{x}})$*, polynomial matrices* $\boldsymbol{F}_{jl}(\tilde{\boldsymbol{x}})$*,* $\boldsymbol{L}_{il}(\tilde{\boldsymbol{x}})$*, positive definite polynomial matrices* $\boldsymbol{\Pi}_{ijl}(\tilde{\boldsymbol{x}})$ *such that the following conditions are satisfied with* $\alpha < 0$ *and the estimation error via the polynomial fuzzy observer tends to zero.*

$$\boldsymbol{v}_1^T(\boldsymbol{Y}_l(\tilde{\boldsymbol{x}}) - \boldsymbol{\epsilon}_1(\tilde{\boldsymbol{x}}))\boldsymbol{v}_1 \in \mathcal{S} \tag{5.10}$$

$$-\Big(\sum_{m=1}^{r}\hat{h}_m^2\Big)^{\mu}\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i^2\hat{h}_j^2\boldsymbol{x}_v^T\boldsymbol{\Lambda}_{ijl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\boldsymbol{x}_v \in \mathcal{S} \tag{5.11}$$

$$\lambda_{ijsl}(\tilde{\boldsymbol{x}}) \in \mathcal{S} \tag{5.12}$$

where μ is a nonnegative integer, $\boldsymbol{v}_1$ is an independent vector, $i,j \in 1,2,\cdots,r$ and $s,l \in 1,2,\cdots,K$ for rules number r and PPLF number K. $\boldsymbol{\epsilon}_1(\tilde{\boldsymbol{x}})$ is a predefined positive definite polynomial matrix. $\boldsymbol{\Lambda}_{ijl}(\boldsymbol{x},\tilde{\boldsymbol{x}})$ are defined as

$$\boldsymbol{\Lambda}_{ijl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = \mathcal{H}\big(\boldsymbol{Y}_l(\tilde{\boldsymbol{x}})\boldsymbol{\mathcal{M}}_{ijl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\big) - \alpha\boldsymbol{\Pi}_{ijl}(\tilde{\boldsymbol{x}}) + \sum_{s=1}^{K}\lambda_{ijsl}(\tilde{\boldsymbol{x}})\{\boldsymbol{Y}_s(\tilde{\boldsymbol{x}}) - \boldsymbol{Y}_l(\tilde{\boldsymbol{x}})\}. \tag{5.13}$$

Proof. Define $\boldsymbol{x}_v = \begin{bmatrix}\tilde{\boldsymbol{x}}^T & \boldsymbol{e}^T\end{bmatrix}$, and the estimation error $\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$ by the observer. The error system is represented as

$$\begin{aligned}\dot{\boldsymbol{e}} = &\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\boldsymbol{z})h_j(\boldsymbol{z})\times \\ &\big\{\big(\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{A}_i(\tilde{\boldsymbol{x}}) - (\boldsymbol{B}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\tilde{\boldsymbol{x}}))\boldsymbol{F}_j(\tilde{\boldsymbol{x}})\big)\tilde{\boldsymbol{x}} + (\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{L}_i(\tilde{\boldsymbol{x}})\boldsymbol{C}_j)\boldsymbol{e}\big\}.\end{aligned} \tag{5.14}$$

We obtain the following augmented system:

$$\dot{\boldsymbol{x}}_v = \sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{\mathcal{M}}_{ijl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\boldsymbol{x}_v \tag{5.15}$$

where

$$\boldsymbol{\mathcal{M}}_{ijl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = \begin{bmatrix}\boldsymbol{\mathcal{M}}_{ijl}^{11}(\tilde{\boldsymbol{x}}) & \boldsymbol{\mathcal{M}}_{ijl}^{12}(\tilde{\boldsymbol{x}}) \\ \boldsymbol{\mathcal{M}}_{ijl}^{21}(\boldsymbol{x},\tilde{\boldsymbol{x}}) & \boldsymbol{\mathcal{M}}_{ijl}^{22}(\boldsymbol{x},\tilde{\boldsymbol{x}})\end{bmatrix}, \tag{5.16}$$

$$\boldsymbol{\mathcal{M}}_{ijl}^{11}(\tilde{\boldsymbol{x}}) = \boldsymbol{A}_i(\tilde{\boldsymbol{x}}) - \boldsymbol{B}_i(\tilde{\boldsymbol{x}})\boldsymbol{F}_{jl}(\tilde{\boldsymbol{x}}) \tag{5.17}$$

$$\boldsymbol{\mathcal{M}}_{ijl}^{12}(\tilde{\boldsymbol{x}}) = \boldsymbol{L}_{il}(\tilde{\boldsymbol{x}})\boldsymbol{C}_j \tag{5.18}$$

$$\boldsymbol{\mathcal{M}}_{ijl}^{21}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = \boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{A}_i(\tilde{\boldsymbol{x}}) - (\boldsymbol{B}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\tilde{\boldsymbol{x}}))\boldsymbol{F}_{jl}(\tilde{\boldsymbol{x}}) \tag{5.19}$$

$$\boldsymbol{\mathcal{M}}_{ijl}^{22}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = \boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{L}_{il}(\tilde{\boldsymbol{x}})\boldsymbol{C}_j \tag{5.20}$$

Now consider the candidate of minimum PPLF as

$$V(\boldsymbol{x}_\upsilon) = \min_{1\le l\le K}\left\{\boldsymbol{x}_\upsilon^T \mathbf{Y}_K(\tilde{\boldsymbol{x}})\boldsymbol{x}_\upsilon\right\} \tag{5.21}$$

where K denotes the number of PPLF. Hence, the chosen Lyapunov function $V_l(\boldsymbol{x}_\upsilon)$ can be described as

$$V_l(\boldsymbol{x}_\upsilon) = \boldsymbol{x}_\upsilon^T \mathbf{Y}_l(\tilde{\boldsymbol{x}})\boldsymbol{x}_\upsilon \tag{5.22}$$

The time derivatives of $V_l(\boldsymbol{x}_\upsilon)$ are represented as follows.

$$\dot{V}_l(\boldsymbol{x}_\upsilon) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{x}_\upsilon^T \mathcal{H}\big(\mathbf{Y}_l(\tilde{\boldsymbol{x}})\boldsymbol{\mathcal{M}}_{ijl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\big)\boldsymbol{x}_\upsilon \tag{5.23}$$

In order to guarantee $\dot{V}(\boldsymbol{x}_\upsilon) < 0$ at $\boldsymbol{x} \neq \mathbf{0}$, a scalar $\alpha < 0$ and positive definite polynomial matrices $\mathbf{\Pi}_{ijl}(\tilde{\boldsymbol{x}})$ are introduces satisfying $\dot{V}_l(\boldsymbol{x}_\upsilon) \le \alpha \boldsymbol{x}_\upsilon^T \mathbf{\Pi}_{ijl}(\tilde{\boldsymbol{x}})\boldsymbol{x}_\upsilon$, that is,

$$\dot{V}_l(\boldsymbol{x}_\upsilon) - \alpha \boldsymbol{x}_\upsilon^T \mathbf{\Pi}_{ijl}(\tilde{\boldsymbol{x}})\boldsymbol{x}_\upsilon \le 0 \tag{5.24}$$

which is equivalent to

$$\begin{aligned}&\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\times\\ &\boldsymbol{x}_\upsilon^T\Big(\mathcal{H}\big(\mathbf{Y}_l(\tilde{\boldsymbol{x}})\boldsymbol{\mathcal{M}}_{ijl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\big) - \alpha\mathbf{\Pi}_{ijl}(\tilde{\boldsymbol{x}})\Big)\boldsymbol{x}_\upsilon \le 0\end{aligned} \tag{5.25}$$

Since we consider minimum type PPLF, the following condition must be satisfied for $s, l \in 1, 2, \cdots, K$

$$V_l(\boldsymbol{x}_\upsilon) - V_s(\boldsymbol{x}_\upsilon) \le 0 \tag{5.26}$$

which is equivalent to

$$\boldsymbol{x}_\upsilon^T\big(\mathbf{Y}_l(\tilde{\boldsymbol{x}}) - \mathbf{Y}_s(\tilde{\boldsymbol{x}})\big)\boldsymbol{x}_\upsilon \le 0 \tag{5.27}$$

Now, recall Lemma 2.5.1 and define sets L_1 and L_2 as (5.24) and (5.27) respectively. If there exist $\lambda_{ijsl}(\tilde{\boldsymbol{x}}) \in \mathcal{P}^{0+}$ such that

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{x}_v^T\boldsymbol{\Lambda}_{ijl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\boldsymbol{x}_v \leq 0 \tag{5.28}$$

for all $\boldsymbol{x}$ then $L_2 \subseteq L_1$. In other words, condition (5.24) is satisfied only if the following condition is satisfied.

$$\sum_{s=1}^{K} \lambda_{ijsl}(\tilde{\boldsymbol{x}})\boldsymbol{x}_v^T\{\boldsymbol{Y}_s(\tilde{\boldsymbol{x}}) - \boldsymbol{Y}_l(\tilde{\boldsymbol{x}})\}\boldsymbol{x}_v \in \mathcal{P}^{0+} \tag{5.29}$$

Hence, by applying $\mathcal{S}$-procedure, (5.25) becomes (5.28). According to the relation between PSD and SOS polynomials, it can be represented as

$$-\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{x}_v^T\boldsymbol{\Lambda}_{ijsl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\boldsymbol{x}_v \in \mathcal{S}. \tag{5.30}$$

Now, define $h_i(\boldsymbol{z})$ as $\hat{h}_i^2$. By applying copositive relaxation, we arrive at the following condition.

$$-\Big(\sum_{m=1}^{r} \hat{h}_m^2\Big)^{\mu}\sum_{i=1}^{r}\sum_{j=1}^{r} \hat{h}_i^2\hat{h}_j^2\boldsymbol{x}_v^T\boldsymbol{\Lambda}_{ijl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\boldsymbol{x}_v \in \mathcal{S} \tag{5.31}$$

□

Remark 7. Theorem 5.1.1 provides SOS conditions for a complicated class of observer and controller design, i.e. $\boldsymbol{A}_i(\boldsymbol{x})$ and $\boldsymbol{B}_i(\boldsymbol{x})$ are dependent on the states $\boldsymbol{x}$. The SOS conditions in Theorem 5.1.1 will be applied for design example Class III. However, note that the derivation process of those SOS conditions are applicable for other classes, i.e. Class I and Class II, discussed in the later section.

5.1.2 Class III

Consider the following polynomial fuzzy model:

$$
\begin{aligned}
\boldsymbol{A}_1(\boldsymbol{x}) &= \begin{bmatrix} 1 & -0.3x_2 \\ -1.5 & -2-x_2^2 \end{bmatrix}, \\
\boldsymbol{A}_2(\boldsymbol{x}) &= \begin{bmatrix} -0.2172 & -0.3x_2 \\ -1.5 & -2-x_2^2 \end{bmatrix}, \\
\boldsymbol{B}_1(\boldsymbol{x}) = \boldsymbol{B}_2(\boldsymbol{x}) &= \begin{bmatrix} x_2^2+1 \\ 0 \end{bmatrix}, \\
\boldsymbol{C}_1 = \boldsymbol{C}_2 &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
h_1(\boldsymbol{z}) &= \frac{\sin x_1 + 0.2172x_1}{1.2172x_1}, \quad h_2(\boldsymbol{z}) = \frac{x_1 - \sin x_1}{1.2172x_1}
\end{aligned}
$$

By solving Theorem 5.1.1, the feasible solutions are obtained as follows:

$$
\boldsymbol{Y}_1 = \begin{bmatrix} 0.084872546217 & -0.015893873678 & 0.006571227939 & -0.01690072455 \\ -0.015893873678 & 21.463101758 & -0.04094578095 & 7.5150851859 \\ 0.00657122799 & -0.040945780950 & 0.036208017307 & 0.005459749095 \\ -0.0169007245 & 7.51508518597 & 0.00545974909 & 20.28222696643 \end{bmatrix}, \tag{5.32}
$$

$$
\boldsymbol{Y}_2 = \begin{bmatrix} 0.071454510942 & -0.019681749295 & 0.006571227939 & -0.01690072455 \\ -0.019681749295 & 19.240128926 & -0.04094578095 & 7.5150851859 \\ 0.00657122799 & -0.040945780950 & 0.036208017307 & 0.005459749095 \\ -0.0169007245 & 7.51508518597 & 0.00545974909 & 20.28222696643 \end{bmatrix}, \tag{5.33}
$$

$$\boldsymbol{F}_{11}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 278.410286\tilde{x}_1^2 - 118.26161\tilde{x}_1\tilde{x}_2 + 273.092568\tilde{x}_2^2 + 6488.52165 \\ -118.262876\tilde{x}_1^2 + 273.092419\tilde{x}_1\tilde{x}_2 - 272.761954\tilde{x}_2^2 - 10182.781400 \end{bmatrix}^T, \quad (5.34)$$

$$\boldsymbol{F}_{12}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 5.222694\tilde{x}_1^2 - 374.837366\tilde{x}_1\tilde{x}_2 + 102.559856\tilde{x}_2^2 + 6263.1893820 \\ -374.836916\tilde{x}_1^2 + 102.559897\tilde{x}_1\tilde{x}_2 - 483.556070\tilde{x}_2^2 - 13481.704452 \end{bmatrix}^T, \quad (5.35)$$

$$\boldsymbol{F}_{21}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 278.412299\tilde{x}_1^2 - 118.239005\tilde{x}_1\tilde{x}_2 + 273.070982\tilde{x}_2^2 + 6488.127035 \\ -118.240083\tilde{x}_1^2 + 273.070894\tilde{x}_1\tilde{x}_2 - 272.732144\tilde{x}_2^2 - 10176.218720 \end{bmatrix}^T, \quad (5.36)$$

$$\boldsymbol{F}_{22}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 5.22123\tilde{x}_1^2 - 374.57046\tilde{x}_1\tilde{x}_2 + 102.534294\tilde{x}_2^2 + 6259.523026 \\ -374.573278\tilde{x}_1^2 + 102.534281\tilde{x}_1\tilde{x}_2 - 483.486272\tilde{x}_2^2 - 13464.058652 \end{bmatrix}^T, \quad (5.37)$$

$$\boldsymbol{L}_{11}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 13.330447\tilde{x}_1^2 + 40.730665\tilde{x}_1\tilde{x}_2 + 148.212518\tilde{x}_2^2 + 399.0613619 \\ -0.011576\tilde{x}_1^2 - 0.094286\tilde{x}_1\tilde{x}_2 + 0.016883\tilde{x}_2^2 + 0.170604 \end{bmatrix}, \quad (5.38)$$

$$\boldsymbol{L}_{12}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 24.778289\tilde{x}_1^2 + 101.942773\tilde{x}_1\tilde{x}_2 - 406.985115\tilde{x}_2^2 + 2191.737402 \\ -0.010774\tilde{x}_1^2 - 0.102280\tilde{x}_1\tilde{x}_2 + 0.430972\tilde{x}_2^2 - 1.220117 \end{bmatrix}, \quad (5.39)$$

$$\boldsymbol{L}_{21}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 13.344929\tilde{x}_1^2 + 40.668152\tilde{x}_1\tilde{x}_2 + 148.295050\tilde{x}_2^2 + 399.005053 \\ -0.011574\tilde{x}_1^2 - 0.094264\tilde{x}_1\tilde{x}_2 + 0.016842\tilde{x}_2^2 + 0.171024 \end{bmatrix}, \quad (5.40)$$

$$\boldsymbol{L}_{22}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 24.796897\tilde{x}_1^2 + 101.535972\tilde{x}_1\tilde{x}_2 - 406.559168\tilde{x}_2^2 + 2190.50042 \\ -0.0107837\tilde{x}_1^2 - 0.1020518\tilde{x}_1\tilde{x}_2 + 0.430687\tilde{x}_2^2 - 1.220407 \end{bmatrix}. \quad (5.41)$$

The feasible solutions of other decision variables can be seen in Appendix A.

Figure 5.1 shows the output $\boldsymbol{y}$ and estimated output $\tilde{\boldsymbol{y}}$ by the designed switching observer and controller. In the simulation, the initial states are $\boldsymbol{x}(0) = \begin{bmatrix} 5 & 5 \end{bmatrix}^T$ and $\tilde{\boldsymbol{x}} = \begin{bmatrix} -5 & -5 \end{bmatrix}^T$. The control results of design example class III can be seen in Figure 5.2. By using the estimated state $\tilde{\boldsymbol{x}}$, the switching polynomial fuzzy controll can stabilize the system and all the initial states converge to the equilibrium point.

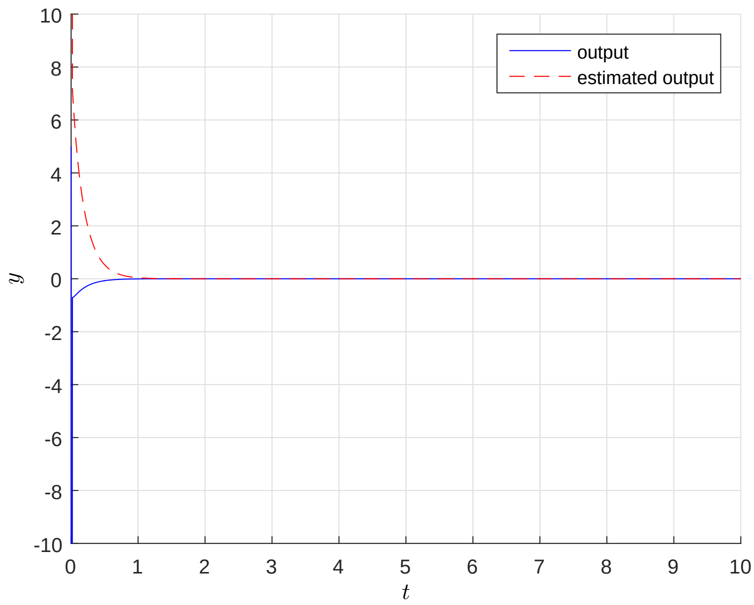

Fig. 5.1:Output and estimated output.

5.1.3 Class II

Consider the following polynomial fuzzy system for Class II:

$$\dot{\boldsymbol{x}} = \sum_{i}^{r} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{\eta})\boldsymbol{u}\} \tag{5.42}$$

$$\boldsymbol{y} = \sum_{i}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\boldsymbol{x} \tag{5.43}$$

where $\boldsymbol{y} \in \mathbb{R}^p$ denotes the output. The switching polynomial fuzzy observer and controller are designed as follows.

$$\dot{\tilde{\boldsymbol{x}}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}} + \boldsymbol{B}_i(\boldsymbol{\eta})\boldsymbol{u}_l + \boldsymbol{L}_{il}(\tilde{\boldsymbol{x}})(\boldsymbol{y} - \hat{\boldsymbol{y}})\} \tag{5.44}$$

$$\hat{\boldsymbol{y}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\tilde{\boldsymbol{x}} \tag{5.45}$$

$$\boldsymbol{u} = -\sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{F}_{il}(\tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}} \tag{5.46}$$

where $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$ is the state vector estimated by the switching observer and $\hat{\boldsymbol{y}} \in \mathbb{R}^p$ denotes the estimated output. The switching controller is constructed with the state feedback that is

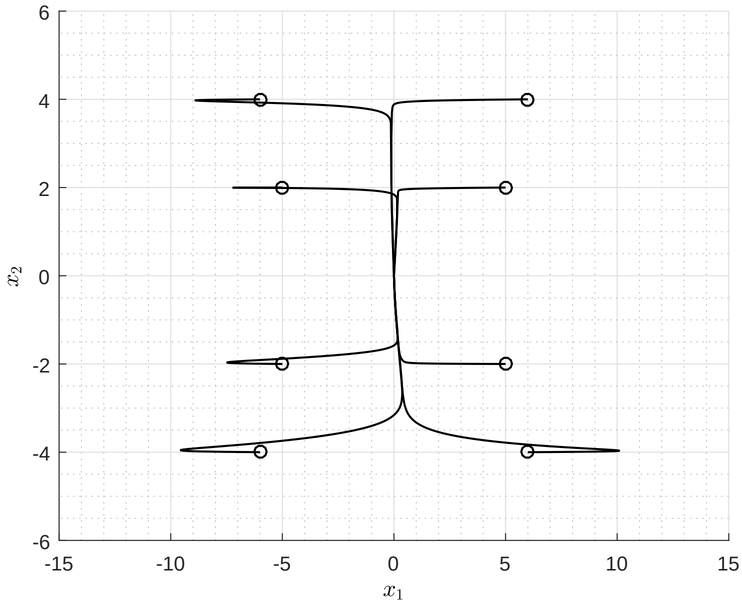

Fig. 5.2:Controlled behavior of design example 5.1.2.

estimated by the switching observer. SOS conditions for the observer and conditions designs are presented in Corollary 5.1.2.

Corollary 5.1.2. *The polynomial fuzzy system is stabilized by the polynomial fuzzy controller if there exist positive definite polynomial matrices* $\mathbf{Y}_l(\tilde{\boldsymbol{x}})$*, polynomial matrices* $\boldsymbol{F}_{jl}(\tilde{\boldsymbol{x}})$*,* $\boldsymbol{L}_{il}(\tilde{\boldsymbol{x}})$*,and positive definite polynomial matrices* $\boldsymbol{\Pi}_{ijl}(\boldsymbol{x}, \boldsymbol{\eta}, \tilde{\boldsymbol{x}})$ *such that the following conditions are satisfied with* $\alpha < 0$ *and the estimation error via the polynomial fuzzy observer tends to zero.*

$$\boldsymbol{\upsilon}_1^T(\mathbf{Y}_l(\tilde{\boldsymbol{x}}) - \boldsymbol{\epsilon}_1(\tilde{\boldsymbol{x}}))\boldsymbol{\upsilon}_1 \in \mathcal{S} \tag{5.47}$$

$$-\Big(\sum_{k=1}^{r}\hat{h}_k^2\Big)^{\mu}\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i^2\hat{h}_j^2\boldsymbol{x}_\upsilon^T\boldsymbol{\Lambda}_{ijl}(\boldsymbol{x}, \boldsymbol{\eta}, \tilde{\boldsymbol{x}})\boldsymbol{x}_\upsilon \in \mathcal{S} \tag{5.48}$$

$$\lambda_{ijml}(\tilde{\boldsymbol{x}}) \in \mathcal{S} \tag{5.49}$$

where μ *is a nonnegative integer,* $\boldsymbol{\upsilon}_1$ *is an independent vector,* $i, j, k \in 1, 2, \cdots, r$ *and* $m, l \in 1, 2, \cdots, N$ *for rules number* r *and PPLF number* K*.* $\boldsymbol{\epsilon}_1(\tilde{\boldsymbol{x}})$ *is a predefined positive definite* *polynomial matrices as slack variable.* $\boldsymbol{\Lambda}_{ijl}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}})$ *are defined as*

$$\begin{aligned}\boldsymbol{\Lambda}_{ijl}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) =& \mathcal{H}\big(\boldsymbol{Y}_l(\tilde{\boldsymbol{x}})\boldsymbol{\mathcal{M}}_{ijl}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}})\big) - \alpha\boldsymbol{\Pi}_{ijl}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) \\ &+ \sum_{m=1}^{N}\lambda_{ijml}(\tilde{\boldsymbol{x}})\{\boldsymbol{Y}_m(\tilde{\boldsymbol{x}}) - \boldsymbol{Y}_l(\tilde{\boldsymbol{x}})\}. \end{aligned} \tag{5.50}$$

where

$$\boldsymbol{\mathcal{M}}_{ijl}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = \begin{bmatrix}\boldsymbol{\mathcal{M}}^{11}_{ijl}(\boldsymbol{\eta},\tilde{\boldsymbol{x}}) & \boldsymbol{\mathcal{M}}^{12}_{ijl}(\tilde{\boldsymbol{x}}) \\ \boldsymbol{0} & \boldsymbol{\mathcal{M}}^{22}_{ijl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\end{bmatrix}, \tag{5.51}$$

$$\boldsymbol{\mathcal{M}}^{11}_{ijl}(\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = \boldsymbol{A}_i(\tilde{\boldsymbol{x}}) - \boldsymbol{B}_i(\boldsymbol{\eta})\boldsymbol{F}_{jl}(\tilde{\boldsymbol{x}}), \tag{5.52}$$

$$\boldsymbol{\mathcal{M}}^{12}_{ijl}(\tilde{\boldsymbol{x}}) = \boldsymbol{L}_{il}(\tilde{\boldsymbol{x}})\boldsymbol{C}_j, \tag{5.53}$$

$$\boldsymbol{\mathcal{M}}^{22}_{ijl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = \hat{\boldsymbol{A}}_i(\boldsymbol{x},\tilde{\boldsymbol{x}}) - \boldsymbol{L}_{il}(\tilde{\boldsymbol{x}})\boldsymbol{C}_j, \tag{5.54}$$

and $\hat{\boldsymbol{A}}_i(\boldsymbol{x},\tilde{\boldsymbol{x}})$ is defined as $\hat{\boldsymbol{A}}_i(\boldsymbol{x},\tilde{\boldsymbol{x}})\boldsymbol{e} = \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}_i(\tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}}$.

In order to show the effectiveness of the proposed design, a design example that was also used in [16] is also performed. Consider the following nonlinear system where x_1 is measurable and $y = x_1$ [16]:

$$\begin{aligned}\dot{x}_1 &= \sin x_1 - 0.3x_2 + (x_1^2 + 1)u \\ \dot{x}_2 &= -1.5x_1 - 2x_2 - x_2^2. \end{aligned} \tag{5.55}$$

The dynamics of the nonlinear system can be represented as the polynomial fuzzy system where $r = 2$, $\boldsymbol{z} = \boldsymbol{\eta} = y$. In order to compare the proposed SOS conditions with other approach, we set parameter a and b in the matrices. The polynomial fuzzy model representation for the nonlinear system is as follows.

$$
\begin{aligned}
\boldsymbol{A}_1(\boldsymbol{x}) &= \begin{bmatrix} 1 & -0.3x_2 + b \\ -1.5 & -2 - ax_2^2 \end{bmatrix}, \\
\boldsymbol{A}_2(\boldsymbol{x}) &= \begin{bmatrix} -0.2172 & -0.3x_2 + b \\ -1.5 & -2 - ax_2^2 \end{bmatrix}, \\
\boldsymbol{B}_1(\boldsymbol{\eta}) = \boldsymbol{B}_2(\boldsymbol{\eta}) &= \begin{bmatrix} y^2 + 1 \\ 0 \end{bmatrix}, \\
\boldsymbol{C}_1 = \boldsymbol{C}_2 &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
h_1(\boldsymbol{z}) &= \frac{\sin y + 0.2172y}{1.2172y}, \quad h_2(\boldsymbol{z}) = \frac{y - \sin y}{1.2172y}, \\
\hat{\boldsymbol{A}}_1(\boldsymbol{x}, \tilde{\boldsymbol{x}})\boldsymbol{e} &= \boldsymbol{A}_1(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}_1(\tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}} \\
&= \begin{bmatrix} 1 & -0.3(x_2 + \tilde{x}_2) + b \\ -1.5 & -2 - ax_2^2 - x_2\tilde{x}_2 - a\tilde{x}_2^2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \\
\hat{\boldsymbol{A}}_2(\boldsymbol{x}, \tilde{\boldsymbol{x}})\boldsymbol{e} &= \boldsymbol{A}_2(\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}_2(\tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}} \\
&= \begin{bmatrix} -0.2172 & -0.3(x_2 + \tilde{x}_2) + b \\ -1.5 & -2 - ax_2^2 - x_2\tilde{x}_2 - a\tilde{x}_2^2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.
\end{aligned}
$$

In this design example, $\lambda_{ijml}(\tilde{\boldsymbol{x}})$ and $\boldsymbol{Y}_l(\tilde{\boldsymbol{x}})$ are set to be zero order polynomials and zero order polynomial matrices in $\tilde{\boldsymbol{x}}$ respectively. Therefore, $\boldsymbol{\epsilon}_1(\tilde{\boldsymbol{x}})$ are set as positive definite matrices instead of polynomial matrices and $\mu = 0$.

The feasible area of the proposed design and those in [16] can be seen in Figure 5.3. From the figure, it can be seen that the proposed design provides more relaxed results compared with Theorem 2 in [16]. The estimated results by the switching polynomial fuzzy observer is showed in Figure 5.5. The figure shows the estimation error converge to zero. The controlled behavior of the nonlinear system is given in Figure 5.6. The switching polynomial fuzzy controller that depends on the estimated state $\tilde{\boldsymbol{x}}$ has successfully stabilized the system, i.e., all the initial states go to the equilibrium point.

For $a = 3.7$ and $b = 1050$, the obtained feasible solutions are presented in (5.56)-(5.66). The domain area of the chosen Lyapunov function can be seen in Figure 5.4.

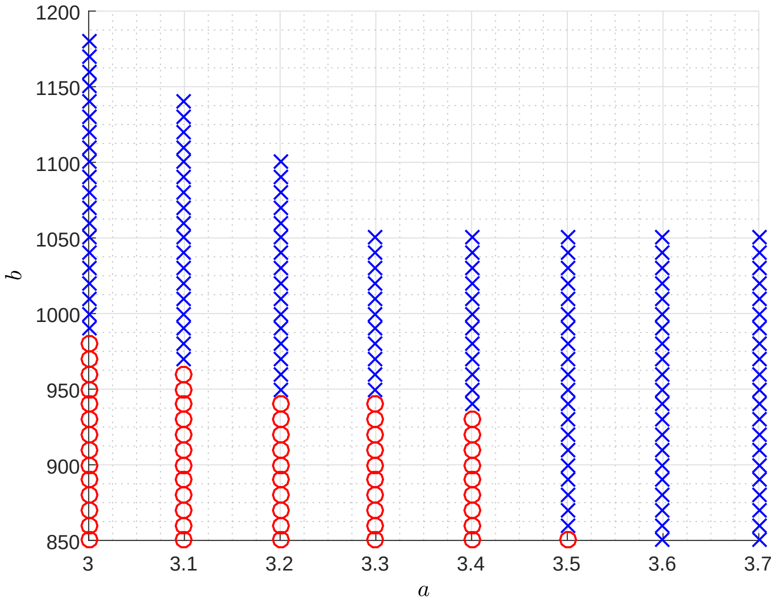

Fig. 5.3:Feasible area of a and b ($\circ$ for Theorem 2 [16] and and $\times$ for Theorem 5.1.2).

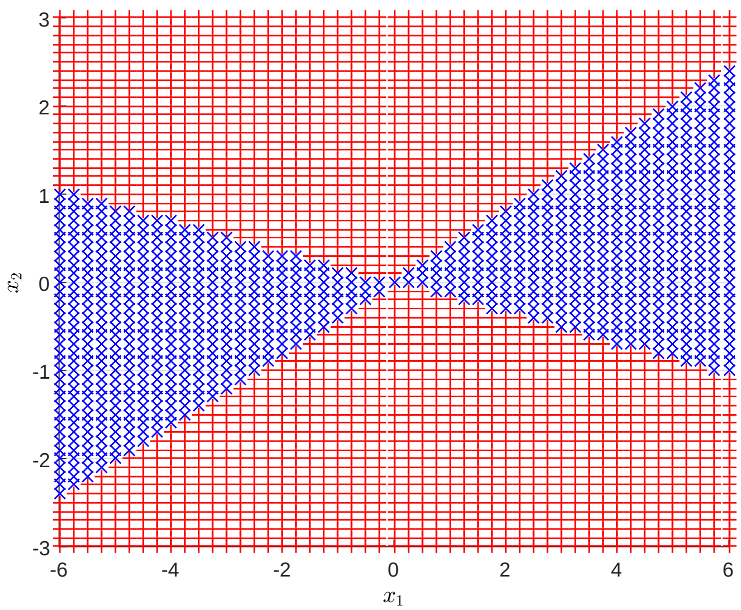

Fig. 5.4:Domain area of chosen Lyapunov functions: $\times$ for $V(\boldsymbol{x}_\upsilon) = V_1(\boldsymbol{x}_\upsilon)$ and $+$ for $V(\boldsymbol{x}_\upsilon) = V_2(\boldsymbol{x}_\upsilon)$.

Feasible solutions for design example class II:

$$\alpha = -0.011609298758377, \tag{5.56}$$

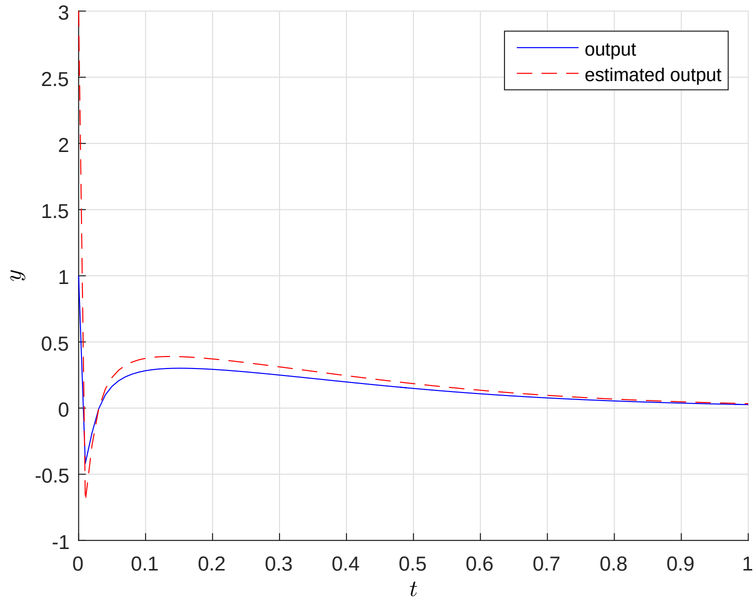

Fig. 5.5:Control and estimation results.

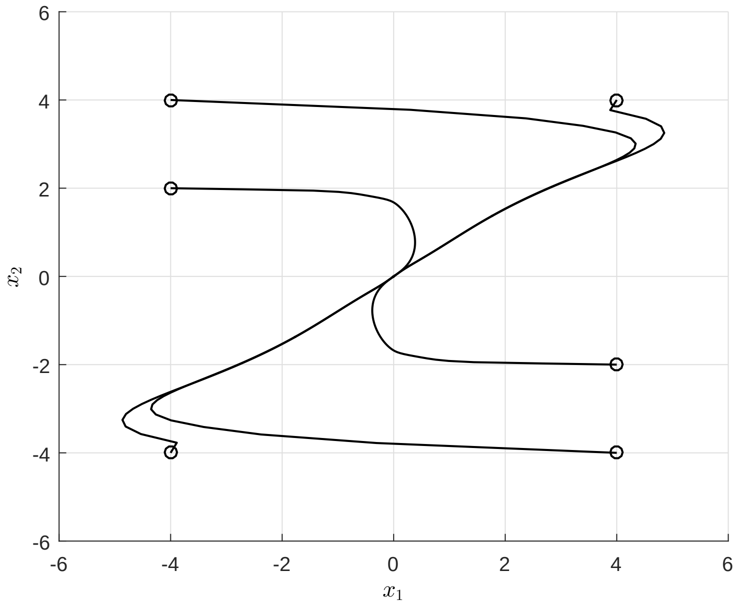

Fig. 5.6:Controlled behavior of design example class II.

$$\boldsymbol{Y}_1(\boldsymbol{\eta}) = \begin{bmatrix} 0.12063721937297 & 0.074535107397063 & 0 & 0 \\ 0.074535107397063 & 76.897608062766 & 0 & 0 \\ 0 & 0 & 0.0165750092 & 0 \\ 0 & 0 & 0 & 6.640939209 \end{bmatrix}, \quad (5.57)$$

$$\boldsymbol{Y}_2(\boldsymbol{\eta}) = \begin{bmatrix} 0.1208809520173 & 0.07493125377689 & 0 & 0 \\ 0.07493125377689 & 76.894238293000 & 0 & 0 \\ 0 & 0 & 0.0165750092 & 0 \\ 0 & 0 & 0 & 6.640939209 \end{bmatrix}, \tag{5.58}$$

$$\boldsymbol{F}_{11}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 512.554321286\tilde{x}_1^2 - 8.37491314278\tilde{x}_1\tilde{x}_2 + 98.5137302779\tilde{x}_2^2 + 607.99119602 \\ 8.37491314278\tilde{x}_1^2 + 236.246469313\tilde{x}_1\tilde{x}_2 + 375.681133067\tilde{x}_2^2 + 5.2408 \times 10^{-6} \end{bmatrix}^T, \tag{5.59}$$

$$\boldsymbol{F}_{21}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 512.554320347\tilde{x}_1^2 - 8.37492204601\tilde{x}_1\tilde{x}_2 + 98.513763418\tilde{x}_2^2 + 600.180814897 \\ 8.37492204601\tilde{x}_1^2 + 98.513763418\tilde{x}_1\tilde{x}_2 + 236.246503809\tilde{x}_2^2 + 370.859430816 \end{bmatrix}^T, \tag{5.60}$$

$$\boldsymbol{F}_{12}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 512.374859495\tilde{x}_1^2 - 8.70599544496\tilde{x}_1\tilde{x}_2 + 97.8937293439\tilde{x}_2^2 + 609.133690126 \\ 8.70599544496\tilde{x}_1^2 + 97.8937293439\tilde{x}_1\tilde{x}_2 + 236.735429289\tilde{x}_2^2 + 377.6212915886 \end{bmatrix}^T, \tag{5.61}$$

$$\boldsymbol{F}_{22}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 512.374853510\tilde{x}_1^2 - 8.70598003631\tilde{x}_1\tilde{x}_2 + 97.8936817073\tilde{x}_2^2 + 601.258007411 \\ 8.70598003631\tilde{x}_1^2 + 97.8936817073\tilde{x}_1\tilde{x}_2 + 236.735380057\tilde{x}_2^2 + 372.744431475 \end{bmatrix}^T, \tag{5.62}$$

$$\boldsymbol{L}_{11}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 106.69961649\tilde{x}_1^2 - 1.8208264639\tilde{x}_1\tilde{x}_2 + 274.83769552\tilde{x}_2^2 + 158.90628583 \\ -0.04655638101\tilde{x}_1^2 - 0.6095116665\tilde{x}_1\tilde{x}_2 - 0.2552126242\tilde{x}_2^2 + 0.03082695247 \end{bmatrix}, \tag{5.63}$$

$$\boldsymbol{L}_{21}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 106.699616035\tilde{x}_1^2 - 1.82082682346\tilde{x}_1\tilde{x}_2 + 274.837695285\tilde{x}_2^2 + 157.059963456 \\ -0.04655637841\tilde{x}_1^2 - 0.60951166233\tilde{x}_1\tilde{x}_2 - 0.255212623\tilde{x}_2^2 + 0.0306023115 \end{bmatrix},, \tag{5.64}$$

$$\boldsymbol{L}_{12}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 106.49256857898\tilde{x}_1^2 - 1.83096875379\tilde{x}_1\tilde{x}_2 + 274.83263256694\tilde{x}_2^2 + 158.83084456528 \\ -0.04661557424\tilde{x}_1^2 - 0.6102986303\tilde{x}_1\tilde{x}_2 - 0.2565548294\tilde{x}_2^2 + 0.03025763564 \end{bmatrix}, \tag{5.65}$$

$$\boldsymbol{L}_{22}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 106.49256751936\tilde{x}_1^2 - 1.83096868959\tilde{x}_1\tilde{x}_2 + 274.83263441102\tilde{x}_2^2 + 156.97222945267 \\ -0.046615578\tilde{x}_1^2 - 0.6102986385\tilde{x}_1\tilde{x}_2 - 0.2565548323\tilde{x}_2^2 + 0.03003235435 \end{bmatrix}. \tag{5.66}$$

In order to maintain readability, other solutions of the decision variables can be seen in Appendix B.

5.1.4 Class I

This section provides a less complicated class of switching observer and controller design. By defining $\boldsymbol{A}_i(\boldsymbol{\eta})$ and $\boldsymbol{B}_i(\boldsymbol{\eta})$ are polynomial matrices in $\boldsymbol{\eta}$, the polynomial fuzzy system is described as follows.

$$\dot{\boldsymbol{x}} = \sum_{i}^{r} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{\eta})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{\eta})\boldsymbol{u}\} \tag{5.67}$$

$$\boldsymbol{y} = \sum_{i}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\boldsymbol{x} \tag{5.68}$$

A switching observer and controller are designed as follows.

$$\dot{\tilde{\boldsymbol{x}}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{\eta})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{\eta})\boldsymbol{u} + \boldsymbol{L}_{il}(\boldsymbol{\eta})(\boldsymbol{y} - \hat{\boldsymbol{y}})\} \tag{5.69}$$

$$\hat{\boldsymbol{y}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i\tilde{\boldsymbol{x}} \tag{5.70}$$

$$\boldsymbol{u} = -\sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{F}_{il}(\boldsymbol{\eta})\tilde{\boldsymbol{x}} \tag{5.71}$$

The SOS conditions for above observer and controller design are described as Corollary 5.1.3.

Corollary 5.1.3. *The polynomial fuzzy system is stabilized by the polynomial fuzzy controller if there exist positive definite polynomial matrices* $\mathbf{Y}_l(\boldsymbol{\eta})$*, polynomial matrices* $\boldsymbol{F}_{jl}(\boldsymbol{\eta})$*,* $\boldsymbol{L}_{il}(\boldsymbol{\eta})$*,and positive definite polynomial matrices* $\mathbf{\Pi}_{ijl}(\boldsymbol{\eta})$ *such that the following conditions are satisfied with* $\alpha < 0$ *and the estimation error via the polynomial fuzzy observer tends to zero.*

$$\boldsymbol{v}_1^T(\boldsymbol{Y}_l(\boldsymbol{\eta}) - \boldsymbol{\epsilon}_1(\boldsymbol{\eta}))\boldsymbol{v}_1 \in \mathcal{S} \tag{5.72}$$

$$-\Big(\sum_{m=1}^{r} \hat{h}_m^2\Big)^{\mu} \sum_{i=1}^{r}\sum_{j=1}^{r} \hat{h}_i^2\hat{h}_j^2\boldsymbol{x}_v^T\mathbf{\Lambda}_{ijl}(\boldsymbol{\eta})\boldsymbol{x}_v \in \mathcal{S} \tag{5.73}$$

$$\lambda_{ijml} \in \mathcal{S} \tag{5.74}$$

where μ is a nonnegative integer, $\boldsymbol{v}_1$ is an independent vector, $i, j \in 1, 2, \cdots, r$ and $s, l \in 1, 2, \cdots, K$ for rules number r and PPLF number K. $\boldsymbol{\epsilon}_1(\boldsymbol{\eta})$ is a predefined positive definite polynomial matrix. $\boldsymbol{\Lambda}_{ijl}(\boldsymbol{x}, \boldsymbol{\eta}, \tilde{\boldsymbol{x}})$ are defined as

$$\begin{aligned}\boldsymbol{\Lambda}_{ijl}(\boldsymbol{\eta}) =& \mathcal{H}\big(\boldsymbol{Y}_l(\boldsymbol{\eta})\boldsymbol{\mathcal{M}}_{ijl}(\boldsymbol{\eta})\big) - \alpha\boldsymbol{\Pi}_{ijl}(\boldsymbol{\eta}) \\ &+ \sum_{m=1}^{K} \lambda_{ijml}\{\boldsymbol{Y}_s(\boldsymbol{\eta}) - \boldsymbol{Y}_l(\boldsymbol{\eta}))\}. \end{aligned} \tag{5.75}$$

Proof. Define the error dynamics as follows.

$$\dot{\boldsymbol{e}} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{\eta}) - \boldsymbol{L}_{il}(\boldsymbol{\eta})\boldsymbol{C}_j\}\boldsymbol{e}. \tag{5.76}$$

The augmented system consisting of the switching observer and controller is represented in (5.77).

$$\dot{\boldsymbol{x}}_v = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\boldsymbol{\mathcal{M}}_{ijl}(\boldsymbol{\eta})\boldsymbol{x}_v \tag{5.77}$$

where

$$\boldsymbol{\mathcal{M}}_{ijl}(\boldsymbol{\eta}) = \begin{bmatrix} \boldsymbol{\mathcal{M}}_{ijl}^{11}(\boldsymbol{\eta}) & \boldsymbol{\mathcal{M}}_{ijl}^{12}(\boldsymbol{\eta}) \\ \boldsymbol{0} & \boldsymbol{\mathcal{M}}_{ijl}^{22}(\boldsymbol{\eta}) \end{bmatrix}, \tag{5.78}$$

$$\boldsymbol{\mathcal{M}}_{ijl}^{11}(\boldsymbol{\eta}) = \boldsymbol{A}_i(\boldsymbol{\eta}) - \boldsymbol{B}_i(\boldsymbol{\eta})\boldsymbol{F}_{jl}(\boldsymbol{\eta}) \tag{5.79}$$

$$\boldsymbol{\mathcal{M}}_{ijl}^{12}(\boldsymbol{\eta}) = \boldsymbol{L}_{il}(\boldsymbol{\eta})\boldsymbol{C}_j \tag{5.80}$$

$$\boldsymbol{\mathcal{M}}_{ijl}^{22}(\boldsymbol{\eta}) = \boldsymbol{A}_i(\boldsymbol{\eta}) - \boldsymbol{L}_{il}(\boldsymbol{\eta})\boldsymbol{C}_j \tag{5.81}$$

The derivation process is the same as presented in Theorem 5.1.2. □

Now, consider the following dynamic polynomial fuzzy system [16] where $r = 2$ and $\boldsymbol{\eta} = y$.

$$\boldsymbol{A}_1(\boldsymbol{\eta}) = \begin{bmatrix} 0.1y^2 + b & -a \\ 1 & -y^2 \end{bmatrix}$$

$$\boldsymbol{A}_2(\boldsymbol{\eta}) = \begin{bmatrix} 0.1y^2 + b & -a \\ -0.2172 & -y^2 \end{bmatrix}$$

$$\boldsymbol{B}_1(\boldsymbol{\eta}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{B}_2(\boldsymbol{\eta}) = \boldsymbol{B}_1(\boldsymbol{\eta})$$

$$\boldsymbol{C}_1 = \boldsymbol{C}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

In this design example, we set $2 \leq a \leq 9$ with interval 0.5 and try to find maximum b. By using these parameters, the feasible area from Corollary 5.1.3 and from [16] are compared in Figure 5.7.

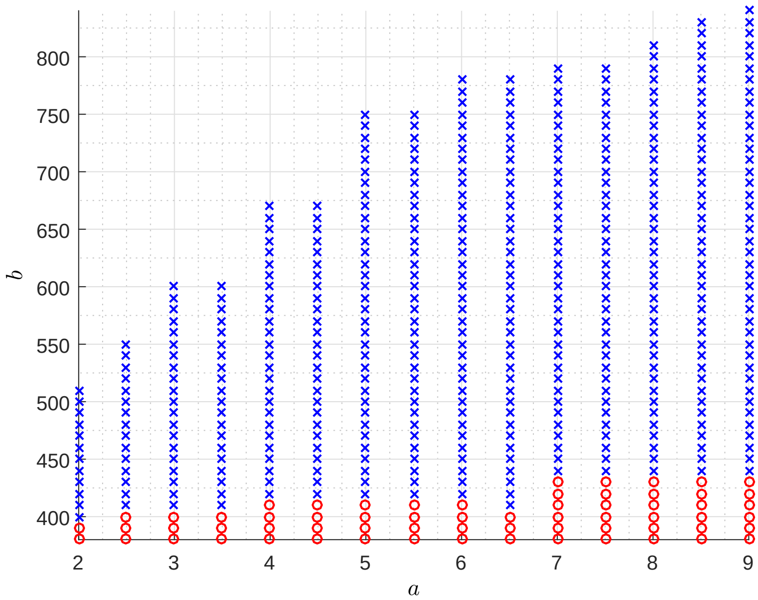

Fig. 5.7:Feasible area of a and b (○ for Theorem 1 [16] and × for Corollary 5.1.3).

For $a > 4.0$, the value of maximum b is still under calculation which means the feaisble area of Corollary 5.1.3 might be wider than those in the Figure 5.7.

For $a = 4.0$ and $b = 670$, the feasible solutions are:

$$\alpha = -0.216378962994441, \tag{5.82}$$

$$\boldsymbol{Y}_1 = \begin{bmatrix} 8.679 \times 10^{-7} & 1.205 \times 10^{-8} & 3.781 \times 10^{-6} & -1.716 \times 10^{-6} \\ 1.205 \times 10^{-8} & 9.013 \times 10^{-6} & 5.663 \times 10^{-6} & 1.323 \times 10^{-5} \\ 3.781 \times 10^{-5} & 5.663 \times 10^{-6} & 0.5714266615488365 & 0.09034608102209585 \\ -1.716 \times 10^{-6} & 1.323 \times 10^{-5} & 0.09034608102209585 & 0.1073956448474586 \end{bmatrix}, \tag{5.83}$$

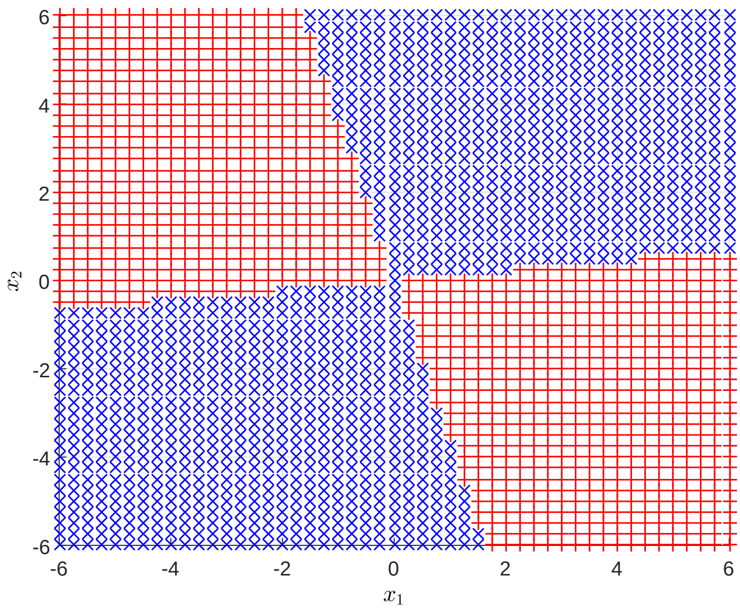

Fig. 5.8:Chosen Lyapunov functions: $\times$ for $V(\boldsymbol{x}_v) = V_1(\boldsymbol{x}_v)$ and $+$ for $V(\boldsymbol{x}_v) = V_2(\boldsymbol{x}_v)$.

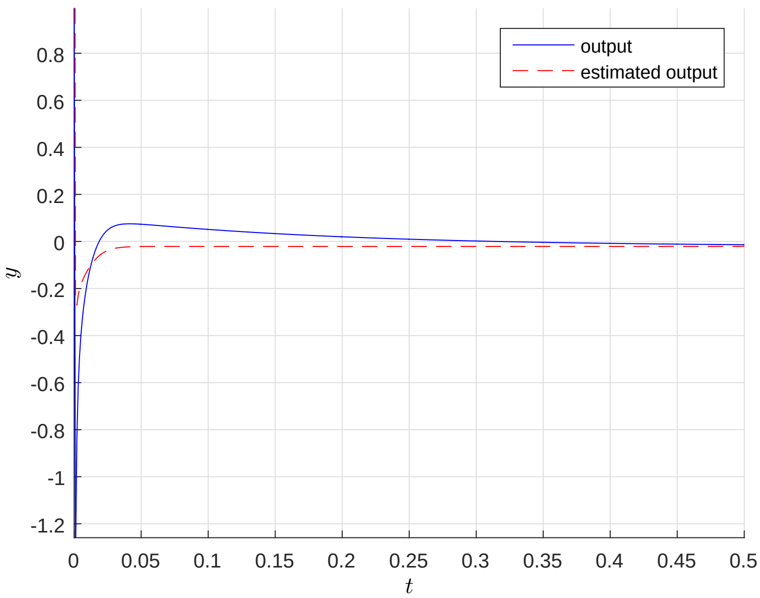

Fig. 5.9:Control and result for $a = 4.0$ and $b = 670$.

$$\boldsymbol{Y}_2 = \begin{bmatrix} 8.474 \times 10^{-7} & 9.720 \times 10^{-8} & 3.781 \times 10^{-5} & -1.716 \times 10^{-6} \\ 9.720 \times 10^{-8} & 9.061 \times 10^{-6} & 5.663 \times 10^{-6} & 1.323 \times 10^{-5} \\ 3.781 \times 10^{-5} & 5.663 \times 10^{-6} & 0.5714266615488365 & 0.09034608102209585 \\ -1.716 \times 10^{-6} & 1.323 \times 10^{-5} & 0.09034608102209585 & 0.1073956448474586 \end{bmatrix}, \quad (5.84)$$

$$\boldsymbol{F}_{11}(\boldsymbol{\eta}) = \begin{bmatrix} 308.82986093443y^2 + 643.4693312809 \\ 0.26589218346769y^2 + 22.755185602646 \end{bmatrix}^T, \tag{5.85}$$

$$\boldsymbol{F}_{21}(\boldsymbol{\eta}) = \begin{bmatrix} 308.8300137908y^2 + 582.21706641301 \\ 0.26593345933302y^2 - 7.1973237595371 \end{bmatrix}^T, \tag{5.86}$$

$$\boldsymbol{F}_{12}(\boldsymbol{\eta}) = \begin{bmatrix} 308.80969875718y^2 + 642.95914175229 \\ 0.4239431701425y^2 + 20.521905206253 \end{bmatrix}^T, \tag{5.87}$$

$$\boldsymbol{F}_{22}(\boldsymbol{\eta}) = \begin{bmatrix} 308.8098511724y^2 + 582.63348795194 \\ 0.42398774111363y^2 - 6.5535617398279 \end{bmatrix}^T, \tag{5.88}$$

$$\boldsymbol{L}_{11}(\boldsymbol{\eta}) = \begin{bmatrix} 388.4292091484068y^2 + 640.002098143941 \\ -278.3733185009212y^2 - 17.50801836252333 \end{bmatrix}, \tag{5.89}$$

$$\boldsymbol{L}_{21}(\boldsymbol{\eta}) = \begin{bmatrix} 388.3915196527874y^2 + 581.7093496549514 \\ -278.3467870540393y^2 - 67.15141610586224 \end{bmatrix}, \tag{5.90}$$

$$\boldsymbol{L}_{12}(\boldsymbol{\eta}) = \begin{bmatrix} 388.436372083897y^2 + 634.8942397136793 \\ -278.3913290146102y^2 - 21.23235991871676 \end{bmatrix}, \tag{5.91}$$

$$\boldsymbol{L}_{22}(\boldsymbol{\eta}) = \begin{bmatrix} 388.3986718685013y^2 + 583.8601422200068 \\ -278.3648104692165y^2 - 66.13437525723143 \end{bmatrix}, \tag{5.92}$$

$$\mathbf{\Pi}_{111} = \begin{bmatrix} 0.03366283418 & 1.92 \times 10^{-5} & -0.00146967397 & -0.00121725964 \\ 1.92 \times 10^{-5} & 8.23 \times 10^{-8} & 2.54 \times 10^{-7} & 4.25 \times 10^{-6} \\ -0.00146967397 & 2.54 \times 10^{-7} & 1.95344817346 & 0.053420809298 \\ -0.00121725964 & 4.25 \times 10^{-6} & 0.053420809298 & 0.015769935976 \end{bmatrix}, \tag{5.93}$$

$$\mathbf{\Pi}_{112} = \begin{bmatrix} 0.037095776548 & 7.67 \times 10^{-6} & 0.0004928154045 & -8.73 \times 10^{-6} \\ 7.67 \times 10^{-5} & 1.94 \times 10^{-7} & 1.65 \times 10^{-7} & -3.56 \times 10^{-6} \\ 0.0004928154045 & 1.65 \times 10^{-7} & 1.95367678337 & 0.05074564828 \\ -8.73 \times 10^{-6} & -3.56 \times 10^{-6} & 0.05074564828 & 0.03269751748 \end{bmatrix}, \tag{5.94}$$

$$\mathbf{\Pi}_{121} = \begin{bmatrix} 0.053485016936 & 4.46 \times 10^{-5} & -8.08 \times 10^{-5} & -0.002202634383 \\ 4.46 \times 10^{-5} & 8.52 \times 10^{-8} & 1.07 \times 10^{-6} & -3.78 \times 10^{-6} \\ -8.08 \times 10^{-5} & 1.07 \times 10^{-6} & 1.95521546980 & 0.04091187136 \\ -0.002202634383 & -3.78 \times 10^{-6} & 0.04091187136 & 0.249164158979 \end{bmatrix}, \quad (5.95)$$

$$\mathbf{\Pi}_{122} = \begin{bmatrix} 0.05807093018 & 0.000102115 & 0.0014248422 & -0.0038939159 \\ 0.00010211552 & 2.20 \times 10^{-7} & 3.68 \times 10^{-6} & -9.61 \times 10^{-6} \\ 0.00142484225 & 3.68 \times 10^{-6} & 1.955672861 & 0.03758674717 \\ -0.0038939159 & -9.61 \times 10^{-6} & 0.03758674717 & 0.2595696846 \end{bmatrix}, \quad (5.96)$$

$$\mathbf{\Pi}_{211} = \begin{bmatrix} 0.053485017602 & 4.46 \times 10^{-5} & -8.08 \times 10^{-5} & -0.00220263557 \\ 4.46 \times 10^{-5} & 8.52 \times 10^{-8} & 1.07 \times 10^{-6} & -3.78 \times 10^{-6} \\ -8.08 \times 10^{-5} & 1.07 \times 10^{-6} & 1.9552155951 & 0.040911877332 \\ -0.00220263557 & -3.78 \times 10^{-6} & 0.040911877332 & 0.24916416324 \end{bmatrix}, \quad (5.97)$$

$$\mathbf{\Pi}_{212} = \begin{bmatrix} 0.058070930718 & 0.000102115 & 0.0014248471 & -0.003893915718 \\ 0.0001021155 & 2.20 \times 10^{-7} & 3.68 \times 10^{-6} & -9.61 \times 10^{-6} \\ 0.0014248471 & 3.68 \times 10^{-6} & 1.95567290512 & 0.037586750794 \\ -0.003893915718 & -9.61 \times 10^{-6} & 0.037586750794 & 0.25956968646 \end{bmatrix}, \quad (5.98)$$

$$\mathbf{\Pi}_{221} = \begin{bmatrix} 0.017786451347 & 1.66 \times 10^{-5} & -0.005299013941 & -0.001580184684 \\ 1.66 \times 10^{-5} & 2.60 \times 10^{-8} & -9.64 \times 10^{-6} & -2.64 \times 10^{-6} \\ -0.005299013941 & -9.64 \times 10^{-6} & 1.9082116944 & -0.03459880875 \\ -0.001580184684 & -2.64 \times 10^{-6} & -0.03459880875 & 0.16279190378 \end{bmatrix}, \quad (5.99)$$

$$\mathbf{\Pi}_{222} = \begin{bmatrix} 0.019920724566 & 3.24 \times 10^{-5} & -0.004915540307 & -0.001091150549 \\ 3.24 \times 10^{-5} & 6.45 \times 10^{-8} & -7.13 \times 10^{-6} & -2.37 \times 10^{-6} \\ -0.004915540307 & -7.13 \times 10^{-6} & 1.940469357 & -0.008332974410 \\ -0.001091150549 & -2.37 \times 10^{-6} & -0.008332974410 & 0.16923354534 \end{bmatrix}, \quad (5.100)$$

Figure 5.10 describes the control results by the switching polynomial fuzzy observer and controller designed in Corollary 5.1.3. From the figure, it can be seen that all initial states converge to zero

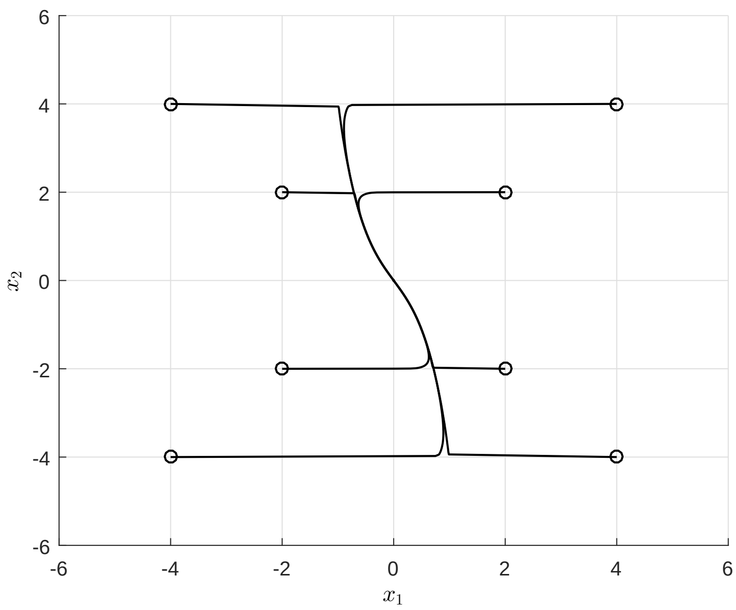

Fig. 5.10:Control results by the switching polynomial fuzzy observer and controller

5.2 Switching Polynomial Fuzzy Observer Design with Unmeasurable Premise Variables

Consider the following polynomial fuzzy model representation:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{x})\{\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) + \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{u}\}, \tag{5.101}$$

$$\boldsymbol{y} = h_i(\boldsymbol{x})\boldsymbol{C}_i(\boldsymbol{x})\boldsymbol{x} \tag{5.102}$$

where $\boldsymbol{x}$ is the state vector , y is the output vector, $\boldsymbol{A}_i(\boldsymbol{x})$ and $\boldsymbol{B}_i(\boldsymbol{x})$ are given polynomial matrices. $\boldsymbol{C}_i(\boldsymbol{x})$ are the polynomial matrices output. As performed in Chapter 3 and Chapter 4, in this chapter the polynomial fuzzy observer is designed as a switching observer according to the information of the switching index l. The switching polynomial fuzzy observer to estimate the state $\boldsymbol{x}$ is represented as follows.

$$\dot{\tilde{\boldsymbol{x}}} = \omega_i(\tilde{\boldsymbol{x}})\{\boldsymbol{A}_i(\tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}} + \boldsymbol{B}_i(\tilde{\boldsymbol{x}})\boldsymbol{u} + \boldsymbol{L}_{il}(\tilde{\boldsymbol{x}})(\boldsymbol{y} - \tilde{\boldsymbol{y}})\}, \tag{5.103}$$

$$\tilde{\boldsymbol{y}} = \sum_{i=1}^{r} \omega_i(\tilde{\boldsymbol{x}})\boldsymbol{C}_i(\tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}} \tag{5.104}$$

where $\tilde{\boldsymbol{x}}$ is the estimated state and $\boldsymbol{L}_{il}(\tilde{\boldsymbol{x}})$ are the switching observer gain.

Remark 8. In this case, we consider unmeasurable premise variables in the polynomial fuzzy model. Therefore, the membership functions of the polynomial fuzzy observer depend on the estimated state $\tilde{\boldsymbol{x}}$.

By using the obtained estimated state $\tilde{\boldsymbol{x}}$, the switching polynomial fuzzy controller is represented as:

$$\boldsymbol{u} = -\sum_{i=1}^{r} \omega_i(\tilde{\boldsymbol{x}})\boldsymbol{F}_{jl}(\tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}} \tag{5.105}$$

where $\boldsymbol{F}_{jl}(\tilde{\boldsymbol{x}})$ are the chosen controller gain according to the switching index l.

Theorem 5.2.1. *The polynomial fuzzy system is stabilized by the polynomial fuzzy controller if there exist positive definite polynomial matrices $\mathbf{Y}_l(\tilde{\boldsymbol{x}})$, polynomial matrices $\boldsymbol{F}_{jl}(\tilde{\boldsymbol{x}})$, $\boldsymbol{L}_{kl}(\tilde{\boldsymbol{x}})$, positive definite polynomial matrices $\boldsymbol{\Pi}_{ijkl}(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ such that the following conditions are satisfied with $\alpha < 0$ and the estimation error via the polynomial fuzzy observer tends to zero.*

$$\boldsymbol{v}_1^T(\mathbf{Y}_l(\tilde{\boldsymbol{x}}) - \boldsymbol{\epsilon}_1(\tilde{\boldsymbol{x}}))\boldsymbol{v}_1 \in \mathcal{S} \tag{5.106}$$

$$-\Big(\sum_{m=1}^{r} \hat{h}_m^2\Big)^{\mu} \sum_{j=1}^{r}\sum_{k=1}^{r} \hat{h}_j^2 \hat{h}_k^2 \boldsymbol{x}_v^T \boldsymbol{\Lambda}_{ijkl}(\boldsymbol{x}, \tilde{\boldsymbol{x}})\boldsymbol{x}_v \in \mathcal{S}\ \forall i \tag{5.107}$$

$$\lambda_{ijksl}(\tilde{\boldsymbol{x}}) \in \mathcal{S} \tag{5.108}$$

where μ is a nonnegative integer, $\boldsymbol{v}_1$ is an independent vector, $i, j \in 1, 2, \cdots, r$ and $s, l \in 1, 2, \cdots, K$ for rules number r and PPLF number K. $\boldsymbol{\epsilon}_1(\tilde{\boldsymbol{x}})$ is a predefined positive definite *polynomial matrix.* $\boldsymbol{\Lambda}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}})$ *are defined as*

$$\begin{aligned}\boldsymbol{\Lambda}_{ijl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) &= \mathcal{H}\big(\boldsymbol{Y}_l(\tilde{\boldsymbol{x}})\boldsymbol{\mathcal{M}}_{ikjl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\big) - \alpha\boldsymbol{\Pi}_{ikjl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) \\ &\quad + \sum_{s=1}^{K}\lambda_{ijksl}(\tilde{\boldsymbol{x}})\{\boldsymbol{Y}_s(\tilde{\boldsymbol{x}}) - \boldsymbol{Y}_l(\tilde{\boldsymbol{x}})\}, \end{aligned} \tag{5.109}$$

$$\boldsymbol{\mathcal{M}}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = \begin{bmatrix}\boldsymbol{\mathcal{M}}^{11}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) & \boldsymbol{\mathcal{M}}^{12}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) \\ \boldsymbol{\mathcal{M}}^{21}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) & \boldsymbol{\mathcal{M}}^{22}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\end{bmatrix}, \tag{5.110}$$

$$\begin{aligned}\boldsymbol{\mathcal{M}}^{11}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) &= \boldsymbol{A}_k(\tilde{\boldsymbol{x}}) - \boldsymbol{B}_k(\tilde{\boldsymbol{x}})\boldsymbol{F}_{jl}(\tilde{\boldsymbol{x}}) \\ &\quad + \boldsymbol{L}_{kl}(\tilde{\boldsymbol{x}})(\boldsymbol{C}_i(\boldsymbol{x}) - \boldsymbol{C}_j(\tilde{\boldsymbol{x}}))\end{aligned} \tag{5.111}$$

$$\boldsymbol{\mathcal{M}}^{12}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = \boldsymbol{L}_{kl}(\tilde{\boldsymbol{x}})\boldsymbol{C}_i(\boldsymbol{x}) \tag{5.112}$$

$$\begin{aligned}\boldsymbol{\mathcal{M}}^{21}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) &= \boldsymbol{A}_k(\tilde{\boldsymbol{x}}) - \boldsymbol{A}_i(\boldsymbol{x}) - (\boldsymbol{B}_k(\tilde{\boldsymbol{x}}) - \boldsymbol{B}_i(\boldsymbol{x}))\boldsymbol{F}_{jl}(\tilde{\boldsymbol{x}}) \\ &\quad - \boldsymbol{L}_{kl}(\tilde{\boldsymbol{x}})(\boldsymbol{C}_i(\boldsymbol{x}) - \boldsymbol{C}_j(\tilde{\boldsymbol{x}}))\end{aligned} \tag{5.113}$$

$$\boldsymbol{\mathcal{M}}^{22}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = \boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{L}_{kl}(\tilde{\boldsymbol{x}})\boldsymbol{C}_i(\boldsymbol{x}) \tag{5.114}$$

Proof. Define $\boldsymbol{x}_v = \begin{bmatrix}\tilde{\boldsymbol{x}}^T & \boldsymbol{e}^T\end{bmatrix}$, and the estimation error $\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$ by the observer. The error system is represented as

$$\begin{aligned}\dot{\boldsymbol{e}} &= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j=1}^{r}\omega_i(\boldsymbol{x})h_j(\tilde{\boldsymbol{x}})h_k(\tilde{\boldsymbol{x}})\times \\ &\Big(\big\{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{A}_k(\tilde{\boldsymbol{x}}) - (\boldsymbol{B}_i(\boldsymbol{x}) - \boldsymbol{B}_k(\tilde{\boldsymbol{x}}))\boldsymbol{F}_{jl}(\tilde{\boldsymbol{x}}) + \boldsymbol{L}_{kl}(\tilde{\boldsymbol{x}})(\boldsymbol{C}_j(\tilde{\boldsymbol{x}}) - \boldsymbol{C}_i(\boldsymbol{x}))\big\}\tilde{\boldsymbol{x}} \\ &+ \big\{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{L}_{kl}(\tilde{\boldsymbol{x}})\boldsymbol{C}_i(\boldsymbol{x})\big\}\boldsymbol{e}\Big).\end{aligned} \tag{5.115}$$

The augmented systems is described as follows.

$$\dot{\boldsymbol{x}}_v = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j=1}^{r}\omega_i(\boldsymbol{x})h_j(\tilde{\boldsymbol{x}})h_k(\tilde{\boldsymbol{x}})\boldsymbol{\mathcal{M}}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\boldsymbol{x}_v \tag{5.116}$$

where

$$\boldsymbol{\mathcal{M}}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = \begin{bmatrix} \boldsymbol{\mathcal{M}}^{11}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) & \boldsymbol{\mathcal{M}}^{12}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) \\ \boldsymbol{\mathcal{M}}^{21}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) & \boldsymbol{\mathcal{M}}^{22}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) \end{bmatrix}, \tag{5.117}$$

$$\boldsymbol{\mathcal{M}}^{11}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = \boldsymbol{A}_k(\tilde{\boldsymbol{x}}) - \boldsymbol{B}_k(\tilde{\boldsymbol{x}})\boldsymbol{F}_{jl}(\tilde{\boldsymbol{x}}) + \boldsymbol{L}_{kl}(\tilde{\boldsymbol{x}})(\boldsymbol{C}_i(\boldsymbol{x}) - \boldsymbol{C}_j(\tilde{\boldsymbol{x}})) \tag{5.118}$$

$$\boldsymbol{\mathcal{M}}^{12}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = \boldsymbol{L}_{kl}(\tilde{\boldsymbol{x}})\boldsymbol{C}_i(\boldsymbol{x}) \tag{5.119}$$

$$\begin{aligned}\boldsymbol{\mathcal{M}}^{21}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& \boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{A}_k(\tilde{\boldsymbol{x}}) - (\boldsymbol{B}_i(\boldsymbol{x}) - \boldsymbol{B}_k(\tilde{\boldsymbol{x}}))\boldsymbol{F}_{jl}(\tilde{\boldsymbol{x}}) \\ & - \boldsymbol{L}_{kl}(\tilde{\boldsymbol{x}})(\boldsymbol{C}_i(\boldsymbol{x}) - \boldsymbol{C}_j(\tilde{\boldsymbol{x}}))\end{aligned} \tag{5.120}$$

$$\boldsymbol{\mathcal{M}}^{22}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = \boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{L}_{kl}(\tilde{\boldsymbol{x}})\boldsymbol{C}_i(\boldsymbol{x}) \tag{5.121}$$

The chosen Lyapunov function is described as

$$V_l(\boldsymbol{x}_\upsilon) = \boldsymbol{x}_\upsilon^T \boldsymbol{Y}_l(\tilde{\boldsymbol{x}})\boldsymbol{x}_\upsilon \tag{5.122}$$

where the time derivatives are

$$\dot{V}_l(\boldsymbol{x}_\upsilon) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j=1}^{r} \omega_i(\boldsymbol{x}) h_j(\tilde{\boldsymbol{x}}) h_k(\tilde{\boldsymbol{x}}) \Big(\boldsymbol{x}_\upsilon^T \mathcal{H}\big(\boldsymbol{Y}_l(\tilde{\boldsymbol{x}})\boldsymbol{\mathcal{M}}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\big)\boldsymbol{x}_\upsilon \Big) \tag{5.123}$$

In order to guarantee $\dot{V}_l(\boldsymbol{x}_\upsilon) < 0$ at $\boldsymbol{x} \neq \boldsymbol{0}$, a scalar $\alpha < 0$ and positive definite polynomial matrices $\boldsymbol{\Pi}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}})$ are introduces satisfying $\dot{V}_l(\boldsymbol{x}_\upsilon) \leq \alpha \boldsymbol{x}_\upsilon^T \boldsymbol{\Pi}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\boldsymbol{x}_\upsilon$, that is,

$$\dot{V}_l(\boldsymbol{x}_\upsilon) - \alpha \boldsymbol{x}_\upsilon^T \boldsymbol{\Pi}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\boldsymbol{x}_\upsilon \leq 0 \tag{5.124}$$

which is equivalent to

$$\begin{aligned}&\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j=1}^{r} \omega_i(\boldsymbol{x}) h_j(\tilde{\boldsymbol{x}}) h_k(\tilde{\boldsymbol{x}}) \times \\ &\boldsymbol{x}_\upsilon^T \Big(\mathcal{H}\big(\boldsymbol{Y}_l(\tilde{\boldsymbol{x}})\boldsymbol{\mathcal{M}}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\big) - \alpha \boldsymbol{\Pi}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}}) \Big)\boldsymbol{x}_\upsilon \leq 0\end{aligned} \tag{5.125}$$

Since we consider minimum type PPLF, the following condition must be satisfied for $s, l \in 1, 2, \cdots, K$

$$V_l(\boldsymbol{x}_\upsilon) - V_s(\boldsymbol{x}_\upsilon) \leq 0 \tag{5.126}$$

which is equivalent to

$$\boldsymbol{x}_\upsilon^T \big(\boldsymbol{Y}_l(\tilde{\boldsymbol{x}}) - \boldsymbol{Y}_s(\tilde{\boldsymbol{x}})\big)\boldsymbol{x}_\upsilon \leq 0 \tag{5.127}$$

Now, recall Lemma 2.5.1 and define sets L_1 and L_2 as (5.124) and (5.127) respectively. If there exist $\lambda_{ijsl}(\tilde{\boldsymbol{x}}) \in \mathcal{P}^{0+}$ such that

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j=1}^{r}\omega_i(\boldsymbol{x})h_j(\tilde{\boldsymbol{x}})h_k(\tilde{\boldsymbol{x}})\boldsymbol{x}_v^T\boldsymbol{\Lambda}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\boldsymbol{x}_v \leq 0 \tag{5.128}$$

for all $\boldsymbol{x}$ then $L_2 \subseteq L_1$. In other words, condition (5.24) is satisfied only if the following condition is satisfied.

$$\sum_{s=1}^{K}\lambda_{ijksl}(\tilde{\boldsymbol{x}})\boldsymbol{x}_v^T\{\mathbf{Y}_s(\tilde{\boldsymbol{x}}) - \mathbf{Y}_l(\tilde{\boldsymbol{x}})\}\boldsymbol{x}_v \in \mathcal{P}^{0+} \tag{5.129}$$

Hence, by applying $\mathcal{S}$-procedure, (5.125) becomes (5.128). According to the relation between SOS and PSD polynomials, the following condition is a sufficient condition of (5.128).

$$-\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j=1}^{r}\omega_i(\boldsymbol{x})h_j(\tilde{\boldsymbol{x}})h_k(\tilde{\boldsymbol{x}})\boldsymbol{x}_v^T\boldsymbol{\Lambda}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\boldsymbol{x}_v \in \mathcal{S}. \tag{5.130}$$

Since all the membership functions are nonnegative, we can define $\omega_i(\boldsymbol{x}) = \hat{\omega}_i^2$,$h_j(\tilde{\boldsymbol{x}}) = \hat{h}_j^2$, and $h_h(\tilde{\boldsymbol{x}}) = \hat{w}_k^2$. By applying copositive relaxation, we arrive at the following condition.

$$-\Big(\sum_{m=1}^{r}\hat{h}_m^2\Big)^{\mu}\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\hat{\omega}_i^2\hat{h}_j^2\hat{h}_k^2\boldsymbol{x}_v^T\boldsymbol{\Lambda}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\boldsymbol{x}_v \in \mathcal{S} \tag{5.131}$$

Condition (5.131) is equivalent to

$$-\Big(\sum_{m=1}^{r}\hat{h}_m^2\Big)^{\mu}\sum_{j=1}^{r}\sum_{k=1}^{r}\hat{h}_j^2\hat{h}_k^2\boldsymbol{x}_v^T\boldsymbol{\Lambda}_{ijkl}(\boldsymbol{x},\tilde{\boldsymbol{x}})\boldsymbol{x}_v \in \mathcal{S}\ \forall i \tag{5.132}$$

□

5.2.1 Design Example

Consider the following polynomial fuzzy model:

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} 1 & 1 \\ -1.5 & -2 - x_2^2 \end{bmatrix}, \qquad \boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} 1 & -0.2172 \\ -1.5 & -2 - x_2^2 \end{bmatrix},$$
$$\boldsymbol{A}_1(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 1 & 1 \\ -1.5 & -2 - \tilde{x}_2^2 \end{bmatrix}, \qquad \boldsymbol{A}_2(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 1 & -0.2172 \\ -1.5 & -2 - \tilde{x}_2^2 \end{bmatrix},$$
$$\boldsymbol{B}_1(\boldsymbol{x}) = \boldsymbol{B}_2(\boldsymbol{x}) = \begin{bmatrix} x_2^2 + 1 \\ 0 \end{bmatrix},$$
$$\boldsymbol{B}_1(\tilde{\boldsymbol{x}}) = \boldsymbol{B}_2(\tilde{\boldsymbol{x}}) = \begin{bmatrix} \tilde{x}_2^2 + 1 \\ 0 \end{bmatrix},$$
$$\boldsymbol{C}_1(\boldsymbol{x}) = \boldsymbol{C}_2(\boldsymbol{x}) = \begin{bmatrix} 0.1x_2 + 1 & 0 \end{bmatrix},$$
$$\boldsymbol{C}_1(\tilde{\boldsymbol{x}}) = \boldsymbol{C}_2(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 0.1\tilde{x}_2 + 1 & 0 \end{bmatrix}.$$

The membership functions with unmeasurable premise variable x_2 are given as follows:

$$h_1(x_2) = \frac{\sin x_2 + 0.2172x_2}{1.2172x_2}, \quad h_2(\boldsymbol{z}) = \frac{x_2 - \sin x_2}{1.2172x_2}$$

By solving the SOS conditions in Theorem 5.2.1, the feasible solutions obtained are as follows.

$$\boldsymbol{Y}_1 = \begin{bmatrix} 83.619570 & 86.148409 & -1.0917213 & -0.3246260 \\ 86.148409 & 4051.332701 & -9.5786605 & -2469.296519 \\ -1.0917213 & -9.5786605 & 2.8566668 & -1.4063507 \\ -0.3246260 & -2469.296519 & -1.4063507 & 3101.863217 \end{bmatrix} \tag{5.133}$$

$$\boldsymbol{Y}_2 = \begin{bmatrix} 14.509313 & 9.2623232 & -1.0917213 & -0.3246260 \\ 9.2623232 & 3856.109286 & -9.5786605 & -2469.296519 \\ -1.0917213 & -9.5786605 & 2.8566668 & -1.4063507 \\ -0.3246260 & -2469.296519 & -1.4063507 & 3101.863217 \end{bmatrix} \tag{5.134}$$

$$\boldsymbol{F}_{11}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 0.3047531\tilde{x}_1^2 - 0.1182763\tilde{x}_1\tilde{x}_2 + 2.668091\tilde{x}_2^2 + 237.375739 \\ -0.118699\tilde{x}_1^2 + 2.668531\tilde{x}_1\tilde{x}_2 + 5.435706\tilde{x}_2^2 + 70.008868 \end{bmatrix}^T \tag{5.135}$$

$$\boldsymbol{F}_{12}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 0.0580521\tilde{x}_1^2 + 0.0164858\tilde{x}_1\tilde{x}_2 + 0.344749\tilde{x}_2^2 + 196.501705 \\ 0.0143320\tilde{x}_1^2 + 0.342859\tilde{x}_1\tilde{x}_2 + 0.7206258\tilde{x}_2^2 - 21.474960 \end{bmatrix}^T \tag{5.136}$$

$$\boldsymbol{F}_{21}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 0.493557\tilde{x}_1^2 - 0.0565624\tilde{x}_1\tilde{x}_2 + 2.9856989\tilde{x}_2^2 + 287.0107846 \\ -0.056720\tilde{x}_1^2 + 2.9861839\tilde{x}_1\tilde{x}_2 + 6.139381\tilde{x}_2^2 + 90.2251491 \end{bmatrix}^T \tag{5.137}$$

$$\boldsymbol{F}_{22}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 0.0581702\tilde{x}_1^2 + 0.0157470\tilde{x}_1\tilde{x}_2 + 0.343319\tilde{x}_2^2 + 196.494352 \\ 0.0151091\tilde{x}_1^2 + 0.3440863\tilde{x}_1\tilde{x}_2 + 0.720499\tilde{x}_2^2 - 21.580022 \end{bmatrix}^T \tag{5.138}$$

$$\boldsymbol{L}_{11}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 15.602788\tilde{x}_1^2 - 7.634681\tilde{x}_1\tilde{x}_2 + 83.961430\tilde{x}_2^2 - 360.227634 \\ 0.0119639\tilde{x}_1^2 - 0.310138\tilde{x}_1\tilde{x}_2 - 0.3661904\tilde{x}_2^2 + 0.90083 \end{bmatrix} \tag{5.139}$$

$$\boldsymbol{L}_{12}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 2.638264\tilde{x}_1^2 + 3.086183\tilde{x}_1\tilde{x}_2 + 24.001923\tilde{x}_2^2 + 81.156223 \\ -0.0150319\tilde{x}_1^2 - 0.101156\tilde{x}_1\tilde{x}_2 - 0.112034\tilde{x}_2^2 - 0.455324 \end{bmatrix} \tag{5.140}$$

$$\boldsymbol{L}_{21}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 14.792057\tilde{x}_1^2 - 4.802265\tilde{x}_1\tilde{x}_2 + 96.618624\tilde{x}_2^2 - 209.4245491 \\ 0.006232\tilde{x}_1^2 - 0.373299\tilde{x}_1\tilde{x}_2 - 0.422054\tilde{x}_2^2 + 0.869989 \end{bmatrix} \tag{5.141}$$

$$\boldsymbol{L}_{22}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} 2.643308\tilde{x}_1^2 + 3.077547\tilde{x}_1\tilde{x}_2 + 24.003241\tilde{x}_2^2 + 81.060344 \\ -0.0150120\tilde{x}_1^2 - 0.101137\tilde{x}_1\tilde{x}_2 - 0.1120935\tilde{x}_2^2 - 0.455670 \end{bmatrix} \tag{5.142}$$

Other feasible solutions of decision variables, i.e., $\lambda_{ijksl}(\tilde{\boldsymbol{x}}), \boldsymbol{\Pi}_{ijkl}(\boldsymbol{x}, \tilde{\boldsymbol{x}})$, are shown in the Appendix C. The output and the estimation output by the PPLF-based observer are shown in Figure 5.11. From the figure, it can be seen that the estimation error tends to zero. In Figure 5.12, the control trajectory results are provided with several initial states. From the results, PPLF-based observer that depends on the estimated states has successfully stabilized the system, i.e. all the states converge to the equilibrium point.

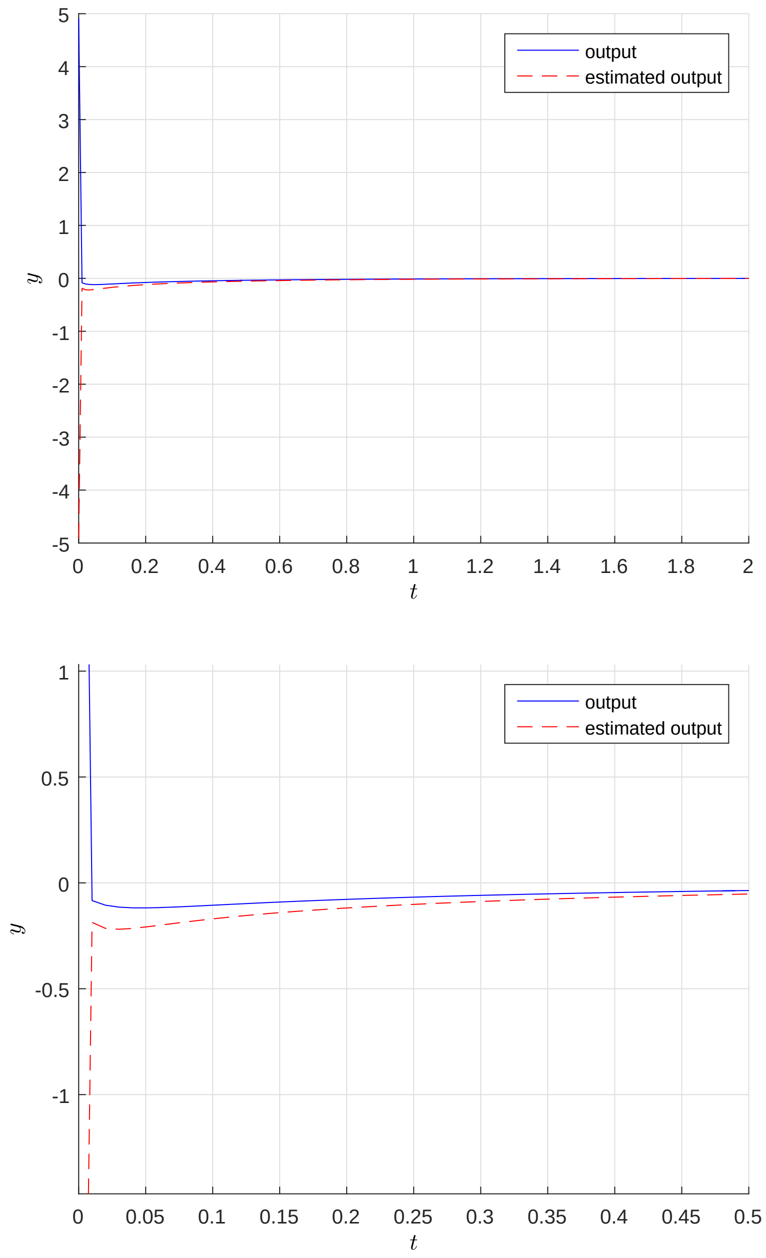

Fig. 5.11:Output and estimated output.

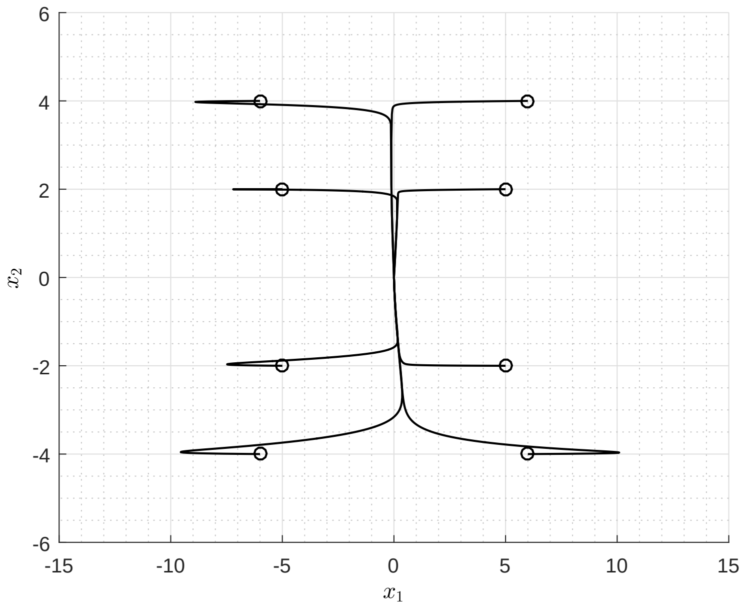

Fig. 5.12:Control behavior of the design example with unmeasurable premise variable.

Chapter 6.

Conclusions and Future Works

6.1 Conclusions

This thesis proposes a minimum-type piecewise polynomial Lyapunov function-based (PPLF-based) approach to reduce the conservativeness in the stability analysis of nonlinear system which are represented as polynomial fuzzy systems. The strength of the proposed PPLF-based approach has been demonstrated in the stabilization, robust stabilization, and observer design for polynomial fuzzy systems.

PPLF-based approach provides several polynomial Lyapunov functions (PLFs) according to the number of PPLF. This is different to that of other approaches which commonly use only one Lyapunov function. In PPLF-based approach, the chosen PLF can be switched simultaneously according to a switching index which was defined to declare the minimum PLF at the time. In order to fully utilize the strength of this approach, a switching controller has been designed based on a parallel distributed compensation (PDC) concept. The switching index decides the switching feedback gain simultaneously when the state is on the switching boundaries of minimum-type PPLF. This technique leads to a wider stability region which has been demonstrated through several design examples.

In regards of stabilization, the proposed PPLF-based approach was utilized to derive stabilization conditions by considering two relaxations: positivstellensatz (P-satz) and copositive relaxation. In order to solve the SOS conditions consisting of nonconvex term, a path following algorithm was presented (Chapter 3). A benchmark design example was used for stabilization conditions based of both P-satz and copositive relaxation. It was found that copositive relaxation resulted in a reduced conservativeness in comparison to that of P-satz relaxation. The effectiveness of the proposed design described in Theorem 3.3.1 has been demonstrated through two benchmark design examples, i.e. a benchmark T-S fuzzy and a polynomial fuzzy design example. Comparison to other existing methods for stabilization of polynomial fuzzy systems showed that the proposed design yields a more relaxed result by achieving a wider feasible areas. Example I in Section 3.3.2, the maximum value of b for second order PPLF_2 is 7.0 while PPLF_1 (or PLF) yields $b_{\max} = 6.5$ while the other existing results are $b_{\max} = 6.5$ [36, 37], $b_{\max} = 6.0$ [33, 35, 38], $b_{\max} = 5.0$ [39], $b_{\max} = 2.5$ [42].

The proposed PPLF-based approach was then employed in a robust control design for polynomial fuzzy systems. The SOS conditions are presented for two cases. In the first case, uncertainties appeared both in the system and in the input term. In the second case, the uncertainty appeared only in the system. In order to solve the SOS conditions in Theorem 4.2.1 and Corollary 4.2.2, a path following algorithm was utilized in a similar manner as the previously explained stabilization case. The results have been compared to two design examples used by other approaches. It was demonstrated that the proposed design yields a wider uncertainties parameter region. For instance, in design example 4.4.2, a much wider feasible region can be obtained with uncertainties satisfying $|\Delta_a(t) \leq 0.35$ and $|\Delta_b(t)| \leq 0.55$ while the results from other existing approaches are $|\Delta_a(t) \leq 0.09$, $|\Delta_b(t)| \leq 0.25$ [61] and $|\Delta_a(t) \leq 0.03$, $|\Delta_b(t)| \leq 0.05$ [44]. Moreover, the calculation time to find the feasible solutions for design example 4.4.2 has also been presented. Although the complexity and the required calculation time are slightly inferior as compared to the other approaches, these demerits are well compensated by the advantage of the proposed design i.e. a wider robust stabilization region.

Finally, the proposed PPLF-based approach was also employed to design a polynomial fuzzy observer (Chapter 5). By using the switching information on the PPLF, a switching polynomial fuzzy observer was designed according to the information of the estimated states obtained by the switching polynomial fuzzy observer. The PPLF-based approach fuzzy observer and controller were designed for several cases, i.e. Class I, Class II, and Class III, according to the dependencies of polynomial matrices in the system and/or input ($\boldsymbol{A}_i$ and $\boldsymbol{B}_i$). This brings more efficiency on designing polynomial fuzzy observer-controller compared with other existing work in [16] that proposed an observer design for each class. Comparison with three design examples used in other existing approaches, this thesis has demonstrated that the proposed observer design was able to obtain a wider feasible area. Moreover, a polynomial fuzzy observer design with unmeasurable premise variables has also been presented.

According to the results, it can be concluded that the proposed designs are effective to reduce the conservativeness in the stabilization, robust stabilization, and observer design for a class of nonlinear systems. This is crucial to improve the stabilizability in practical applications such as aerial vehicle applications (unmanned aerial vehicle, powered paragliding, micro helicopter).

6.2 Future Works

Numerous investigations have been addressed to accomplish stability analysis and design of polynomial fuzzy systems via SOS framework. The stability analysis based on Lyapunov stability theory can be reduced to the existence of positive definite polynomial such that its partial derivative is negative definite along the trajectories. The stability/stabilization conditions are derived and formulated as SOS optimization problems which then be solved by an SOS solver. During the past decade, a lot of research efforts have been put on reducing conservativeness in the derivation process of stability/stabilization conditions. One of the main sources causing conservativeness is selection of Lyapunov function candidate form. This thesis proposed a PPLF-based approach to overcome the problem and it has been showed that the PPLF-based approach has successfully produced more relaxed results compared to other existing approaches. For further improvement, the following aspects need to be considered in the future. Firstly, further optimization in the path-following algorithm to select initial conditions shall be addressed. In this thesis, the initial conditions are given by either using random generation (see Chapter 3.2.1) or grid search method (see Chapter 3.3.1). Since the success of path-following algorithm depends on given initial conditions, a method to select a good initial condition can be improved. The improvement of this aspect may lead to a more relaxed result. One of the ideas is by using the solution from local convex stabilization conditions. For example, we have 2-rules polynomial fuzzy system. By deriving convex stabilization conditions, we can solve the problems for each local system. The solutions obtained for each local system can be used to create a cone. Then, grid search method can be performed to select good initial conditions inside the cone. This method may possibly bring better initial conditions causing a less conservative result. Another aspect that can be improved is the perturbation value, e.g. $\epsilon_v, \epsilon_\xi, \epsilon_f$ in Chapter 3.3.1. In this thesis, the perturbation value was usually equal to 0.001. Small change of the perturbation value may lead to different results which can be more or less conservative results. Therefore, a method to choose proper perturbation values shall be investigated in order to improve path-following algorithm.

Secondly, an alternative method to reduce the gap which exists between SOS and PSD polynomial forms shall be explored further. This thesis has proposed a PPLF-based approach to reduce conservativeness existing in the stability analysis and design of polynomial fuzzy systems that has been applied in three key features: stabilization, robust control, and observer design. However, conservativeness issue in the SOS design framework has not been deeply investigated. Therefore, improvement on this aspect may possibly give significant contribution on stability analysis not only in the fuzzy system but also in the other classes of nonlinear systems. One of the alternative methods that can be used to overcome this problem is by using Handelman's theorem instead of SOS to guarantee the existence of nonnegative polynomials. Stability analysis of nonlinear systems by using Handelman's theorem has been investigated in [26]. The results in [26] showed their proposed approach can be used to analyze the stability of nonlinear systems with polynomial vector field where the Lyapunov functions were represented in the Handelman basis. Complexity comparison between SOS approach with Polya theorem and linear program (LP) with Handelman representation has also been discussed. Stability analysis represented in Handelman basis can be applied for polynomial fuzzy system.

Thirdly, simulation of design examples of a real system shall be considered in order to show the practical ability of the proposed designs. The design examples used in this thesis were mostly benchmark design examples in the fuzzy control areas. Simulation of design examples of a real system becomes important to demonstrate an expanded usability of the proposed approach. Some potential practical applications include powered paraglider (PPG), UAV system, among others. As demonstrated in the robust stabilization design, the proposed piecewise polynomial Lyapunov function (PPLF) based design successfully obtained larger upper bounds of the uncertain terms as compared to the existing results [44] [61]. Therefore, it is highly potential that the proposed robust control can yield a significant improvement in the control system design of practical applications that have many unknown parameters such as moment of inertia, drag coefficient, etc. A representative example in this case is powered paraglider (PPG), in which a problem related to the drag coefficient still remains. The drag coefficient may be slightly changed even near the considered trim equilibrium [61]. [73] offered a solution to this problem by considering parameter uncertainties in the constructed lateral model. A robust controller to stabilize the lateral model was designed based on quadratic Lyapunov function approach. In this aspect, considering that the form of quadratic Lyapunov function is more conservative than PPLF, the proposed robust control design in this thesis work allows a significantly better solution for the problem and there is a possibility to expand for other unknown parameters, e.g., wind disturbance, in the PPG robust control.

Finally, as the proposed PPLF-based design is effective for the stabilization, robust control, and observer design, promising results can be expected in applying the proposed PPLF-based approach to other fields of control theory. For instance, it can be utilized in designing H_∞ control, guaranteed cost control, optimal control, stochastic control, and adaptive control, among others.

Appendices

6.3 Appendix A

The obtained feasible solutions of λ_{ijsl} and $\mathbf{\Pi}_{ijl}(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ for design example class III in Chapter 5.1.2 are as follows.

$$\lambda_{1111} = 0.6090895542693308 \qquad \lambda_{1112} = 0.6090895626845496$$
$$\lambda_{1121} = 0.6090895489549435 \qquad \lambda_{1122} = 0.6090895585473359$$
$$\lambda_{1211} = 0.6090895541207217 \qquad \lambda_{1212} = 0.6090895594138583$$
$$\lambda_{1221} = 0.6090895573739531 \qquad \lambda_{1222} = 0.6090895583839942$$
$$\lambda_{2111} = 0.6090895604949692 \qquad \lambda_{2112} = 0.6090895601780549$$
$$\lambda_{2121} = 0.6090895550411645 \qquad \lambda_{2122} = 0.6090895574885047$$
$$\lambda_{2211} = 0.6090895576425222 \qquad \lambda_{2212} = 0.6090895551894614$$
$$\lambda_{2221} = 0.6090895554171446 \qquad \lambda_{2222} = 0.6090895621370879$$

$$\begin{aligned}
\Pi_{111}^{11}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 4.437x_2^4 + 0.043x_2^3\tilde{x}_1 + 0.223x_2^3\tilde{x}_2 + 3.407x_2^2\tilde{x}_1^2 + 0.016x_2^2\tilde{x}_1\tilde{x}_2 + 2.717x_2^2\tilde{x}_2^2 + \\
& 0.03x_2\tilde{x}_1^3 + 0.148x_2\tilde{x}_1^2\tilde{x}_2 + 0.026x_2\tilde{x}_1\tilde{x}_2^2 + 0.096x_2\tilde{x}_2^3 + 6.269\tilde{x}_1^4 + 0.013\tilde{x}_1^3\tilde{x}_2 + \\
& 2.715\tilde{x}_1^2\tilde{x}_2^2 + 0.044\tilde{x}_1\tilde{x}_2^3 + 3.817\tilde{x}_2^4 \\
\Pi_{111}^{21}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.063x_2^4 + 0.235x_2^3\tilde{x}_1 + 0.076x_2^3\tilde{x}_2 + 0.015x_2^2\tilde{x}_1^2 + 0.261x_2^2\tilde{x}_1\tilde{x}_2 + 0.099x_2^2\tilde{x}_2^2 + \\
& 0.127x_2\tilde{x}_1^3 + 0.026x_2\tilde{x}_1^2\tilde{x}_2 + 0.118x_2\tilde{x}_1\tilde{x}_2^2 + 0.091x_2\tilde{x}_2^3 + 0.01\tilde{x}_1^4 + 0.247\tilde{x}_1^3\tilde{x}_2 + \\
& 0.047\tilde{x}_1^2\tilde{x}_2^2 + 0.32\tilde{x}_1\tilde{x}_2^3 + 0.159\tilde{x}_2^4 \\
\Pi_{111}^{31}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.474x_2^4 + 0.016x_2^3\tilde{x}_1 + 0.066x_2^3\tilde{x}_2 + 0.517x_2^2\tilde{x}_1^2 + 0.004x_2^2\tilde{x}_1\tilde{x}_2 + 0.088x_2^2\tilde{x}_2^2 + \\
& 0.008x_2\tilde{x}_1^3 + 0.042x_2\tilde{x}_1^2\tilde{x}_2 - 0.008x_2\tilde{x}_1\tilde{x}_2^2 - 0.008x_2\tilde{x}_2^3 - 0.098\tilde{x}_1^4 - 0.002\tilde{x}_1^3\tilde{x}_2 - \\
& 0.081\tilde{x}_1^2\tilde{x}_2^2 - 0.028\tilde{x}_1\tilde{x}_2^3 - 0.117\tilde{x}_2^4 \\
\Pi_{111}^{41}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.026x_2^4 + 0.839x_2^3\tilde{x}_1 - 0.003x_2^3\tilde{x}_2 + 0.034x_2^2\tilde{x}_1^2 + 0.004x_2^2\tilde{x}_1\tilde{x}_2 - 0.014x_2^2\tilde{x}_2^2 + \\
& 0.722x_2\tilde{x}_1^3 - 0.01x_2\tilde{x}_1^2\tilde{x}_2 + 0.126x_2\tilde{x}_1\tilde{x}_2^2 - 0.001x_2\tilde{x}_2^3 + 0.016\tilde{x}_1^4 - 0.12\tilde{x}_1^3\tilde{x}_2 - \\
& 0.023\tilde{x}_1^2\tilde{x}_2^2 - 0.217\tilde{x}_1\tilde{x}_2^3 - 0.084\tilde{x}_2^4 \\
\Pi_{111}^{12}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.063x_2^4 + 0.235x_2^3\tilde{x}_1 + 0.076x_2^3\tilde{x}_2 + 0.015x_2^2\tilde{x}_1^2 + 0.261x_2^2\tilde{x}_1\tilde{x}_2 + 0.099x_2^2\tilde{x}_2^2 + \\
& 0.127x_2\tilde{x}_1^3 + 0.026x_2\tilde{x}_1^2\tilde{x}_2 + 0.118x_2\tilde{x}_1\tilde{x}_2^2 + 0.091x_2\tilde{x}_2^3 + 0.01\tilde{x}_1^4 + 0.247\tilde{x}_1^3\tilde{x}_2 + \\
& 0.047\tilde{x}_1^2\tilde{x}_2^2 + 0.32\tilde{x}_1\tilde{x}_2^3 + 0.159\tilde{x}_2^4 \\
\Pi_{111}^{22}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 4.693x_2^4 + 0.075x_2^3\tilde{x}_1 + 0.766x_2^3\tilde{x}_2 + 2.711x_2^2\tilde{x}_1^2 + 0.1x_2^2\tilde{x}_1\tilde{x}_2 + 3.528x_2^2\tilde{x}_2^2 + \\
& 0.019x_2\tilde{x}_1^3 + 0.105x_2\tilde{x}_1^2\tilde{x}_2 + 0.104x_2\tilde{x}_1\tilde{x}_2^2 + 0.721x_2\tilde{x}_2^3 + 3.754\tilde{x}_1^4 + 0.036\tilde{x}_1^3\tilde{x}_2 + \\
& 2.815\tilde{x}_1^2\tilde{x}_2^2 + 0.186\tilde{x}_1\tilde{x}_2^3 + 5.166\tilde{x}_2^4
\end{aligned}$$

$$\begin{aligned}
\Pi_{111}^{32}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -0.057x_2^4 + 0.070x_2^3\tilde{x}_1 - 0.005x_2^3\tilde{x}_2 + 0.005x_2^2\tilde{x}_1^2 + 0.064x_2^2\tilde{x}_1\tilde{x}_2 - 0.013x_2^2\tilde{x}_2^2 + \\
& 0.039x_2\tilde{x}_1^3 - 0.008x_2\tilde{x}_1^2\tilde{x}_2 - 0.014x_2\tilde{x}_1\tilde{x}_2^2 - 0.007x_2\tilde{x}_2^3 - 0.001\tilde{x}_1^4 - 0.088\tilde{x}_1^3\tilde{x}_2 - \\
& 0.032\tilde{x}_1^2\tilde{x}_2^2 - 0.179\tilde{x}_1\tilde{x}_2^3 + 0.015\tilde{x}_2^4 \\
\Pi_{111}^{42}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.391x_2^4 - 0.002x_2^3\tilde{x}_1 + 0.593x_2^3\tilde{x}_2 + 0.03x_2^2\tilde{x}_1^2 - 0.012x_2^2\tilde{x}_1\tilde{x}_2 - 0.178x_2^2\tilde{x}_2^2 - \\
& 0.008x_2\tilde{x}_1^3 + 0.13x_2\tilde{x}_1^2\tilde{x}_2 + 0.21x_2\tilde{x}_2^3 - 0.06\tilde{x}_1^4 - 0.017\tilde{x}_1^3\tilde{x}_2 - 0.201\tilde{x}_1^2\tilde{x}_2^2 - \\
& 0.097\tilde{x}_1\tilde{x}_2^3 - 0.955\tilde{x}_2^4
\end{aligned}$$

$$
\begin{aligned}
\Pi^{13}_{111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.474x_2^4 + 0.016x_2^3\tilde{x}_1 + 0.066x_2^3\tilde{x}_2 + 0.517x_2^2\tilde{x}_1^2 + 0.004x_2^2\tilde{x}_1\tilde{x}_2 + 0.088x_2^2\tilde{x}_2^2 + \\
& 0.008x_2\tilde{x}_1^3 + 0.042x_2\tilde{x}_1^2\tilde{x}_2 - 0.008x_2\tilde{x}_1\tilde{x}_2^2 - 0.008x_2\tilde{x}_2^3 - 0.098\tilde{x}_1^4 - 0.002\tilde{x}_1^3\tilde{x}_2 - \\
& 0.081\tilde{x}_1^2\tilde{x}_2^2 - 0.028\tilde{x}_1\tilde{x}_2^3 - 0.117\tilde{x}_2^4 \\
\Pi^{23}_{111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -0.057x_2^4 + 0.070x_2^3\tilde{x}_1 - 0.005x_2^3\tilde{x}_2 + 0.005x_2^2\tilde{x}_1^2 + 0.064x_2^2\tilde{x}_1\tilde{x}_2 - 0.013x_2^2\tilde{x}_2^2 + \\
& 0.039x_2\tilde{x}_1^3 - 0.008x_2\tilde{x}_1^2\tilde{x}_2 - 0.014x_2\tilde{x}_1\tilde{x}_2^2 - 0.007x_2\tilde{x}_2^3 - 0.001\tilde{x}_1^4 - 0.088\tilde{x}_1^3\tilde{x}_2 - \\
& 0.032\tilde{x}_1^2\tilde{x}_2^2 - 0.179\tilde{x}_1\tilde{x}_2^3 + 0.015\tilde{x}_2^4 \\
\Pi^{33}_{111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 6.488x_2^4 + 0.006x_2^3\tilde{x}_1 - 0.014x_2^3\tilde{x}_2 + 4.014x_2^2\tilde{x}_1^2 - 0.013x_2^2\tilde{x}_1\tilde{x}_2 + 3.979x_2^2\tilde{x}_2^2 + \\
& 0.003x_2\tilde{x}_1^3 + 0.038x_2\tilde{x}_1^2\tilde{x}_2 + 0.008x_2\tilde{x}_1\tilde{x}_2^2 - 0.002x_2\tilde{x}_2^3 + 6.495\tilde{x}_1^4 - 0.014\tilde{x}_1^3\tilde{x}_2 + \\
& 4.209\tilde{x}_1^2\tilde{x}_2^2 + 0.014\tilde{x}_1\tilde{x}_2^3 + 6.12\tilde{x}_2^4 \\
\Pi^{43}_{111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.02x_2^4 + 0.409x_2^3\tilde{x}_1 - 0.059x_2^3\tilde{x}_2 + 0.012x_2^2\tilde{x}_1^2 + 0.008x_2^2\tilde{x}_1\tilde{x}_2 + 0.008x_2^2\tilde{x}_2^2 + \\
& 0.38x_2\tilde{x}_1^3 + 0.012x_2\tilde{x}_1^2\tilde{x}_2 + 0.074x_2\tilde{x}_1\tilde{x}_2^2 - 0.005x_2\tilde{x}_2^3 + 0.01\tilde{x}_1^4 + 0.129\tilde{x}_1^3\tilde{x}_2 + \\
& 0.024\tilde{x}_1^2\tilde{x}_2^2 + 0.168\tilde{x}_1\tilde{x}_2^3 - 0.021\tilde{x}_2^4 \\
\Pi^{14}_{111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.026x_2^4 + 0.839x_2^3\tilde{x}_1 - 0.003x_2^3\tilde{x}_2 + 0.034x_2^2\tilde{x}_1^2 + 0.004x_2^2\tilde{x}_1\tilde{x}_2 - 0.014x_2^2\tilde{x}_2^2 + \\
& 0.722x_2\tilde{x}_1^3 - 0.01x_2\tilde{x}_1^2\tilde{x}_2 + 0.126x_2\tilde{x}_1\tilde{x}_2^2 - 0.001x_2\tilde{x}_2^3 + 0.016\tilde{x}_1^4 - 0.12\tilde{x}_1^3\tilde{x}_2 - \\
& 0.023\tilde{x}_1^2\tilde{x}_2^2 - 0.217\tilde{x}_1\tilde{x}_2^3 - 0.084\tilde{x}_2^4 \\
\Pi^{24}_{111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.391x_2^4 - 0.002x_2^3\tilde{x}_1 + 0.593x_2^3\tilde{x}_2 + 0.03x_2^2\tilde{x}_1^2 - 0.012x_2^2\tilde{x}_1\tilde{x}_2 - 0.178x_2^2\tilde{x}_2^2 - \\
& 0.008x_2\tilde{x}_1^3 + 0.13x_2\tilde{x}_1^2\tilde{x}_2 + 0.21x_2\tilde{x}_2^3 - 0.06\tilde{x}_1^4 - 0.017\tilde{x}_1^3\tilde{x}_2 - 0.201\tilde{x}_1^2\tilde{x}_2^2 - \\
& 0.097\tilde{x}_1\tilde{x}_2^3 - 0.955\tilde{x}_2^4 \\
\Pi^{34}_{111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.02x_2^4 + 0.409x_2^3\tilde{x}_1 - 0.059x_2^3\tilde{x}_2 + 0.012x_2^2\tilde{x}_1^2 + 0.008x_2^2\tilde{x}_1\tilde{x}_2 + 0.008x_2^2\tilde{x}_2^2 + \\
& 0.38x_2\tilde{x}_1^3 + 0.012x_2\tilde{x}_1^2\tilde{x}_2 + 0.074x_2\tilde{x}_1\tilde{x}_2^2 - 0.005x_2\tilde{x}_2^3 + 0.01\tilde{x}_1^4 + 0.129\tilde{x}_1^3\tilde{x}_2 + \\
& 0.024\tilde{x}_1^2\tilde{x}_2^2 + 0.168\tilde{x}_1\tilde{x}_2^3 - 0.021\tilde{x}_2^4 \\
\Pi^{44}_{111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 6.071x_2^4 + 0.03x_2^3\tilde{x}_1 - 0.236x_2^3\tilde{x}_2 + 3.351x_2^2\tilde{x}_1^2 + 0.008x_2^2\tilde{x}_1\tilde{x}_2 + 3.271x_2^2\tilde{x}_2^2 + \\
& 0.036x_2\tilde{x}_1^3 - 0.13x_2\tilde{x}_1^2\tilde{x}_2 - 0.016x_2\tilde{x}_1\tilde{x}_2^2 - 0.411x_2\tilde{x}_2^3 + 4.202\tilde{x}_1^4 + 0.008\tilde{x}_1^3\tilde{x}_2 + \\
& 2.813\tilde{x}_1^2\tilde{x}_2^2 + 0.091\tilde{x}_1\tilde{x}_2^3 + 4.655\tilde{x}_2^4
\end{aligned}
$$

$$\Pi^{11}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 4.535x_2^4 + 0.038x_2^3\tilde{x}_1 + 0.343x_2^3\tilde{x}_2 + 3.695x_2^2\tilde{x}_1^2 + 0.035x_2^2\tilde{x}_1\tilde{x}_2 + 2.911x_2^2\tilde{x}_2^2 + 0.034x_2\tilde{x}_1^3 + 0.398x_2\tilde{x}_1^2\tilde{x}_2 + 0.033x_2\tilde{x}_1\tilde{x}_2^2 + 0.217x_2\tilde{x}_2^3 + 6.845\tilde{x}_1^4 + 0.035\tilde{x}_1^3\tilde{x}_2 + 3.31\tilde{x}_1^2\tilde{x}_2^2 + 0.038\tilde{x}_1\tilde{x}_2^3 + 4.035\tilde{x}_2^4$$

$$\Pi^{21}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -0.006x_2^4 + 0.355x_2^3\tilde{x}_1 + 0.025x_2^3\tilde{x}_2 + 0.035x_2^2\tilde{x}_1^2 + 0.468x_2^2\tilde{x}_1\tilde{x}_2 + 0.033x_2^2\tilde{x}_2^2 + 0.347x_2\tilde{x}_1^3 + 0.034x_2\tilde{x}_1^2\tilde{x}_2 + 0.262x_2\tilde{x}_1\tilde{x}_2^2 + 0.017x_2\tilde{x}_2^3 + 0.031\tilde{x}_1^4 + 0.789\tilde{x}_1^3\tilde{x}_2 + 0.045\tilde{x}_1^2\tilde{x}_2^2 + 0.605\tilde{x}_1\tilde{x}_2^3 + 0.016\tilde{x}_2^4$$

$$\Pi^{31}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.314x_2^4 - 0.008x_2^3\tilde{x}_1 + 0.035x_2^3\tilde{x}_2 + 0.214x_2^2\tilde{x}_1^2 - 0.049x_2^2\tilde{x}_1\tilde{x}_2 + 0.079x_2^2\tilde{x}_2^2 + 0.02x_2\tilde{x}_1^3 + 0.015x_2\tilde{x}_1^2\tilde{x}_2 + 0.015x_2\tilde{x}_1\tilde{x}_2^2 + 0.015x_2\tilde{x}_2^3 + 0.208\tilde{x}_1^4 + 0.028\tilde{x}_1^3\tilde{x}_2 + 0.095\tilde{x}_1^2\tilde{x}_2^2 + 0.076\tilde{x}_1\tilde{x}_2^3 + 0.111\tilde{x}_2^4$$

$$\Pi^{41}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.001x_2^4 + 0.847x_2^3\tilde{x}_1 - 0.029x_2^3\tilde{x}_2 + 0.01x_2^2\tilde{x}_1^2 - 0.064x_2^2\tilde{x}_1\tilde{x}_2 - 0.004x_2^2\tilde{x}_2^2 + 0.769x_2\tilde{x}_1^3 + 0.002x_2\tilde{x}_1^2\tilde{x}_2 + 0.194x_2\tilde{x}_1\tilde{x}_2^2 + 0.011x_2\tilde{x}_2^3 - 0.001\tilde{x}_1^4 - 0.442\tilde{x}_1^3\tilde{x}_2 - 0.012\tilde{x}_1^2\tilde{x}_2^2 - 0.38\tilde{x}_1\tilde{x}_2^3 - 0.001\tilde{x}_2^4$$

$$\Pi^{12}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -0.006x_2^4 + 0.355x_2^3\tilde{x}_1 + 0.025x_2^3\tilde{x}_2 + 0.035x_2^2\tilde{x}_1^2 + 0.468x_2^2\tilde{x}_1\tilde{x}_2 + 0.033x_2^2\tilde{x}_2^2 + 0.347x_2\tilde{x}_1^3 + 0.034x_2\tilde{x}_1^2\tilde{x}_2 + 0.262x_2\tilde{x}_1\tilde{x}_2^2 + 0.017x_2\tilde{x}_2^3 + 0.031\tilde{x}_1^4 + 0.789\tilde{x}_1^3\tilde{x}_2 + 0.045\tilde{x}_1^2\tilde{x}_2^2 + 0.605\tilde{x}_1\tilde{x}_2^3 + 0.016\tilde{x}_2^4$$

$$\Pi^{22}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 4.686x_2^4 + 0.025x_2^3\tilde{x}_1 + 0.746x_2^3\tilde{x}_2 + 2.908x_2^2\tilde{x}_1^2 + 0.034x_2^2\tilde{x}_1\tilde{x}_2 + 3.501x_2^2\tilde{x}_2^2 + 0.029x_2\tilde{x}_1^3 + 0.254x_2\tilde{x}_1^2\tilde{x}_2 + 0.02x_2\tilde{x}_1\tilde{x}_2^2 + 0.674x_2\tilde{x}_2^3 + 4.173\tilde{x}_1^4 + 0.038\tilde{x}_1^3\tilde{x}_2 + 3.145\tilde{x}_1^2\tilde{x}_2^2 + 0.018\tilde{x}_1\tilde{x}_2^3 + 5.098\tilde{x}_2^4$$

$$\Pi^{32}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -0.115x_2^4 + 0.037x_2^3\tilde{x}_1 - 0.021x_2^3\tilde{x}_2 - 0.049x_2^2\tilde{x}_1^2 + 0.051x_2^2\tilde{x}_1\tilde{x}_2 - 0.039x_2^2\tilde{x}_2^2 + 0.016x_2\tilde{x}_1^3 + 0.015x_2\tilde{x}_1^2\tilde{x}_2 + 0.013x_2\tilde{x}_1\tilde{x}_2^2 + 0.024x_2\tilde{x}_2^3 + 0.024\tilde{x}_1^4 + 0.06\tilde{x}_1^3\tilde{x}_2 + 0.083\tilde{x}_1^2\tilde{x}_2^2 + 0.08\tilde{x}_1\tilde{x}_2^3 + 0.223\tilde{x}_2^4$$

$$\Pi^{42}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.385x_2^4 - 0.029x_2^3\tilde{x}_1 + 0.611x_2^3\tilde{x}_2 - 0.033x_2^2\tilde{x}_1^2 - 0.004x_2^2\tilde{x}_1\tilde{x}_2 - 0.174x_2^2\tilde{x}_2^2 + 0.001x_2\tilde{x}_1^3 + 0.201x_2\tilde{x}_1^2\tilde{x}_2 + 0.014x_2\tilde{x}_1\tilde{x}_2^2 + 0.233x_2\tilde{x}_2^3 - 0.32\tilde{x}_1^4 - 0.01\tilde{x}_1^3\tilde{x}_2 - 0.391\tilde{x}_1^2\tilde{x}_2^2 - 0.001\tilde{x}_1\tilde{x}_2^3 - 0.931\tilde{x}_2^4$$

$$\begin{aligned}
\Pi^{13}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.314x_2^4 - 0.008x_2^3\tilde{x}_1 + 0.035x_2^3\tilde{x}_2 + 0.214x_2^2\tilde{x}_1^2 - 0.049x_2^2\tilde{x}_1\tilde{x}_2 + 0.079x_2^2\tilde{x}_2^2 + \\
& 0.02x_2\tilde{x}_1^3 + 0.015x_2\tilde{x}_1^2\tilde{x}_2 + 0.015x_2\tilde{x}_1\tilde{x}_2^2 + 0.015x_2\tilde{x}_2^3 + 0.208\tilde{x}_1^4 + 0.028\tilde{x}_1^3\tilde{x}_2 + \\
& 0.095\tilde{x}_1^2\tilde{x}_2^2 + 0.076\tilde{x}_1\tilde{x}_2^3 + 0.111\tilde{x}_2^4 \\
\Pi^{23}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -0.115x_2^4 + 0.037x_2^3\tilde{x}_1 - 0.021x_2^3\tilde{x}_2 - 0.049x_2^2\tilde{x}_1^2 + 0.051x_2^2\tilde{x}_1\tilde{x}_2 - 0.039x_2^2\tilde{x}_2^2 + \\
& 0.016x_2\tilde{x}_1^3 + 0.015x_2\tilde{x}_1^2\tilde{x}_2 + 0.013x_2\tilde{x}_1\tilde{x}_2^2 + 0.024x_2\tilde{x}_2^3 + 0.024\tilde{x}_1^4 + 0.06\tilde{x}_1^3\tilde{x}_2 + \\
& 0.083\tilde{x}_1^2\tilde{x}_2^2 + 0.08\tilde{x}_1\tilde{x}_2^3 + 0.223\tilde{x}_2^4 \\
\Pi^{33}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 6.238x_2^4 - 0.008x_2^3\tilde{x}_1 - 0.018x_2^3\tilde{x}_2 + 4.202x_2^2\tilde{x}_1^2 - 0.025x_2^2\tilde{x}_1\tilde{x}_2 + 3.967x_2^2\tilde{x}_2^2 + \\
& 0.012x_2\tilde{x}_1^3 + 0.031x_2\tilde{x}_1^2\tilde{x}_2 + 0.005x_2\tilde{x}_1\tilde{x}_2^2 - 0.003x_2\tilde{x}_2^3 + 6.034\tilde{x}_1^4 - 0.007\tilde{x}_1^3\tilde{x}_2 + \\
& 4.016\tilde{x}_1^2\tilde{x}_2^2 - 0.008\tilde{x}_1\tilde{x}_2^3 + 5.912\tilde{x}_2^4 \\
\Pi^{43}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.034x_2^4 + 0.259x_2^3\tilde{x}_1 - 0.106x_2^3\tilde{x}_2 + 0.04x_2^2\tilde{x}_1^2 - 0.016x_2^2\tilde{x}_1\tilde{x}_2 + 0.017x_2^2\tilde{x}_2^2 + \\
& 0.146x_2\tilde{x}_1^3 - 0.062x_2\tilde{x}_1^2\tilde{x}_2 + 0.039x_2\tilde{x}_1\tilde{x}_2^2 - 0.05x_2\tilde{x}_2^3 - 0.006\tilde{x}_1^4 - 0.044\tilde{x}_1^3\tilde{x}_2 - \\
& 0.068\tilde{x}_1^2\tilde{x}_2^2 - 0.065\tilde{x}_1\tilde{x}_2^3 - 0.199\tilde{x}_2^4 \\
\Pi^{14}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.001x_2^4 + 0.847x_2^3\tilde{x}_1 - 0.029x_2^3\tilde{x}_2 + 0.01x_2^2\tilde{x}_1^2 - 0.064x_2^2\tilde{x}_1\tilde{x}_2 - 0.004x_2^2\tilde{x}_2^2 + \\
& 0.769x_2\tilde{x}_1^3 + 0.002x_2\tilde{x}_1^2\tilde{x}_2 + 0.194x_2\tilde{x}_1\tilde{x}_2^2 + 0.011x_2\tilde{x}_2^3 - 0.001\tilde{x}_1^4 - 0.442\tilde{x}_1^3\tilde{x}_2 - \\
& 0.012\tilde{x}_1^2\tilde{x}_2^2 - 0.38\tilde{x}_1\tilde{x}_2^3 - 0.001\tilde{x}_2^4 \\
\Pi^{24}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.385x_2^4 - 0.029x_2^3\tilde{x}_1 + 0.611x_2^3\tilde{x}_2 - 0.033x_2^2\tilde{x}_1^2 - 0.004x_2^2\tilde{x}_1\tilde{x}_2 - 0.174x_2^2\tilde{x}_2^2 + \\
& 0.001x_2\tilde{x}_1^3 + 0.201x_2\tilde{x}_1^2\tilde{x}_2 + 0.014x_2\tilde{x}_1\tilde{x}_2^2 + 0.233x_2\tilde{x}_2^3 - 0.32\tilde{x}_1^4 - 0.01\tilde{x}_1^3\tilde{x}_2 - \\
& 0.391\tilde{x}_1^2\tilde{x}_2^2 - 0.001\tilde{x}_1\tilde{x}_2^3 - 0.931\tilde{x}_2^4 \\
\Pi^{34}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.034x_2^4 + 0.259x_2^3\tilde{x}_1 - 0.106x_2^3\tilde{x}_2 + 0.04x_2^2\tilde{x}_1^2 - 0.016x_2^2\tilde{x}_1\tilde{x}_2 + 0.017x_2^2\tilde{x}_2^2 + \\
& 0.146x_2\tilde{x}_1^3 - 0.062x_2\tilde{x}_1^2\tilde{x}_2 + 0.039x_2\tilde{x}_1\tilde{x}_2^2 - 0.05x_2\tilde{x}_2^3 - 0.006\tilde{x}_1^4 - 0.044\tilde{x}_1^3\tilde{x}_2 - \\
& 0.068\tilde{x}_1^2\tilde{x}_2^2 - 0.065\tilde{x}_1\tilde{x}_2^3 - 0.199\tilde{x}_2^4 \\
\Pi^{44}_{112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 5.958x_2^4 + 0.029x_2^3\tilde{x}_1 - 0.264x_2^3\tilde{x}_2 + 3.423x_2^2\tilde{x}_1^2 - 0.028x_2^2\tilde{x}_1\tilde{x}_2 + 3.286x_2^2\tilde{x}_2^2 + \\
& 0.008x_2\tilde{x}_1^3 - 0.276x_2\tilde{x}_1^2\tilde{x}_2 - 0.019x_2\tilde{x}_1\tilde{x}_2^2 - 0.43x_2\tilde{x}_2^3 + 4.478\tilde{x}_1^4 + 0.008\tilde{x}_1^3\tilde{x}_2 + \\
& 3.052\tilde{x}_1^2\tilde{x}_2^2 + 0.011\tilde{x}_1\tilde{x}_2^3 + 4.646\tilde{x}_2^4
\end{aligned}$$

$$\Pi^{11}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 3.882x_2^4 + 0.017x_2^3\tilde{x}_1 + 0.074x_2^3\tilde{x}_2 + 2.794x_2^2\tilde{x}_1^2 + 0.004x_2^2\tilde{x}_1\tilde{x}_2 + 2.575x_2^2\tilde{x}_2^2 + 0.012x_2\tilde{x}_1^3 + 0.049x_2\tilde{x}_1^2\tilde{x}_2 + 0.008x_2\tilde{x}_1\tilde{x}_2^2 + 0.029x_2\tilde{x}_2^3 + 4.466\tilde{x}_1^4 + 0.003\tilde{x}_1^3\tilde{x}_2 + 2.536\tilde{x}_1^2\tilde{x}_2^2 + 0.013\tilde{x}_1\tilde{x}_2^3 + 3.66\tilde{x}_2^4$$

$$\Pi^{21}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.021x_2^4 + 0.076x_2^3\tilde{x}_1 + 0.028x_2^3\tilde{x}_2 + 0.002x_2^2\tilde{x}_1^2 + 0.069x_2^2\tilde{x}_1\tilde{x}_2 + 0.034x_2^2\tilde{x}_2^2 + 0.04x_2\tilde{x}_1^3 + 0.008x_2\tilde{x}_1^2\tilde{x}_2 + 0.036x_2\tilde{x}_1\tilde{x}_2^2 + 0.032x_2\tilde{x}_2^3 + 0.002\tilde{x}_1^4 + 0.04\tilde{x}_1^3\tilde{x}_2 + 0.012\tilde{x}_1^2\tilde{x}_2^2 + 0.079\tilde{x}_1\tilde{x}_2^3 + 0.051\tilde{x}_2^4$$

$$\Pi^{31}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.2x_2^4 + 0.008x_2^3\tilde{x}_1 + 0.031x_2^3\tilde{x}_2 + 0.252x_2^2\tilde{x}_1^2 + 0.004x_2^2\tilde{x}_1\tilde{x}_2 + 0.044x_2^2\tilde{x}_2^2 + 0.006x_2\tilde{x}_1^3 + 0.019x_2\tilde{x}_1^2\tilde{x}_2 - 0.004x_2\tilde{x}_1\tilde{x}_2^2 - 0.001x_2\tilde{x}_2^3 + 0.023\tilde{x}_1^4 + 0.001\tilde{x}_1^3\tilde{x}_2 - 0.005\tilde{x}_1^2\tilde{x}_2^2 - 0.011\tilde{x}_1\tilde{x}_2^3 - 0.026\tilde{x}_2^4$$

$$\Pi^{41}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.011x_2^4 + 0.275x_2^3\tilde{x}_1 - 0.002x_2^3\tilde{x}_2 + 0.016x_2^2\tilde{x}_1^2 + 0.019x_2^2\tilde{x}_1\tilde{x}_2 - 0.003x_2^2\tilde{x}_2^2 + 0.223x_2\tilde{x}_1^3 - 0.005x_2\tilde{x}_1^2\tilde{x}_2 + 0.026x_2\tilde{x}_1\tilde{x}_2^2 + 0.008\tilde{x}_1^4 + 0.001\tilde{x}_1^3\tilde{x}_2 - 0.005\tilde{x}_1^2\tilde{x}_2^2 - 0.049\tilde{x}_1\tilde{x}_2^3 - 0.025\tilde{x}_2^4$$

$$\Pi^{12}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.021x_2^4 + 0.076x_2^3\tilde{x}_1 + 0.028x_2^3\tilde{x}_2 + 0.002x_2^2\tilde{x}_1^2 + 0.069x_2^2\tilde{x}_1\tilde{x}_2 + 0.034x_2^2\tilde{x}_2^2 + 0.04x_2\tilde{x}_1^3 + 0.008x_2\tilde{x}_1^2\tilde{x}_2 + 0.036x_2\tilde{x}_1\tilde{x}_2^2 + 0.032x_2\tilde{x}_2^3 + 0.002\tilde{x}_1^4 + 0.04\tilde{x}_1^3\tilde{x}_2 + 0.012\tilde{x}_1^2\tilde{x}_2^2 + 0.079\tilde{x}_1\tilde{x}_2^3 + 0.051\tilde{x}_2^4$$

$$\Pi^{22}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 3.948x_2^4 + 0.028x_2^3\tilde{x}_1 + 0.251x_2^3\tilde{x}_2 + 2.571x_2^2\tilde{x}_1^2 + 0.037x_2^2\tilde{x}_1\tilde{x}_2 + 2.834x_2^2\tilde{x}_2^2 + 0.006x_2\tilde{x}_1^3 + 0.029x_2\tilde{x}_1^2\tilde{x}_2 + 0.038x_2\tilde{x}_1\tilde{x}_2^2 + 0.239x_2\tilde{x}_2^3 + 3.627\tilde{x}_1^4 + 0.01\tilde{x}_1^3\tilde{x}_2 + 2.591\tilde{x}_1^2\tilde{x}_2^2 + 0.063\tilde{x}_1\tilde{x}_2^3 + 4.056\tilde{x}_2^4$$

$$\Pi^{32}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -0.021x_2^4 + 0.032x_2^3\tilde{x}_1 - 0.002x_2^3\tilde{x}_2 + 0.005x_2^2\tilde{x}_1^2 + 0.042x_2^2\tilde{x}_1\tilde{x}_2 - 0.003x_2^2\tilde{x}_2^2 + 0.018x_2\tilde{x}_1^3 - 0.004x_2\tilde{x}_1^2\tilde{x}_2 - 0.003x_2\tilde{x}_1\tilde{x}_2^2 - 0.004x_2\tilde{x}_2^3 + 0.001\tilde{x}_1^4 - 0.008\tilde{x}_1^3\tilde{x}_2 - 0.012\tilde{x}_1^2\tilde{x}_2^2 - 0.046\tilde{x}_1\tilde{x}_2^3 + 0.005\tilde{x}_2^4$$

$$\Pi^{42}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.148x_2^4 - 0.002x_2^3\tilde{x}_1 + 0.179x_2^3\tilde{x}_2 + 0.023x_2^2\tilde{x}_1^2 - 0.003x_2^2\tilde{x}_1\tilde{x}_2 - 0.035x_2^2\tilde{x}_2^2 - 0.004x_2\tilde{x}_1^3 + 0.029x_2\tilde{x}_1^2\tilde{x}_2 + 0.048x_2\tilde{x}_2^3 + 0.01\tilde{x}_1^4 - 0.004\tilde{x}_1^3\tilde{x}_2 - 0.039\tilde{x}_1^2\tilde{x}_2^2 - 0.031\tilde{x}_1\tilde{x}_2^3 - 0.253\tilde{x}_2^4$$

$$\Pi^{13}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.2x_2^4 + 0.008x_2^3\tilde{x}_1 + 0.031x_2^3\tilde{x}_2 + 0.252x_2^2\tilde{x}_1^2 + 0.004x_2^2\tilde{x}_1\tilde{x}_2 + 0.044x_2^2\tilde{x}_2^2 + 0.006x_2\tilde{x}_1^3 + 0.019x_2\tilde{x}_1^2\tilde{x}_2 - 0.004x_2\tilde{x}_1\tilde{x}_2^2 - 0.001x_2\tilde{x}_2^3 + 0.023\tilde{x}_1^4 + 0.001\tilde{x}_1^3\tilde{x}_2 - 0.005\tilde{x}_1^2\tilde{x}_2^2 - 0.011\tilde{x}_1\tilde{x}_2^3 - 0.026\tilde{x}_2^4$$

$$\Pi^{23}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -0.021x_2^4 + 0.032x_2^3\tilde{x}_1 - 0.002x_2^3\tilde{x}_2 + 0.005x_2^2\tilde{x}_1^2 + 0.042x_2^2\tilde{x}_1\tilde{x}_2 - 0.003x_2^2\tilde{x}_2^2 + 0.018x_2\tilde{x}_1^3 - 0.004x_2\tilde{x}_1^2\tilde{x}_2 - 0.003x_2\tilde{x}_1\tilde{x}_2^2 - 0.004x_2\tilde{x}_2^3 + 0.001\tilde{x}_1^4 - 0.008\tilde{x}_1^3\tilde{x}_2 - 0.012\tilde{x}_1^2\tilde{x}_2^2 - 0.046\tilde{x}_1\tilde{x}_2^3 + 0.005\tilde{x}_2^4$$

$$\Pi^{33}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 4.559x_2^4 + 0.004x_2^3\tilde{x}_1 - 0.005x_2^3\tilde{x}_2 + 2.989x_2^2\tilde{x}_1^2 - 0.007x_2^2\tilde{x}_1\tilde{x}_2 + 2.962x_2^2\tilde{x}_2^2 + 0.002x_2\tilde{x}_1^3 + 0.017x_2\tilde{x}_1^2\tilde{x}_2 + 0.005x_2\tilde{x}_1\tilde{x}_2^2 - 0.003x_2\tilde{x}_2^3 + 4.536\tilde{x}_1^4 - 0.007\tilde{x}_1^3\tilde{x}_2 + 3.069\tilde{x}_1^2\tilde{x}_2^2 + 0.008\tilde{x}_1\tilde{x}_2^3 + 4.399\tilde{x}_2^4$$

$$\Pi^{43}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.008x_2^4 + 0.191x_2^3\tilde{x}_1 - 0.023x_2^3\tilde{x}_2 + 0.005x_2^2\tilde{x}_1^2 - 0.008x_2^2\tilde{x}_1\tilde{x}_2 + 0.001x_2^2\tilde{x}_2^2 + 0.19x_2\tilde{x}_1^3 + 0.008x_2\tilde{x}_1^2\tilde{x}_2 + 0.039x_2\tilde{x}_1\tilde{x}_2^2 + 0.001x_2\tilde{x}_2^3 + 0.004\tilde{x}_1^4 + 0.026\tilde{x}_1^3\tilde{x}_2 + 0.008\tilde{x}_1^2\tilde{x}_2^2 + 0.045\tilde{x}_1\tilde{x}_2^3 - 0.008\tilde{x}_2^4$$

$$\Pi^{14}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.011x_2^4 + 0.275x_2^3\tilde{x}_1 - 0.002x_2^3\tilde{x}_2 + 0.016x_2^2\tilde{x}_1^2 + 0.019x_2^2\tilde{x}_1\tilde{x}_2 - 0.003x_2^2\tilde{x}_2^2 + 0.223x_2\tilde{x}_1^3 - 0.005x_2\tilde{x}_1^2\tilde{x}_2 + 0.026x_2\tilde{x}_1\tilde{x}_2^2 + 0.008\tilde{x}_1^4 + 0.001\tilde{x}_1^3\tilde{x}_2 - 0.005\tilde{x}_1^2\tilde{x}_2^2 - 0.049\tilde{x}_1\tilde{x}_2^3 - 0.025\tilde{x}_2^4$$

$$\Pi^{24}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.148x_2^4 - 0.002x_2^3\tilde{x}_1 + 0.179x_2^3\tilde{x}_2 + 0.023x_2^2\tilde{x}_1^2 - 0.003x_2^2\tilde{x}_1\tilde{x}_2 - 0.035x_2^2\tilde{x}_2^2 - 0.004x_2\tilde{x}_1^3 + 0.029x_2\tilde{x}_1^2\tilde{x}_2 + 0.048x_2\tilde{x}_2^3 + 0.01\tilde{x}_1^4 - 0.004\tilde{x}_1^3\tilde{x}_2 - 0.039\tilde{x}_1^2\tilde{x}_2^2 - 0.031\tilde{x}_1\tilde{x}_2^3 - 0.253\tilde{x}_2^4$$

$$\Pi^{34}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.008x_2^4 + 0.191x_2^3\tilde{x}_1 - 0.023x_2^3\tilde{x}_2 + 0.005x_2^2\tilde{x}_1^2 - 0.008x_2^2\tilde{x}_1\tilde{x}_2 + 0.001x_2^2\tilde{x}_2^2 + 0.19x_2\tilde{x}_1^3 + 0.008x_2\tilde{x}_1^2\tilde{x}_2 + 0.039x_2\tilde{x}_1\tilde{x}_2^2 + 0.001x_2\tilde{x}_2^3 + 0.004\tilde{x}_1^4 + 0.026\tilde{x}_1^3\tilde{x}_2 + 0.008\tilde{x}_1^2\tilde{x}_2^2 + 0.045\tilde{x}_1\tilde{x}_2^3 - 0.008\tilde{x}_2^4$$

$$\Pi^{44}_{121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 4.342x_2^4 + 0.014x_2^3\tilde{x}_1 - 0.023x_2^3\tilde{x}_2 + 2.779x_2^2\tilde{x}_1^2 + 0.001x_2^2\tilde{x}_1\tilde{x}_2 + 2.727x_2^2\tilde{x}_2^2 + 0.018x_2\tilde{x}_1^3 - 0.012x_2\tilde{x}_1^2\tilde{x}_2 - 0.005x_2\tilde{x}_1\tilde{x}_2^2 - 0.094x_2\tilde{x}_2^3 + 3.767\tilde{x}_1^4 + 2.582\tilde{x}_1^2\tilde{x}_2^2 + 0.03\tilde{x}_1\tilde{x}_2^3 + 3.884\tilde{x}_2^4$$

$$\Pi^{11}_{122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 3.904x_2^4 + 0.016x_2^3\tilde{x}_1 + 0.116x_2^3\tilde{x}_2 + 2.893x_2^2\tilde{x}_1^2 + 0.015x_2^2\tilde{x}_1\tilde{x}_2 + 2.644x_2^2\tilde{x}_2^2 + 0.014x_2\tilde{x}_1^3 + 0.142x_2\tilde{x}_1^2\tilde{x}_2 + 0.013x_2\tilde{x}_1\tilde{x}_2^2 + 0.07x_2\tilde{x}_2^3 + 4.619\tilde{x}_1^4 + 0.014\tilde{x}_1^3\tilde{x}_2 + 2.755\tilde{x}_1^2\tilde{x}_2^2 + 0.014\tilde{x}_1\tilde{x}_2^3 + 3.729\tilde{x}_2^4$$

$$\Pi^{21}_{122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -0.005x_2^4 + 0.119x_2^3\tilde{x}_1 + 0.01x_2^3\tilde{x}_2 + 0.015x_2^2\tilde{x}_1^2 + 0.145x_2^2\tilde{x}_1\tilde{x}_2 + 0.013x_2^2\tilde{x}_2^2 + 0.12x_2\tilde{x}_1^3 + 0.013x_2\tilde{x}_1^2\tilde{x}_2 + 0.086x_2\tilde{x}_1\tilde{x}_2^2 + 0.005x_2\tilde{x}_2^3 + 0.011\tilde{x}_1^4 + 0.237\tilde{x}_1^3\tilde{x}_2 + 0.016\tilde{x}_1^2\tilde{x}_2^2 + 0.17\tilde{x}_1\tilde{x}_2^3$$

$$\Pi^{31}_{122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.137x_2^4 - 0.004x_2^3\tilde{x}_1 + 0.018x_2^3\tilde{x}_2 + 0.104x_2^2\tilde{x}_1^2 - 0.023x_2^2\tilde{x}_1\tilde{x}_2 + 0.039x_2^2\tilde{x}_2^2 + 0.007x_2\tilde{x}_1^3 + 0.007x_2\tilde{x}_1^2\tilde{x}_2 + 0.006x_2\tilde{x}_1\tilde{x}_2^2 + 0.008x_2\tilde{x}_2^3 + 0.135\tilde{x}_1^4 + 0.011\tilde{x}_1^3\tilde{x}_2 + 0.062\tilde{x}_1^2\tilde{x}_2^2 + 0.032\tilde{x}_1\tilde{x}_2^3 + 0.06\tilde{x}_2^4$$

$$\Pi^{41}_{122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.265x_2^3\tilde{x}_1 - 0.015x_2^3\tilde{x}_2 + 0.004x_2^2\tilde{x}_1^2 - 0.006x_2^2\tilde{x}_1\tilde{x}_2 - 0.002x_2^2\tilde{x}_2^2 + 0.234x_2\tilde{x}_1^3 + 0.002x_2\tilde{x}_1^2\tilde{x}_2 + 0.052x_2\tilde{x}_1\tilde{x}_2^2 + 0.006x_2\tilde{x}_2^3 + 0.001\tilde{x}_1^4 - 0.117\tilde{x}_1^3\tilde{x}_2 - 0.003\tilde{x}_1^2\tilde{x}_2^2 - 0.097\tilde{x}_1\tilde{x}_2^3 + 0.004\tilde{x}_2^4$$

$$\Pi^{12}_{122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -0.005x_2^4 + 0.119x_2^3\tilde{x}_1 + 0.01x_2^3\tilde{x}_2 + 0.015x_2^2\tilde{x}_1^2 + 0.145x_2^2\tilde{x}_1\tilde{x}_2 + 0.013x_2^2\tilde{x}_2^2 + 0.12x_2\tilde{x}_1^3 + 0.013x_2\tilde{x}_1^2\tilde{x}_2 + 0.086x_2\tilde{x}_1\tilde{x}_2^2 + 0.005x_2\tilde{x}_2^3 + 0.011\tilde{x}_1^4 + 0.237\tilde{x}_1^3\tilde{x}_2 + 0.016\tilde{x}_1^2\tilde{x}_2^2 + 0.17\tilde{x}_1\tilde{x}_2^3$$

$$\Pi^{22}_{122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 3.944x_2^4 + 0.01x_2^3\tilde{x}_1 + 0.243x_2^3\tilde{x}_2 + 2.642x_2^2\tilde{x}_1^2 + 0.014x_2^2\tilde{x}_1\tilde{x}_2 + 2.818x_2^2\tilde{x}_2^2 + 0.011x_2\tilde{x}_1^3 + 0.083x_2\tilde{x}_1^2\tilde{x}_2 + 0.006x_2\tilde{x}_1\tilde{x}_2^2 + 0.218x_2\tilde{x}_2^3 + 3.782\tilde{x}_1^4 + 0.014\tilde{x}_1^3\tilde{x}_2 + 2.697\tilde{x}_1^2\tilde{x}_2^2 + 0.001\tilde{x}_1\tilde{x}_2^3 + 4.024\tilde{x}_2^4$$

$$\Pi^{32}_{122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -0.045x_2^4 + 0.019x_2^3\tilde{x}_1 - 0.011x_2^3\tilde{x}_2 - 0.021x_2^2\tilde{x}_1^2 + 0.034x_2^2\tilde{x}_1\tilde{x}_2 - 0.018x_2^2\tilde{x}_2^2 + 0.008x_2\tilde{x}_1^3 + 0.006x_2\tilde{x}_1^2\tilde{x}_2 + 0.01x_2\tilde{x}_1\tilde{x}_2^2 + 0.007x_2\tilde{x}_2^3 + 0.008\tilde{x}_1^4 + 0.048\tilde{x}_1^3\tilde{x}_2 + 0.034\tilde{x}_1^2\tilde{x}_2^2 + 0.058\tilde{x}_1\tilde{x}_2^3 + 0.09\tilde{x}_2^4$$

$$\Pi^{42}_{122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.149x_2^4 - 0.014x_2^3\tilde{x}_1 + 0.186x_2^3\tilde{x}_2 + 0.001x_2^2\tilde{x}_1^2 - 0.002x_2^2\tilde{x}_1\tilde{x}_2 - 0.03x_2^2\tilde{x}_2^2 + 0.002x_2\tilde{x}_1^3 + 0.057x_2\tilde{x}_1^2\tilde{x}_2 + 0.008x_2\tilde{x}_1\tilde{x}_2^2 + 0.055x_2\tilde{x}_2^3 - 0.082\tilde{x}_1^4 - 0.003\tilde{x}_1^3\tilde{x}_2 - 0.096\tilde{x}_1^2\tilde{x}_2^2 + 0.004\tilde{x}_1\tilde{x}_2^3 - 0.24\tilde{x}_2^4$$

$$\Pi_{122}^{13}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.137x_2^4 - 0.004x_2^3\tilde{x}_1 + 0.018x_2^3\tilde{x}_2 + 0.104x_2^2\tilde{x}_1^2 - 0.023x_2^2\tilde{x}_1\tilde{x}_2 + 0.039x_2^2\tilde{x}_2^2 + 0.007x_2\tilde{x}_1^3 + 0.007x_2\tilde{x}_1^2\tilde{x}_2 + 0.006x_2\tilde{x}_1\tilde{x}_2^2 + 0.008x_2\tilde{x}_2^3 + 0.135\tilde{x}_1^4 + 0.011\tilde{x}_1^3\tilde{x}_2 + 0.062\tilde{x}_1^2\tilde{x}_2^2 + 0.032\tilde{x}_1\tilde{x}_2^3 + 0.06\tilde{x}_2^4$$

$$\Pi_{122}^{23}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -0.045x_2^4 + 0.019x_2^3\tilde{x}_1 - 0.011x_2^3\tilde{x}_2 - 0.021x_2^2\tilde{x}_1^2 + 0.034x_2^2\tilde{x}_1\tilde{x}_2 - 0.018x_2^2\tilde{x}_2^2 + 0.008x_2\tilde{x}_1^3 + 0.006x_2\tilde{x}_1^2\tilde{x}_2 + 0.01x_2\tilde{x}_1\tilde{x}_2^2 + 0.007x_2\tilde{x}_2^3 + 0.008\tilde{x}_1^4 + 0.048\tilde{x}_1^3\tilde{x}_2 + 0.034\tilde{x}_1^2\tilde{x}_2^2 + 0.058\tilde{x}_1\tilde{x}_2^3 + 0.09\tilde{x}_2^4$$

$$\Pi_{122}^{33}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 4.4x_2^4 - 0.003x_2^3\tilde{x}_1 - 0.012x_2^3\tilde{x}_2 + 3.052x_2^2\tilde{x}_1^2 - 0.011x_2^2\tilde{x}_1\tilde{x}_2 + 2.962x_2^2\tilde{x}_2^2 + 0.007x_2\tilde{x}_1^3 + 0.013x_2\tilde{x}_1^2\tilde{x}_2 + 0.003x_2\tilde{x}_1\tilde{x}_2^2 - 0.002x_2\tilde{x}_2^3 + 4.333\tilde{x}_1^4 + 2.988\tilde{x}_1^2\tilde{x}_2^2 - 0.001\tilde{x}_1\tilde{x}_2^3 + 4.316\tilde{x}_2^4$$

$$\Pi_{122}^{43}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.014x_2^4 + 0.127x_2^3\tilde{x}_1 - 0.042x_2^3\tilde{x}_2 + 0.017x_2^2\tilde{x}_1^2 - 0.015x_2^2\tilde{x}_1\tilde{x}_2 + 0.007x_2^2\tilde{x}_2^2 + 0.077x_2\tilde{x}_1^3 - 0.027x_2\tilde{x}_1^2\tilde{x}_2 + 0.024x_2\tilde{x}_1\tilde{x}_2^2 - 0.019x_2\tilde{x}_2^3 - 0.003\tilde{x}_1^4 - 0.04\tilde{x}_1^3\tilde{x}_2 - 0.029\tilde{x}_1^2\tilde{x}_2^2 - 0.048\tilde{x}_1\tilde{x}_2^3 - 0.083\tilde{x}_2^4$$

$$\Pi_{122}^{14}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.265x_2^3\tilde{x}_1 - 0.015x_2^3\tilde{x}_2 + 0.004x_2^2\tilde{x}_1^2 - 0.006x_2^2\tilde{x}_1\tilde{x}_2 - 0.002x_2^2\tilde{x}_2^2 + 0.234x_2\tilde{x}_1^3 + 0.002x_2\tilde{x}_1^2\tilde{x}_2 + 0.052x_2\tilde{x}_1\tilde{x}_2^2 + 0.006x_2\tilde{x}_2^3 + 0.001\tilde{x}_1^4 - 0.117\tilde{x}_1^3\tilde{x}_2 - 0.003\tilde{x}_1^2\tilde{x}_2^2 - 0.097\tilde{x}_1\tilde{x}_2^3 + 0.004\tilde{x}_2^4$$

$$\Pi_{122}^{24}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.149x_2^4 - 0.014x_2^3\tilde{x}_1 + 0.186x_2^3\tilde{x}_2 + 0.001x_2^2\tilde{x}_1^2 - 0.002x_2^2\tilde{x}_1\tilde{x}_2 - 0.03x_2^2\tilde{x}_2^2 + 0.002x_2\tilde{x}_1^3 + 0.057x_2\tilde{x}_1^2\tilde{x}_2 + 0.008x_2\tilde{x}_1\tilde{x}_2^2 + 0.055x_2\tilde{x}_2^3 - 0.082\tilde{x}_1^4 - 0.003\tilde{x}_1^3\tilde{x}_2 - 0.096\tilde{x}_1^2\tilde{x}_2^2 + 0.004\tilde{x}_1\tilde{x}_2^3 - 0.24\tilde{x}_2^4$$

$$\Pi_{122}^{34}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.014x_2^4 + 0.127x_2^3\tilde{x}_1 - 0.042x_2^3\tilde{x}_2 + 0.017x_2^2\tilde{x}_1^2 - 0.015x_2^2\tilde{x}_1\tilde{x}_2 + 0.007x_2^2\tilde{x}_2^2 + 0.077x_2\tilde{x}_1^3 - 0.027x_2\tilde{x}_1^2\tilde{x}_2 + 0.024x_2\tilde{x}_1\tilde{x}_2^2 - 0.019x_2\tilde{x}_2^3 - 0.003\tilde{x}_1^4 - 0.04\tilde{x}_1^3\tilde{x}_2 - 0.029\tilde{x}_1^2\tilde{x}_2^2 - 0.048\tilde{x}_1\tilde{x}_2^3 - 0.083\tilde{x}_2^4$$

$$\Pi_{122}^{44}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 4.296x_2^4 + 0.014x_2^3\tilde{x}_1 - 0.031x_2^3\tilde{x}_2 + 2.792x_2^2\tilde{x}_1^2 - 0.014x_2^2\tilde{x}_1\tilde{x}_2 + 2.73x_2^2\tilde{x}_2^2 + 0.002x_2\tilde{x}_1^3 - 0.068x_2\tilde{x}_1^2\tilde{x}_2 - 0.01x_2\tilde{x}_1\tilde{x}_2^2 - 0.097x_2\tilde{x}_2^3 + 3.862\tilde{x}_1^4 + 0.004\tilde{x}_1^3\tilde{x}_2 + 2.66\tilde{x}_1^2\tilde{x}_2^2 + 0.001\tilde{x}_1\tilde{x}_2^3 + 3.876\tilde{x}_2^4$$

$$\Pi^{11}_{211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 3.882x_2^4 + 0.017x_2^3\tilde{x}_1 + 0.074x_2^3\tilde{x}_2 + 2.794x_2^2\tilde{x}_1^2 + 0.004x_2^2\tilde{x}_1\tilde{x}_2 + 2.575x_2^2\tilde{x}_2^2 + 0.012x_2\tilde{x}_1^3 + 0.049x_2\tilde{x}_1^2\tilde{x}_2 + 0.008x_2\tilde{x}_1\tilde{x}_2^2 + 0.029x_2\tilde{x}_2^3 + 4.466\tilde{x}_1^4 + 0.003\tilde{x}_1^3\tilde{x}_2 + 2.536\tilde{x}_1^2\tilde{x}_2^2 + 0.013\tilde{x}_1\tilde{x}_2^3 + 3.66\tilde{x}_2^4$$

$$\Pi^{21}_{211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.021x_2^4 + 0.076x_2^3\tilde{x}_1 + 0.028x_2^3\tilde{x}_2 + 0.002x_2^2\tilde{x}_1^2 + 0.069x_2^2\tilde{x}_1\tilde{x}_2 + 0.034x_2^2\tilde{x}_2^2 + 0.04x_2\tilde{x}_1^3 + 0.008x_2\tilde{x}_1^2\tilde{x}_2 + 0.036x_2\tilde{x}_1\tilde{x}_2^2 + 0.032x_2\tilde{x}_2^3 + 0.002\tilde{x}_1^4 + 0.04\tilde{x}_1^3\tilde{x}_2 + 0.012\tilde{x}_1^2\tilde{x}_2^2 + 0.079\tilde{x}_1\tilde{x}_2^3 + 0.051\tilde{x}_2^4$$

$$\Pi^{31}_{211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.2x_2^4 + 0.008x_2^3\tilde{x}_1 + 0.031x_2^3\tilde{x}_2 + 0.252x_2^2\tilde{x}_1^2 + 0.004x_2^2\tilde{x}_1\tilde{x}_2 + 0.044x_2^2\tilde{x}_2^2 + 0.006x_2\tilde{x}_1^3 + 0.019x_2\tilde{x}_1^2\tilde{x}_2 - 0.004x_2\tilde{x}_1\tilde{x}_2^2 - 0.001x_2\tilde{x}_2^3 + 0.023\tilde{x}_1^4 + 0.001\tilde{x}_1^3\tilde{x}_2 - 0.005\tilde{x}_1^2\tilde{x}_2^2 - 0.011\tilde{x}_1\tilde{x}_2^3 - 0.026\tilde{x}_2^4$$

$$\Pi^{41}_{211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.011x_2^4 + 0.275x_2^3\tilde{x}_1 - 0.002x_2^3\tilde{x}_2 + 0.016x_2^2\tilde{x}_1^2 + 0.019x_2^2\tilde{x}_1\tilde{x}_2 - 0.003x_2^2\tilde{x}_2^2 + 0.223x_2\tilde{x}_1^3 - 0.005x_2\tilde{x}_1^2\tilde{x}_2 + 0.026x_2\tilde{x}_1\tilde{x}_2^2 + 0.008\tilde{x}_1^4 + 0.001\tilde{x}_1^3\tilde{x}_2 - 0.005\tilde{x}_1^2\tilde{x}_2^2 - 0.049\tilde{x}_1\tilde{x}_2^3 - 0.025\tilde{x}_2^4$$

$$\Pi^{12}_{211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.021x_2^4 + 0.076x_2^3\tilde{x}_1 + 0.028x_2^3\tilde{x}_2 + 0.002x_2^2\tilde{x}_1^2 + 0.069x_2^2\tilde{x}_1\tilde{x}_2 + 0.034x_2^2\tilde{x}_2^2 + 0.04x_2\tilde{x}_1^3 + 0.008x_2\tilde{x}_1^2\tilde{x}_2 + 0.036x_2\tilde{x}_1\tilde{x}_2^2 + 0.032x_2\tilde{x}_2^3 + 0.002\tilde{x}_1^4 + 0.04\tilde{x}_1^3\tilde{x}_2 + 0.012\tilde{x}_1^2\tilde{x}_2^2 + 0.079\tilde{x}_1\tilde{x}_2^3 + 0.051\tilde{x}_2^4$$

$$\Pi^{22}_{211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 3.948x_2^4 + 0.028x_2^3\tilde{x}_1 + 0.251x_2^3\tilde{x}_2 + 2.571x_2^2\tilde{x}_1^2 + 0.037x_2^2\tilde{x}_1\tilde{x}_2 + 2.834x_2^2\tilde{x}_2^2 + 0.006x_2\tilde{x}_1^3 + 0.029x_2\tilde{x}_1^2\tilde{x}_2 + 0.038x_2\tilde{x}_1\tilde{x}_2^2 + 0.239x_2\tilde{x}_2^3 + 3.627\tilde{x}_1^4 + 0.01\tilde{x}_1^3\tilde{x}_2 + 2.591\tilde{x}_1^2\tilde{x}_2^2 + 0.063\tilde{x}_1\tilde{x}_2^3 + 4.056\tilde{x}_2^4$$

$$\Pi^{32}_{211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -0.021x_2^4 + 0.032x_2^3\tilde{x}_1 - 0.002x_2^3\tilde{x}_2 + 0.005x_2^2\tilde{x}_1^2 + 0.042x_2^2\tilde{x}_1\tilde{x}_2 - 0.003x_2^2\tilde{x}_2^2 + 0.018x_2\tilde{x}_1^3 - 0.004x_2\tilde{x}_1^2\tilde{x}_2 - 0.003x_2\tilde{x}_1\tilde{x}_2^2 - 0.004x_2\tilde{x}_2^3 + 0.001\tilde{x}_1^4 - 0.008\tilde{x}_1^3\tilde{x}_2 - 0.012\tilde{x}_1^2\tilde{x}_2^2 - 0.046\tilde{x}_1\tilde{x}_2^3 + 0.005\tilde{x}_2^4$$

$$\Pi^{42}_{211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.148x_2^4 - 0.002x_2^3\tilde{x}_1 + 0.179x_2^3\tilde{x}_2 + 0.023x_2^2\tilde{x}_1^2 - 0.003x_2^2\tilde{x}_1\tilde{x}_2 - 0.035x_2^2\tilde{x}_2^2 - 0.004x_2\tilde{x}_1^3 + 0.029x_2\tilde{x}_1^2\tilde{x}_2 + 0.048x_2\tilde{x}_2^3 + 0.01\tilde{x}_1^4 - 0.004\tilde{x}_1^3\tilde{x}_2 - 0.039\tilde{x}_1^2\tilde{x}_2^2 - 0.031\tilde{x}_1\tilde{x}_2^3 - 0.253\tilde{x}_2^4$$

$$
\begin{aligned}
\Pi_{211}^{13}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.2x_2^4 + 0.008x_2^3\tilde{x}_1 + 0.031x_2^3\tilde{x}_2 + 0.252x_2^2\tilde{x}_1^2 + 0.004x_2^2\tilde{x}_1\tilde{x}_2 + 0.044x_2^2\tilde{x}_2^2 + \\
& 0.006x_2\tilde{x}_1^3 + 0.019x_2\tilde{x}_1^2\tilde{x}_2 - 0.004x_2\tilde{x}_1\tilde{x}_2^2 - 0.001x_2\tilde{x}_2^3 + 0.023\tilde{x}_1^4 + 0.001\tilde{x}_1^3\tilde{x}_2 - \\
& 0.005\tilde{x}_1^2\tilde{x}_2^2 - 0.011\tilde{x}_1\tilde{x}_2^3 - 0.026\tilde{x}_2^4 \\
\Pi_{211}^{23}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -0.021x_2^4 + 0.032x_2^3\tilde{x}_1 - 0.002x_2^3\tilde{x}_2 + 0.005x_2^2\tilde{x}_1^2 + 0.042x_2^2\tilde{x}_1\tilde{x}_2 - 0.003x_2^2\tilde{x}_2^2 + \\
& 0.018x_2\tilde{x}_1^3 - 0.004x_2\tilde{x}_1^2\tilde{x}_2 - 0.003x_2\tilde{x}_1\tilde{x}_2^2 - 0.004x_2\tilde{x}_2^3 + 0.001\tilde{x}_1^4 - 0.008\tilde{x}_1^3\tilde{x}_2 - \\
& 0.012\tilde{x}_1^2\tilde{x}_2^2 - 0.046\tilde{x}_1\tilde{x}_2^3 + 0.005\tilde{x}_2^4 \\
\Pi_{211}^{33}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 4.559x_2^4 + 0.004x_2^3\tilde{x}_1 - 0.005x_2^3\tilde{x}_2 + 2.989x_2^2\tilde{x}_1^2 - 0.007x_2^2\tilde{x}_1\tilde{x}_2 + 2.962x_2^2\tilde{x}_2^2 + \\
& 0.002x_2\tilde{x}_1^3 + 0.017x_2\tilde{x}_1^2\tilde{x}_2 + 0.005x_2\tilde{x}_1\tilde{x}_2^2 - 0.003x_2\tilde{x}_2^3 + 4.536\tilde{x}_1^4 - 0.007\tilde{x}_1^3\tilde{x}_2 + \\
& 3.069\tilde{x}_1^2\tilde{x}_2^2 + 0.008\tilde{x}_1\tilde{x}_2^3 + 4.399\tilde{x}_2^4 \\
\Pi_{211}^{43}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.008x_2^4 + 0.191x_2^3\tilde{x}_1 - 0.023x_2^3\tilde{x}_2 + 0.005x_2^2\tilde{x}_1^2 - 0.008x_2^2\tilde{x}_1\tilde{x}_2 + 0.001x_2^2\tilde{x}_2^2 + \\
& 0.19x_2\tilde{x}_1^3 + 0.008x_2\tilde{x}_1^2\tilde{x}_2 + 0.039x_2\tilde{x}_1\tilde{x}_2^2 + 0.001x_2\tilde{x}_2^3 + 0.004\tilde{x}_1^4 + 0.026\tilde{x}_1^3\tilde{x}_2 + \\
& 0.008\tilde{x}_1^2\tilde{x}_2^2 + 0.045\tilde{x}_1\tilde{x}_2^3 - 0.008\tilde{x}_2^4 \\
\Pi_{211}^{14}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.011x_2^4 + 0.275x_2^3\tilde{x}_1 - 0.002x_2^3\tilde{x}_2 + 0.016x_2^2\tilde{x}_1^2 + 0.019x_2^2\tilde{x}_1\tilde{x}_2 - 0.003x_2^2\tilde{x}_2^2 + \\
& 0.223x_2\tilde{x}_1^3 - 0.005x_2\tilde{x}_1^2\tilde{x}_2 + 0.026x_2\tilde{x}_1\tilde{x}_2^2 + 0.008\tilde{x}_1^4 + 0.001\tilde{x}_1^3\tilde{x}_2 - \\
& 0.005\tilde{x}_1^2\tilde{x}_2^2 - 0.049\tilde{x}_1\tilde{x}_2^3 - 0.025\tilde{x}_2^4 \\
\Pi_{211}^{24}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.148x_2^4 - 0.002x_2^3\tilde{x}_1 + 0.179x_2^3\tilde{x}_2 + 0.023x_2^2\tilde{x}_1^2 - 0.003x_2^2\tilde{x}_1\tilde{x}_2 - 0.035x_2^2\tilde{x}_2^2 - \\
& 0.004x_2\tilde{x}_1^3 + 0.029x_2\tilde{x}_1^2\tilde{x}_2 + 0.048x_2\tilde{x}_2^3 + 0.01\tilde{x}_1^4 - 0.004\tilde{x}_1^3\tilde{x}_2 - 0.039\tilde{x}_1^2\tilde{x}_2^2 - \\
& 0.031\tilde{x}_1\tilde{x}_2^3 - 0.253\tilde{x}_2^4 \\
\Pi_{211}^{34}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.008x_2^4 + 0.191x_2^3\tilde{x}_1 - 0.023x_2^3\tilde{x}_2 + 0.005x_2^2\tilde{x}_1^2 - 0.008x_2^2\tilde{x}_1\tilde{x}_2 + 0.001x_2^2\tilde{x}_2^2 + \\
& 0.19x_2\tilde{x}_1^3 + 0.008x_2\tilde{x}_1^2\tilde{x}_2 + 0.039x_2\tilde{x}_1\tilde{x}_2^2 + 0.001x_2\tilde{x}_2^3 + 0.004\tilde{x}_1^4 + 0.026\tilde{x}_1^3\tilde{x}_2 + \\
& 0.008\tilde{x}_1^2\tilde{x}_2^2 + 0.045\tilde{x}_1\tilde{x}_2^3 - 0.008\tilde{x}_2^4 \\
\Pi_{211}^{44}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 4.342x_2^4 + 0.014x_2^3\tilde{x}_1 - 0.023x_2^3\tilde{x}_2 + 2.779x_2^2\tilde{x}_1^2 + 0.001x_2^2\tilde{x}_1\tilde{x}_2 + 2.727x_2^2\tilde{x}_2^2 + \\
& 0.018x_2\tilde{x}_1^3 - 0.012x_2\tilde{x}_1^2\tilde{x}_2 - 0.005x_2\tilde{x}_1\tilde{x}_2^2 - 0.094x_2\tilde{x}_2^3 + 3.767\tilde{x}_1^4 + 2.582\tilde{x}_1^2\tilde{x}_2^2 + \\
& 0.03\tilde{x}_1\tilde{x}_2^3 + 3.884\tilde{x}_2^4
\end{aligned}
$$

$$\Pi^{11}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 3.904x_2^4 + 0.016x_2^3\tilde{x}_1 + 0.116x_2^3\tilde{x}_2 + 2.893x_2^2\tilde{x}_1^2 + 0.015x_2^2\tilde{x}_1\tilde{x}_2 + 2.644x_2^2\tilde{x}_2^2 + 0.014x_2\tilde{x}_1^3 + 0.142x_2\tilde{x}_1^2\tilde{x}_2 + 0.013x_2\tilde{x}_1\tilde{x}_2^2 + 0.07x_2\tilde{x}_2^3 + 4.619\tilde{x}_1^4 + 0.014\tilde{x}_1^3\tilde{x}_2 + 2.755\tilde{x}_1^2\tilde{x}_2^2 + 0.014\tilde{x}_1\tilde{x}_2^3 + 3.729\tilde{x}_2^4$$

$$\Pi^{21}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -0.005x_2^4 + 0.119x_2^3\tilde{x}_1 + 0.01x_2^3\tilde{x}_2 + 0.015x_2^2\tilde{x}_1^2 + 0.145x_2^2\tilde{x}_1\tilde{x}_2 + 0.013x_2^2\tilde{x}_2^2 + 0.12x_2\tilde{x}_1^3 + 0.013x_2\tilde{x}_1^2\tilde{x}_2 + 0.086x_2\tilde{x}_1\tilde{x}_2^2 + 0.005x_2\tilde{x}_2^3 + 0.011\tilde{x}_1^4 + 0.237\tilde{x}_1^3\tilde{x}_2 + 0.016\tilde{x}_1^2\tilde{x}_2^2 + 0.17\tilde{x}_1\tilde{x}_2^3$$

$$\Pi^{31}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.137x_2^4 - 0.004x_2^3\tilde{x}_1 + 0.018x_2^3\tilde{x}_2 + 0.104x_2^2\tilde{x}_1^2 - 0.023x_2^2\tilde{x}_1\tilde{x}_2 + 0.039x_2^2\tilde{x}_2^2 + 0.007x_2\tilde{x}_1^3 + 0.007x_2\tilde{x}_1^2\tilde{x}_2 + 0.006x_2\tilde{x}_1\tilde{x}_2^2 + 0.008x_2\tilde{x}_2^3 + 0.135\tilde{x}_1^4 + 0.011\tilde{x}_1^3\tilde{x}_2 + 0.062\tilde{x}_1^2\tilde{x}_2^2 + 0.032\tilde{x}_1\tilde{x}_2^3 + 0.06\tilde{x}_2^4$$

$$\Pi^{41}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.265x_2^3\tilde{x}_1 - 0.015x_2^3\tilde{x}_2 + 0.004x_2^2\tilde{x}_1^2 - 0.006x_2^2\tilde{x}_1\tilde{x}_2 - 0.002x_2^2\tilde{x}_2^2 + 0.234x_2\tilde{x}_1^3 + 0.002x_2\tilde{x}_1^2\tilde{x}_2 + 0.052x_2\tilde{x}_1\tilde{x}_2^2 + 0.006x_2\tilde{x}_2^3 + 0.001\tilde{x}_1^4 - 0.117\tilde{x}_1^3\tilde{x}_2 - 0.003\tilde{x}_1^2\tilde{x}_2^2 - 0.097\tilde{x}_1\tilde{x}_2^3 + 0.004\tilde{x}_2^4$$

$$\Pi^{12}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -0.005x_2^4 + 0.119x_2^3\tilde{x}_1 + 0.01x_2^3\tilde{x}_2 + 0.015x_2^2\tilde{x}_1^2 + 0.145x_2^2\tilde{x}_1\tilde{x}_2 + 0.013x_2^2\tilde{x}_2^2 + 0.12x_2\tilde{x}_1^3 + 0.013x_2\tilde{x}_1^2\tilde{x}_2 + 0.086x_2\tilde{x}_1\tilde{x}_2^2 + 0.005x_2\tilde{x}_2^3 + 0.011\tilde{x}_1^4 + 0.237\tilde{x}_1^3\tilde{x}_2 + 0.016\tilde{x}_1^2\tilde{x}_2^2 + 0.17\tilde{x}_1\tilde{x}_2^3$$

$$\Pi^{22}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 3.944x_2^4 + 0.01x_2^3\tilde{x}_1 + 0.243x_2^3\tilde{x}_2 + 2.642x_2^2\tilde{x}_1^2 + 0.014x_2^2\tilde{x}_1\tilde{x}_2 + 2.818x_2^2\tilde{x}_2^2 + 0.011x_2\tilde{x}_1^3 + 0.083x_2\tilde{x}_1^2\tilde{x}_2 + 0.006x_2\tilde{x}_1\tilde{x}_2^2 + 0.218x_2\tilde{x}_2^3 + 3.782\tilde{x}_1^4 + 0.014\tilde{x}_1^3\tilde{x}_2 + 2.697\tilde{x}_1^2\tilde{x}_2^2 + 0.001\tilde{x}_1\tilde{x}_2^3 + 4.024\tilde{x}_2^4$$

$$\Pi^{32}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -0.045x_2^4 + 0.019x_2^3\tilde{x}_1 - 0.011x_2^3\tilde{x}_2 - 0.021x_2^2\tilde{x}_1^2 + 0.034x_2^2\tilde{x}_1\tilde{x}_2 - 0.018x_2^2\tilde{x}_2^2 + 0.008x_2\tilde{x}_1^3 + 0.006x_2\tilde{x}_1^2\tilde{x}_2 + 0.01x_2\tilde{x}_1\tilde{x}_2^2 + 0.007x_2\tilde{x}_2^3 + 0.008\tilde{x}_1^4 + 0.048\tilde{x}_1^3\tilde{x}_2 + 0.034\tilde{x}_1^2\tilde{x}_2^2 + 0.058\tilde{x}_1\tilde{x}_2^3 + 0.09\tilde{x}_2^4$$

$$\Pi^{42}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.149x_2^4 - 0.014x_2^3\tilde{x}_1 + 0.186x_2^3\tilde{x}_2 + 0.001x_2^2\tilde{x}_1^2 - 0.002x_2^2\tilde{x}_1\tilde{x}_2 - 0.03x_2^2\tilde{x}_2^2 + 0.002x_2\tilde{x}_1^3 + 0.057x_2\tilde{x}_1^2\tilde{x}_2 + 0.008x_2\tilde{x}_1\tilde{x}_2^2 + 0.055x_2\tilde{x}_2^3 - 0.082\tilde{x}_1^4 - 0.003\tilde{x}_1^3\tilde{x}_2 - 0.096\tilde{x}_1^2\tilde{x}_2^2 + 0.004\tilde{x}_1\tilde{x}_2^3 - 0.24\tilde{x}_2^4$$

$$\begin{aligned}\Pi^{13}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&0.137x_2^4 - 0.004x_2^3\tilde{x}_1 + 0.018x_2^3\tilde{x}_2 + 0.104x_2^2\tilde{x}_1^2 - 0.023x_2^2\tilde{x}_1\tilde{x}_2 + 0.039x_2^2\tilde{x}_2^2+\\&0.007x_2\tilde{x}_1^3 + 0.007x_2\tilde{x}_1^2\tilde{x}_2 + 0.006x_2\tilde{x}_1\tilde{x}_2^2 + 0.008x_2\tilde{x}_2^3 + 0.135\tilde{x}_1^4 + 0.011\tilde{x}_1^3\tilde{x}_2+\\&0.062\tilde{x}_1^2\tilde{x}_2^2 + 0.032\tilde{x}_1\tilde{x}_2^3 + 0.06\tilde{x}_2^4\\
\Pi^{23}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&-0.045x_2^4 + 0.019x_2^3\tilde{x}_1 - 0.011x_2^3\tilde{x}_2 - 0.021x_2^2\tilde{x}_1^2 + 0.034x_2^2\tilde{x}_1\tilde{x}_2 - 0.018x_2^2\tilde{x}_2^2+\\&0.008x_2\tilde{x}_1^3 + 0.006x_2\tilde{x}_1^2\tilde{x}_2 + 0.01x_2\tilde{x}_1\tilde{x}_2^2 + 0.007x_2\tilde{x}_2^3 + 0.008\tilde{x}_1^4 + 0.048\tilde{x}_1^3\tilde{x}_2+\\&0.034\tilde{x}_1^2\tilde{x}_2^2 + 0.058\tilde{x}_1\tilde{x}_2^3 + 0.09\tilde{x}_2^4\\
\Pi^{33}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&4.4x_2^4 - 0.003x_2^3\tilde{x}_1 - 0.012x_2^3\tilde{x}_2 + 3.052x_2^2\tilde{x}_1^2 - 0.011x_2^2\tilde{x}_1\tilde{x}_2 + 2.962x_2^2\tilde{x}_2^2+\\&0.007x_2\tilde{x}_1^3 + 0.013x_2\tilde{x}_1^2\tilde{x}_2 + 0.003x_2\tilde{x}_1\tilde{x}_2^2 - 0.002x_2\tilde{x}_2^3 + 4.333\tilde{x}_1^4 + 2.988\tilde{x}_1^2\tilde{x}_2^2-\\&0.001\tilde{x}_1\tilde{x}_2^3 + 4.316\tilde{x}_2^4\\
\Pi^{43}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&0.014x_2^4 + 0.127x_2^3\tilde{x}_1 - 0.042x_2^3\tilde{x}_2 + 0.017x_2^2\tilde{x}_1^2 - 0.015x_2^2\tilde{x}_1\tilde{x}_2 + 0.007x_2^2\tilde{x}_2^2+\\&0.077x_2\tilde{x}_1^3 - 0.027x_2\tilde{x}_1^2\tilde{x}_2 + 0.024x_2\tilde{x}_1\tilde{x}_2^2 - 0.019x_2\tilde{x}_2^3 - 0.003\tilde{x}_1^4 - 0.04\tilde{x}_1^3\tilde{x}_2-\\&0.029\tilde{x}_1^2\tilde{x}_2^2 - 0.048\tilde{x}_1\tilde{x}_2^3 - 0.083\tilde{x}_2^4\\
\Pi^{14}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&0.265x_2^3\tilde{x}_1 - 0.015x_2^3\tilde{x}_2 + 0.004x_2^2\tilde{x}_1^2 - 0.006x_2^2\tilde{x}_1\tilde{x}_2 - 0.002x_2^2\tilde{x}_2^2+\\&0.234x_2\tilde{x}_1^3 + 0.002x_2\tilde{x}_1^2\tilde{x}_2 + 0.052x_2\tilde{x}_1\tilde{x}_2^2 + 0.006x_2\tilde{x}_2^3 + 0.001\tilde{x}_1^4 - 0.117\tilde{x}_1^3\tilde{x}_2-\\&0.003\tilde{x}_1^2\tilde{x}_2^2 - 0.097\tilde{x}_1\tilde{x}_2^3 + 0.004\tilde{x}_2^4\\
\Pi^{24}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&0.149x_2^4 - 0.014x_2^3\tilde{x}_1 + 0.186x_2^3\tilde{x}_2 + 0.001x_2^2\tilde{x}_1^2 - 0.002x_2^2\tilde{x}_1\tilde{x}_2 - 0.03x_2^2\tilde{x}_2^2+\\&0.002x_2\tilde{x}_1^3 + 0.057x_2\tilde{x}_1^2\tilde{x}_2 + 0.008x_2\tilde{x}_1\tilde{x}_2^2 + 0.055x_2\tilde{x}_2^3 - 0.082\tilde{x}_1^4 - 0.003\tilde{x}_1^3\tilde{x}_2-\\&0.096\tilde{x}_1^2\tilde{x}_2^2 + 0.004\tilde{x}_1\tilde{x}_2^3 - 0.24\tilde{x}_2^4\\
\Pi^{34}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&0.014x_2^4 + 0.127x_2^3\tilde{x}_1 - 0.042x_2^3\tilde{x}_2 + 0.017x_2^2\tilde{x}_1^2 - 0.015x_2^2\tilde{x}_1\tilde{x}_2 + 0.007x_2^2\tilde{x}_2^2+\\&0.077x_2\tilde{x}_1^3 - 0.027x_2\tilde{x}_1^2\tilde{x}_2 + 0.024x_2\tilde{x}_1\tilde{x}_2^2 - 0.019x_2\tilde{x}_2^3 - 0.003\tilde{x}_1^4 - 0.04\tilde{x}_1^3\tilde{x}_2-\\&0.029\tilde{x}_1^2\tilde{x}_2^2 - 0.048\tilde{x}_1\tilde{x}_2^3 - 0.083\tilde{x}_2^4\\
\Pi^{44}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&4.296x_2^4 + 0.014x_2^3\tilde{x}_1 - 0.031x_2^3\tilde{x}_2 + 2.792x_2^2\tilde{x}_1^2 - 0.014x_2^2\tilde{x}_1\tilde{x}_2 + 2.73x_2^2\tilde{x}_2^2+\\&0.002x_2\tilde{x}_1^3 - 0.068x_2\tilde{x}_1^2\tilde{x}_2 - 0.01x_2\tilde{x}_1\tilde{x}_2^2 - 0.097x_2\tilde{x}_2^3 + 3.862\tilde{x}_1^4 + 0.004\tilde{x}_1^3\tilde{x}_2+\\&2.66\tilde{x}_1^2\tilde{x}_2^2 + 0.001\tilde{x}_1\tilde{x}_2^3 + 3.876\tilde{x}_2^4\end{aligned}$$

$$\Pi_{221}^{11}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 4.439x_2^4 + 0.043x_2^3\tilde{x}_1 + 0.223x_2^3\tilde{x}_2 + 3.406x_2^2\tilde{x}_1^2 + 0.016x_2^2\tilde{x}_1\tilde{x}_2 + 2.717x_2^2\tilde{x}_2^2 + 0.03x_2\tilde{x}_1^3 + 0.148x_2\tilde{x}_1^2\tilde{x}_2 + 0.026x_2\tilde{x}_1\tilde{x}_2^2 + 0.096x_2\tilde{x}_2^3 + 6.273\tilde{x}_1^4 + 0.013\tilde{x}_1^3\tilde{x}_2 + 2.715\tilde{x}_1^2\tilde{x}_2^2 + 0.043\tilde{x}_1\tilde{x}_2^3 + 3.817\tilde{x}_2^4$$

$$\Pi_{221}^{21}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.063x_2^4 + 0.235x_2^3\tilde{x}_1 + 0.076x_2^3\tilde{x}_2 + 0.015x_2^2\tilde{x}_1^2 + 0.261x_2^2\tilde{x}_1\tilde{x}_2 + 0.099x_2^2\tilde{x}_2^2 + 0.127x_2\tilde{x}_1^3 + 0.026x_2\tilde{x}_1^2\tilde{x}_2 + 0.118x_2\tilde{x}_1\tilde{x}_2^2 + 0.091x_2\tilde{x}_2^3 + 0.01\tilde{x}_1^4 + 0.247\tilde{x}_1^3\tilde{x}_2 + 0.047\tilde{x}_1^2\tilde{x}_2^2 + 0.32\tilde{x}_1\tilde{x}_2^3 + 0.159\tilde{x}_2^4$$

$$\Pi_{221}^{31}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.476x_2^4 + 0.016x_2^3\tilde{x}_1 + 0.066x_2^3\tilde{x}_2 + 0.518x_2^2\tilde{x}_1^2 + 0.004x_2^2\tilde{x}_1\tilde{x}_2 + 0.088x_2^2\tilde{x}_2^2 + 0.008x_2\tilde{x}_1^3 + 0.043x_2\tilde{x}_1^2\tilde{x}_2 - 0.008x_2\tilde{x}_1\tilde{x}_2^2 - 0.01x_2\tilde{x}_2^3 - 0.097\tilde{x}_1^4 - 0.002\tilde{x}_1^3\tilde{x}_2 - 0.081\tilde{x}_1^2\tilde{x}_2^2 - 0.027\tilde{x}_1\tilde{x}_2^3 - 0.117\tilde{x}_2^4$$

$$\Pi_{221}^{41}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.026x_2^4 + 0.841x_2^3\tilde{x}_1 - 0.003x_2^3\tilde{x}_2 + 0.034x_2^2\tilde{x}_1^2 + 0.004x_2^2\tilde{x}_1\tilde{x}_2 - 0.013x_2^2\tilde{x}_2^2 + 0.722x_2\tilde{x}_1^3 - 0.01x_2\tilde{x}_1^2\tilde{x}_2 + 0.126x_2\tilde{x}_1\tilde{x}_2^2 - 0.001x_2\tilde{x}_2^3 + 0.016\tilde{x}_1^4 - 0.12\tilde{x}_1^3\tilde{x}_2 - 0.023\tilde{x}_1^2\tilde{x}_2^2 - 0.217\tilde{x}_1\tilde{x}_2^3 - 0.084\tilde{x}_2^4$$

$$\Pi_{221}^{12}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.063x_2^4 + 0.235x_2^3\tilde{x}_1 + 0.076x_2^3\tilde{x}_2 + 0.015x_2^2\tilde{x}_1^2 + 0.261x_2^2\tilde{x}_1\tilde{x}_2 + 0.099x_2^2\tilde{x}_2^2 + 0.127x_2\tilde{x}_1^3 + 0.026x_2\tilde{x}_1^2\tilde{x}_2 + 0.118x_2\tilde{x}_1\tilde{x}_2^2 + 0.091x_2\tilde{x}_2^3 + 0.01\tilde{x}_1^4 + 0.247\tilde{x}_1^3\tilde{x}_2 + 0.047\tilde{x}_1^2\tilde{x}_2^2 + 0.32\tilde{x}_1\tilde{x}_2^3 + 0.159\tilde{x}_2^4$$

$$\Pi_{221}^{22}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 4.693x_2^4 + 0.075x_2^3\tilde{x}_1 + 0.766x_2^3\tilde{x}_2 + 2.711x_2^2\tilde{x}_1^2 + 0.1x_2^2\tilde{x}_1\tilde{x}_2 + 3.529x_2^2\tilde{x}_2^2 + 0.019x_2\tilde{x}_1^3 + 0.105x_2\tilde{x}_1^2\tilde{x}_2 + 0.104x_2\tilde{x}_1\tilde{x}_2^2 + 0.721x_2\tilde{x}_2^3 + 3.754\tilde{x}_1^4 + 0.035\tilde{x}_1^3\tilde{x}_2 + 2.815\tilde{x}_1^2\tilde{x}_2^2 + 0.186\tilde{x}_1\tilde{x}_2^3 + 5.167\tilde{x}_2^4$$

$$\Pi_{221}^{32}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -0.058x_2^4 + 0.070x_2^3\tilde{x}_1 - 0.005x_2^3\tilde{x}_2 + 0.006x_2^2\tilde{x}_1^2 + 0.064x_2^2\tilde{x}_1\tilde{x}_2 - 0.013x_2^2\tilde{x}_2^2 + 0.039x_2\tilde{x}_1^3 - 0.008x_2\tilde{x}_1^2\tilde{x}_2 - 0.014x_2\tilde{x}_1\tilde{x}_2^2 - 0.007x_2\tilde{x}_2^3 - 0.001\tilde{x}_1^4 - 0.086\tilde{x}_1^3\tilde{x}_2 - 0.032\tilde{x}_1^2\tilde{x}_2^2 - 0.18\tilde{x}_1\tilde{x}_2^3 + 0.014\tilde{x}_2^4$$

$$\Pi_{221}^{42}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.392x_2^4 - 0.002x_2^3\tilde{x}_1 + 0.593x_2^3\tilde{x}_2 + 0.03x_2^2\tilde{x}_1^2 - 0.012x_2^2\tilde{x}_1\tilde{x}_2 - 0.178x_2^2\tilde{x}_2^2 - 0.008x_2\tilde{x}_1^3 + 0.13x_2\tilde{x}_1^2\tilde{x}_2 + 0.21x_2\tilde{x}_2^3 - 0.06\tilde{x}_1^4 - 0.017\tilde{x}_1^3\tilde{x}_2 - 0.201\tilde{x}_1^2\tilde{x}_2^2 - 0.096\tilde{x}_1\tilde{x}_2^3 - 0.956\tilde{x}_2^4$$

$$\begin{aligned}\Pi_{221}^{13}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.476x_2^4 + 0.016x_2^3\tilde{x}_1 + 0.066x_2^3\tilde{x}_2 + 0.518x_2^2\tilde{x}_1^2 + 0.004x_2^2\tilde{x}_1\tilde{x}_2 + 0.088x_2^2\tilde{x}_2^2 + \\ & 0.008x_2\tilde{x}_1^3 + 0.043x_2\tilde{x}_1^2\tilde{x}_2 - 0.008x_2\tilde{x}_1\tilde{x}_2^2 - 0.01x_2\tilde{x}_2^3 - 0.097\tilde{x}_1^4 - 0.002\tilde{x}_1^3\tilde{x}_2 - \\ & 0.081\tilde{x}_1^2\tilde{x}_2^2 - 0.027\tilde{x}_1\tilde{x}_2^3 - 0.117\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi_{221}^{23}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -0.058x_2^4 + 0.070x_2^3\tilde{x}_1 - 0.005x_2^3\tilde{x}_2 + 0.006x_2^2\tilde{x}_1^2 + 0.064x_2^2\tilde{x}_1\tilde{x}_2 - 0.013x_2^2\tilde{x}_2^2 + \\ & 0.039x_2\tilde{x}_1^3 - 0.008x_2\tilde{x}_1^2\tilde{x}_2 - 0.014x_2\tilde{x}_1\tilde{x}_2^2 - 0.007x_2\tilde{x}_2^3 - 0.001\tilde{x}_1^4 - 0.086\tilde{x}_1^3\tilde{x}_2 - \\ & 0.032\tilde{x}_1^2\tilde{x}_2^2 - 0.18\tilde{x}_1\tilde{x}_2^3 + 0.014\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi_{221}^{33}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 6.488x_2^4 + 0.006x_2^3\tilde{x}_1 - 0.013x_2^3\tilde{x}_2 + 4.017x_2^2\tilde{x}_1^2 - 0.013x_2^2\tilde{x}_1\tilde{x}_2 + 3.98x_2^2\tilde{x}_2^2 + \\ & 0.003x_2\tilde{x}_1^3 + 0.037x_2\tilde{x}_1^2\tilde{x}_2 + 0.008x_2\tilde{x}_1\tilde{x}_2^2 - 0.002x_2\tilde{x}_2^3 + 6.49\tilde{x}_1^4 - 0.014\tilde{x}_1^3\tilde{x}_2 + \\ & 4.209\tilde{x}_1^2\tilde{x}_2^2 + 0.013\tilde{x}_1\tilde{x}_2^3 + 6.12\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi_{221}^{43}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.021x_2^4 + 0.411x_2^3\tilde{x}_1 - 0.06x_2^3\tilde{x}_2 + 0.012x_2^2\tilde{x}_1^2 + 0.008x_2^2\tilde{x}_1\tilde{x}_2 + 0.008x_2^2\tilde{x}_2^2 + \\ & 0.381x_2\tilde{x}_1^3 + 0.012x_2\tilde{x}_1^2\tilde{x}_2 + 0.074x_2\tilde{x}_1\tilde{x}_2^2 - 0.005x_2\tilde{x}_2^3 + 0.01\tilde{x}_1^4 + 0.128\tilde{x}_1^3\tilde{x}_2 + \\ & 0.024\tilde{x}_1^2\tilde{x}_2^2 + 0.168\tilde{x}_1\tilde{x}_2^3 - 0.02\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi_{221}^{14}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.026x_2^4 + 0.841x_2^3\tilde{x}_1 - 0.003x_2^3\tilde{x}_2 + 0.034x_2^2\tilde{x}_1^2 + 0.004x_2^2\tilde{x}_1\tilde{x}_2 - 0.013x_2^2\tilde{x}_2^2 + \\ & 0.722x_2\tilde{x}_1^3 - 0.01x_2\tilde{x}_1^2\tilde{x}_2 + 0.126x_2\tilde{x}_1\tilde{x}_2^2 - 0.001x_2\tilde{x}_2^3 + 0.016\tilde{x}_1^4 - 0.12\tilde{x}_1^3\tilde{x}_2 - \\ & 0.023\tilde{x}_1^2\tilde{x}_2^2 - 0.217\tilde{x}_1\tilde{x}_2^3 - 0.084\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi_{221}^{24}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.392x_2^4 - 0.002x_2^3\tilde{x}_1 + 0.593x_2^3\tilde{x}_2 + 0.03x_2^2\tilde{x}_1^2 - 0.012x_2^2\tilde{x}_1\tilde{x}_2 - 0.178x_2^2\tilde{x}_2^2 - \\ & 0.008x_2\tilde{x}_1^3 + 0.13x_2\tilde{x}_1^2\tilde{x}_2 + 0.21x_2\tilde{x}_2^3 - 0.06\tilde{x}_1^4 - 0.017\tilde{x}_1^3\tilde{x}_2 - 0.201\tilde{x}_1^2\tilde{x}_2^2 - \\ & 0.096\tilde{x}_1\tilde{x}_2^3 - 0.956\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi_{221}^{34}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.021x_2^4 + 0.411x_2^3\tilde{x}_1 - 0.06x_2^3\tilde{x}_2 + 0.012x_2^2\tilde{x}_1^2 + 0.008x_2^2\tilde{x}_1\tilde{x}_2 + 0.008x_2^2\tilde{x}_2^2 + \\ & 0.381x_2\tilde{x}_1^3 + 0.012x_2\tilde{x}_1^2\tilde{x}_2 + 0.074x_2\tilde{x}_1\tilde{x}_2^2 - 0.005x_2\tilde{x}_2^3 + 0.01\tilde{x}_1^4 + 0.128\tilde{x}_1^3\tilde{x}_2 + \\ & 0.024\tilde{x}_1^2\tilde{x}_2^2 + 0.168\tilde{x}_1\tilde{x}_2^3 - 0.02\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi_{221}^{44}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 6.073x_2^4 + 0.03x_2^3\tilde{x}_1 - 0.236x_2^3\tilde{x}_2 + 3.352x_2^2\tilde{x}_1^2 + 0.008x_2^2\tilde{x}_1\tilde{x}_2 + 3.271x_2^2\tilde{x}_2^2 + \\ & 0.036x_2\tilde{x}_1^3 - 0.129x_2\tilde{x}_1^2\tilde{x}_2 - 0.016x_2\tilde{x}_1\tilde{x}_2^2 - 0.411x_2\tilde{x}_2^3 + 4.201\tilde{x}_1^4 + 0.008\tilde{x}_1^3\tilde{x}_2 + \\ & 2.813\tilde{x}_1^2\tilde{x}_2^2 + 0.09\tilde{x}_1\tilde{x}_2^3 + 4.656\tilde{x}_2^4\end{aligned}$$

$$
\begin{aligned}
\Pi_{222}^{11}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 4.539x_2^4 + 0.038x_2^3\tilde{x}_1 + 0.343x_2^3\tilde{x}_2 + 3.697x_2^2\tilde{x}_1^2 + 0.036x_2^2\tilde{x}_1\tilde{x}_2 + 2.911x_2^2\tilde{x}_2^2 + \\
& 0.035x_2\tilde{x}_1^3 + 0.398x_2\tilde{x}_1^2\tilde{x}_2 + 0.034x_2\tilde{x}_1\tilde{x}_2^2 + 0.217x_2\tilde{x}_2^3 + 6.847\tilde{x}_1^4 + 0.036\tilde{x}_1^3\tilde{x}_2 + \\
& 3.311\tilde{x}_1^2\tilde{x}_2^2 + 0.039\tilde{x}_1\tilde{x}_2^3 + 4.035\tilde{x}_2^4 \\
\Pi_{222}^{21}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -0.007x_2^4 + 0.356x_2^3\tilde{x}_1 + 0.025x_2^3\tilde{x}_2 + 0.036x_2^2\tilde{x}_1^2 + 0.468x_2^2\tilde{x}_1\tilde{x}_2 + 0.034x_2^2\tilde{x}_2^2 + \\
& 0.348x_2\tilde{x}_1^3 + 0.035x_2\tilde{x}_1^2\tilde{x}_2 + 0.262x_2\tilde{x}_1\tilde{x}_2^2 + 0.018x_2\tilde{x}_2^3 + 0.032\tilde{x}_1^4 + 0.789\tilde{x}_1^3\tilde{x}_2 + \\
& 0.046\tilde{x}_1^2\tilde{x}_2^2 + 0.605\tilde{x}_1\tilde{x}_2^3 + 0.015\tilde{x}_2^4 \\
\Pi_{222}^{31}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.313x_2^4 - 0.008x_2^3\tilde{x}_1 + 0.035x_2^3\tilde{x}_2 + 0.213x_2^2\tilde{x}_1^2 - 0.049x_2^2\tilde{x}_1\tilde{x}_2 + 0.08x_2^2\tilde{x}_2^2 + \\
& 0.019x_2\tilde{x}_1^3 + 0.014x_2\tilde{x}_1^2\tilde{x}_2 + 0.015x_2\tilde{x}_1\tilde{x}_2^2 + 0.014x_2\tilde{x}_2^3 + 0.205\tilde{x}_1^4 + 0.029\tilde{x}_1^3\tilde{x}_2 + \\
& 0.094\tilde{x}_1^2\tilde{x}_2^2 + 0.076\tilde{x}_1\tilde{x}_2^3 + 0.11\tilde{x}_2^4 \\
\Pi_{222}^{41}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.001x_2^4 + 0.85x_2^3\tilde{x}_1 - 0.03x_2^3\tilde{x}_2 + 0.01x_2^2\tilde{x}_1^2 - 0.063x_2^2\tilde{x}_1\tilde{x}_2 - 0.005x_2^2\tilde{x}_2^2 + \\
& 0.771x_2\tilde{x}_1^3 + 0.002x_2\tilde{x}_1^2\tilde{x}_2 + 0.193x_2\tilde{x}_1\tilde{x}_2^2 + 0.011x_2\tilde{x}_2^3 - 0.001\tilde{x}_1^4 - 0.442\tilde{x}_1^3\tilde{x}_2 - \\
& 0.012\tilde{x}_1^2\tilde{x}_2^2 - 0.38\tilde{x}_1\tilde{x}_2^3 - 0.001\tilde{x}_2^4 \\
\Pi_{222}^{12}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -0.007x_2^4 + 0.356x_2^3\tilde{x}_1 + 0.025x_2^3\tilde{x}_2 + 0.036x_2^2\tilde{x}_1^2 + 0.468x_2^2\tilde{x}_1\tilde{x}_2 + 0.034x_2^2\tilde{x}_2^2 + \\
& 0.348x_2\tilde{x}_1^3 + 0.035x_2\tilde{x}_1^2\tilde{x}_2 + 0.262x_2\tilde{x}_1\tilde{x}_2^2 + 0.018x_2\tilde{x}_2^3 + 0.032\tilde{x}_1^4 + 0.789\tilde{x}_1^3\tilde{x}_2 + \\
& 0.046\tilde{x}_1^2\tilde{x}_2^2 + 0.605\tilde{x}_1\tilde{x}_2^3 + 0.015\tilde{x}_2^4 \\
\Pi_{222}^{22}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 4.687x_2^4 + 0.025x_2^3\tilde{x}_1 + 0.747x_2^3\tilde{x}_2 + 2.908x_2^2\tilde{x}_1^2 + 0.035x_2^2\tilde{x}_1\tilde{x}_2 + 3.5x_2^2\tilde{x}_2^2 + \\
& 0.029x_2\tilde{x}_1^3 + 0.255x_2\tilde{x}_1^2\tilde{x}_2 + 0.021x_2\tilde{x}_1\tilde{x}_2^2 + 0.674x_2\tilde{x}_2^3 + 4.174\tilde{x}_1^4 + 0.039\tilde{x}_1^3\tilde{x}_2 + \\
& 3.146\tilde{x}_1^2\tilde{x}_2^2 + 0.018\tilde{x}_1\tilde{x}_2^3 + 5.099\tilde{x}_2^4 \\
\Pi_{222}^{32}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -0.116x_2^4 + 0.038x_2^3\tilde{x}_1 - 0.022x_2^3\tilde{x}_2 - 0.049x_2^2\tilde{x}_1^2 + 0.052x_2^2\tilde{x}_1\tilde{x}_2 - 0.039x_2^2\tilde{x}_2^2 + \\
& 0.016x_2\tilde{x}_1^3 + 0.015x_2\tilde{x}_1^2\tilde{x}_2 + 0.013x_2\tilde{x}_1\tilde{x}_2^2 + 0.025x_2\tilde{x}_2^3 + 0.025\tilde{x}_1^4 + 0.06\tilde{x}_1^3\tilde{x}_2 + \\
& 0.084\tilde{x}_1^2\tilde{x}_2^2 + 0.079\tilde{x}_1\tilde{x}_2^3 + 0.222\tilde{x}_2^4 \\
\Pi_{222}^{42}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 0.386x_2^4 - 0.03x_2^3\tilde{x}_1 + 0.611x_2^3\tilde{x}_2 - 0.032x_2^2\tilde{x}_1^2 - 0.005x_2^2\tilde{x}_1\tilde{x}_2 - 0.173x_2^2\tilde{x}_2^2 + \\
& 0.001x_2\tilde{x}_1^3 + 0.201x_2\tilde{x}_1^2\tilde{x}_2 + 0.014x_2\tilde{x}_1\tilde{x}_2^2 + 0.232x_2\tilde{x}_2^3 - 0.32\tilde{x}_1^4 - 0.01\tilde{x}_1^3\tilde{x}_2 - \\
& 0.391\tilde{x}_1^2\tilde{x}_2^2 - 0.001\tilde{x}_1\tilde{x}_2^3 - 0.931\tilde{x}_2^4
\end{aligned}
$$

$$\Pi_{222}^{13}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.313x_2^4 - 0.008x_2^3\tilde{x}_1 + 0.035x_2^3\tilde{x}_2 + 0.213x_2^2\tilde{x}_1^2 - 0.049x_2^2\tilde{x}_1\tilde{x}_2 + 0.08x_2^2\tilde{x}_2^2 + 0.019x_2\tilde{x}_1^3 + 0.014x_2\tilde{x}_1^2\tilde{x}_2 + 0.015x_2\tilde{x}_1\tilde{x}_2^2 + 0.014x_2\tilde{x}_2^3 + 0.205\tilde{x}_1^4 + 0.029\tilde{x}_1^3\tilde{x}_2 + 0.094\tilde{x}_1^2\tilde{x}_2^2 + 0.076\tilde{x}_1\tilde{x}_2^3 + 0.11\tilde{x}_2^4$$

$$\Pi_{222}^{23}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -0.116x_2^4 + 0.038x_2^3\tilde{x}_1 - 0.022x_2^3\tilde{x}_2 - 0.049x_2^2\tilde{x}_1^2 + 0.052x_2^2\tilde{x}_1\tilde{x}_2 - 0.039x_2^2\tilde{x}_2^2 + 0.016x_2\tilde{x}_1^3 + 0.015x_2\tilde{x}_1^2\tilde{x}_2 + 0.013x_2\tilde{x}_1\tilde{x}_2^2 + 0.025x_2\tilde{x}_2^3 + 0.025\tilde{x}_1^4 + 0.06\tilde{x}_1^3\tilde{x}_2 + 0.084\tilde{x}_1^2\tilde{x}_2^2 + 0.079\tilde{x}_1\tilde{x}_2^3 + 0.222\tilde{x}_2^4$$

$$\Pi_{222}^{33}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 6.225x_2^4 - 0.007x_2^3\tilde{x}_1 - 0.018x_2^3\tilde{x}_2 + 4.201x_2^2\tilde{x}_1^2 - 0.024x_2^2\tilde{x}_1\tilde{x}_2 + 3.971x_2^2\tilde{x}_2^2 + 0.011x_2\tilde{x}_1^3 + 0.03x_2\tilde{x}_1^2\tilde{x}_2 + 0.004x_2\tilde{x}_1\tilde{x}_2^2 - 0.002x_2\tilde{x}_2^3 + 6.035\tilde{x}_1^4 - 0.008\tilde{x}_1^3\tilde{x}_2 + 4.015\tilde{x}_1^2\tilde{x}_2^2 - 0.01\tilde{x}_1\tilde{x}_2^3 + 5.908\tilde{x}_2^4$$

$$\Pi_{222}^{43}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.035x_2^4 + 0.258x_2^3\tilde{x}_1 - 0.106x_2^3\tilde{x}_2 + 0.039x_2^2\tilde{x}_1^2 - 0.016x_2^2\tilde{x}_1\tilde{x}_2 + 0.017x_2^2\tilde{x}_2^2 + 0.146x_2\tilde{x}_1^3 - 0.062x_2\tilde{x}_1^2\tilde{x}_2 + 0.04x_2\tilde{x}_1\tilde{x}_2^2 - 0.051x_2\tilde{x}_2^3 - 0.008\tilde{x}_1^4 - 0.043\tilde{x}_1^3\tilde{x}_2 - 0.068\tilde{x}_1^2\tilde{x}_2^2 - 0.064\tilde{x}_1\tilde{x}_2^3 - 0.198\tilde{x}_2^4$$

$$\Pi_{222}^{14}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.001x_2^4 + 0.85x_2^3\tilde{x}_1 - 0.03x_2^3\tilde{x}_2 + 0.01x_2^2\tilde{x}_1^2 - 0.063x_2^2\tilde{x}_1\tilde{x}_2 - 0.005x_2^2\tilde{x}_2^2 + 0.771x_2\tilde{x}_1^3 + 0.002x_2\tilde{x}_1^2\tilde{x}_2 + 0.193x_2\tilde{x}_1\tilde{x}_2^2 + 0.011x_2\tilde{x}_2^3 - 0.001\tilde{x}_1^4 - 0.442\tilde{x}_1^3\tilde{x}_2 - 0.012\tilde{x}_1^2\tilde{x}_2^2 - 0.38\tilde{x}_1\tilde{x}_2^3 - 0.001\tilde{x}_2^4$$

$$\Pi_{222}^{24}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.386x_2^4 - 0.03x_2^3\tilde{x}_1 + 0.611x_2^3\tilde{x}_2 - 0.032x_2^2\tilde{x}_1^2 - 0.005x_2^2\tilde{x}_1\tilde{x}_2 - 0.173x_2^2\tilde{x}_2^2 + 0.001x_2\tilde{x}_1^3 + 0.201x_2\tilde{x}_1^2\tilde{x}_2 + 0.014x_2\tilde{x}_1\tilde{x}_2^2 + 0.232x_2\tilde{x}_2^3 - 0.32\tilde{x}_1^4 - 0.01\tilde{x}_1^3\tilde{x}_2 - 0.391\tilde{x}_1^2\tilde{x}_2^2 - 0.001\tilde{x}_1\tilde{x}_2^3 - 0.931\tilde{x}_2^4$$

$$\Pi_{222}^{34}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 0.035x_2^4 + 0.258x_2^3\tilde{x}_1 - 0.106x_2^3\tilde{x}_2 + 0.039x_2^2\tilde{x}_1^2 - 0.016x_2^2\tilde{x}_1\tilde{x}_2 + 0.017x_2^2\tilde{x}_2^2 + 0.146x_2\tilde{x}_1^3 - 0.062x_2\tilde{x}_1^2\tilde{x}_2 + 0.04x_2\tilde{x}_1\tilde{x}_2^2 - 0.051x_2\tilde{x}_2^3 - 0.008\tilde{x}_1^4 - 0.043\tilde{x}_1^3\tilde{x}_2 - 0.068\tilde{x}_1^2\tilde{x}_2^2 - 0.064\tilde{x}_1\tilde{x}_2^3 - 0.198\tilde{x}_2^4$$

$$\Pi_{222}^{44}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 5.96x_2^4 + 0.029x_2^3\tilde{x}_1 - 0.264x_2^3\tilde{x}_2 + 3.425x_2^2\tilde{x}_1^2 - 0.028x_2^2\tilde{x}_1\tilde{x}_2 + 3.286x_2^2\tilde{x}_2^2 + 0.007x_2\tilde{x}_1^3 - 0.275x_2\tilde{x}_1^2\tilde{x}_2 - 0.02x_2\tilde{x}_1\tilde{x}_2^2 - 0.429x_2\tilde{x}_2^3 + 4.479\tilde{x}_1^4 + 0.008\tilde{x}_1^3\tilde{x}_2 + 3.052\tilde{x}_1^2\tilde{x}_2^2 + 0.011\tilde{x}_1\tilde{x}_2^3 + 4.646\tilde{x}_2^4$$

6.4 Appendix B

The obtained solutions of decision variables $\Pi_{ijl}(\boldsymbol{x}, \boldsymbol{\eta}, \tilde{\boldsymbol{x}})$ for observer class II design in Chapter 5.1.3 are as follows.

$$
\begin{aligned}
\Pi^{41}_{111}(\boldsymbol{x}, \boldsymbol{\eta}, \tilde{\boldsymbol{x}}) &= 0.030138x_2^2 + 0.179679x_2\hat{x}_2 + 1.7\times10^{-9}x_2y - 0.032945\hat{x}_2^2 \\
&\quad - 1.4\times10^{-7}\hat{x}_2y + 8.0\times10^{-10}y^2 \\
\Pi^{42}_{111}(\boldsymbol{x}, \boldsymbol{\eta}, \tilde{\boldsymbol{x}}) &= -0.283326x_2^2 - 0.735998x_2\hat{x}_2 + 1.0\times10^{-9}x_2y + 0.037576\hat{x}_2^2 \\
&\quad - 8.8\times10^{-8}\hat{x}_2y + 1.3\times10^{-10}y^2 \\
\Pi^{43}_{111}(\boldsymbol{x}, \boldsymbol{\eta}, \tilde{\boldsymbol{x}}) &= 1.5\times10^{-18}x_2^2 + 0.009414x_2\hat{x}_2 - 7.6\times10^{-26}x_2y - 0.078421\hat{x}_2^2 \\
&\quad - 1.2\times10^{-25}\hat{x}_2y - 5.2\times10^{-7}y^2 \\
\Pi^{44}_{111}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) &= 89.731947x_2^2 + 0.247553x_2\hat{x}_2 + 92.043523\hat{x}_2^2, \\
\Pi^{11}_{112}(\boldsymbol{x}, \boldsymbol{\eta}, \tilde{\boldsymbol{x}}) &= 90.634222x_2^2 - 0.442199x_2\hat{x}_2 - 3.8\times10^{-9}x_2y + 92.115911\hat{x}_2^2 \\
&\quad + 9.8\times10^{-9}\hat{x}_2y + 66.363700y^2 \\
\Pi^{21}_{112}(\boldsymbol{x}, \boldsymbol{\eta}, \tilde{\boldsymbol{x}}) &= 0.150077x_2^2 + 0.076463x_2\hat{x}_2 + 1.6\times10^{-9}x_2y + 0.077956hatx_2^2 \\
&\quad - 1.2\times10^{-7}\hat{x}_2y + 41.137331y^2 \\
\Pi^{31}_{112}(\boldsymbol{x}, \boldsymbol{\eta}, \tilde{\boldsymbol{x}}) &= 4.5\times10^{-18}x_2^2 - 0.337857x_2\hat{x}_2 - 2.5\times10^{-26}x_2y - 0.030428\hat{x}_2^2 \\
&\quad + 1.7\times10^{-7}\hat{x}_2y - 3.2\times10^{-13}y^2,
\end{aligned}
$$

$$\Pi_{112}^{41}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.030336x_2^2 + 0.180511x_2\hat{x}_2 - 7.3\times10^{-10}x_2y - 0.033088\hat{x}_2^2 - 1.5\times10^{-8}\hat{x}_2y - 2.6\times10^{-12}y^2$$

$$\Pi_{112}^{12}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.150077x_2^2 + 0.076463x_2\hat{x}_2 + 1.6\times10^{-9}x_2y + 0.077956\hat{x}_2^2 - 1.2\times10^{-7}\hat{x}_2y + 41.137331y^2,$$

$$\Pi_{112}^{22}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 91.942959x_2^2 - 0.236876x_2\hat{x}_2 + 3.4\times10^{-9}x_2y + 92.337919\hat{x}_2^2 - 1.5\times10^{-7}\hat{x}_2y + 25.500119y^2,$$

$$\Pi_{112}^{32}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 3.3\times10^{-19}x_2^2 - 0.084252x_2\hat{x}_2 + 4.5\times10^{-26}x_2y - 0.005402\hat{x}_2^2 + 1.1\times10^{-7}\hat{x}_2y - 6.0\times10^{-13}y^2,$$

$$\Pi_{112}^{42}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -0.283388x_2^2 - 0.736161x_2\hat{x}_2 - 4.5\times10^{-10}x_2y + 0.037611\hat{x}_2^2 - 9.1\times10^{-9}\hat{x}_2y - 1.2\times10^{-12}y^2$$

$$\Pi_{112}^{13}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 4.5\times10^{-18}x_2^2 - 0.337857x_2\hat{x}_2 - 2.5\times10^{-26}x_2y - 0.030428\hat{x}_2^2 + 1.7\times10^{-7}\hat{x}_2y - 3.2\times10^{-13}y^2$$

$$\Pi_{112}^{23}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 3.3\times10^{-19}x_2^2 - 0.084252x_2\hat{x}_2 + 4.5\times10^{-26}x_2y - 0.005402\hat{x}_2^2 + 1.1\times10^{-7}\hat{x}_2y - 6.0\times10^{-13}y^2,$$

$$\Pi_{112}^{33}(\tilde{\boldsymbol{x}}) = 80.6358846\hat{x}_2^2$$

$$\Pi_{112}^{43}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 2.2\times10^{-19}x_2^2 + 0.009472x_2\hat{x}_2 - 2.6\times10^{-25}x_2y - 0.078796\hat{x}_2^2 + 3\times10^{-25}\hat{x}_2y - 5\times10^{-14}y^2$$

$$\Pi_{112}^{14}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.030336x_2^2 + 0.180511x_2\hat{x}_2 - 7.3\times10^{-10}x_2y - 0.033088\hat{x}_2^2 - 1.5\times10^{-8}\hat{x}_2y - 2.6\times10^{-12}y^2$$

$$\Pi_{112}^{24}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -0.283388x_2^2 - 0.736161x_2\hat{x}_2 - 4.5\times10^{-10}x_2y + 0.037611\hat{x}_2^2 - 9.1\times10^{-9}\hat{x}_2y - 1.2\times10^{-12}y^2$$

$$\Pi_{112}^{34}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 2.2\times10^{-19}x_2^2 + 0.009472x_2\hat{x}_2 - 2.6\times10^{-25}x_2y - 0.078796\hat{x}_2^2 + 3\times10^{-25}\hat{x}_2y - 5\times10^{-14}y^2$$

$$\Pi_{112}^{44}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 89.731961x_2^2 + 0.247605x_2\hat{x}_2 + 92.043385\hat{x}_2^2,$$

$$\Pi_{121}^{11}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 91.911283x_2^2 - 0.146212x_2\hat{x}_2 - 1.0\times10^{-8}x_2y + 92.393640\hat{x}_2^2 - 1.2\times10^{-8}\hat{x}_2y + 66.821423y^2,$$

$$\Pi_{121}^{21}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.050466x_2^2 + 0.026066x_2\hat{x}_2 - 5.7\times10^{-9}x_2y + 0.026420\hat{x}_2^2 - 1.4\times10^{-8}\hat{x}_2y + 41.285280y^2,$$

$$\Pi_{121}^{31}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 8\times10^{-18}x_2^2 - 0.126049x_2\hat{x}_2 + 5\times10^{-27}x_2y - 0.005706\hat{x}_2^2 - 4.7\times10^{-9}\hat{x}_2y - 4.9\times10^{-11}y^2,$$

$$\Pi_{121}^{41}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.010687x_2^2 + 0.060349x_2\hat{x}_2 + 5.3\times10^{-9}x_2y - 0.011182\hat{x}_2^2 - 2.3\times10^{-7}\hat{x}_2y - 6.7\times10^{-13}y^2,$$

$$\Pi_{121}^{12}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.050466x_2^2 + 0.026066x_2\hat{x}_2 - 5.7\times10^{-9}x_2y + 0.026420\hat{x}_2^2 - 1.4\times10^{-8}\hat{x}_2y + 41.285280y^2,$$

$$\Pi_{121}^{22}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 92.336492x_2^2 - 0.076820x_2\hat{x}_2 - 3.1\times10^{-9}x_2y + 92.474212\hat{x}_2^2 - 1.2\times10^{-8}\hat{x}_2y + 25.507945y^2,$$

$$\Pi_{121}^{32}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 2.8\times10^{-20}x_2^2 - 0.031930x_2\hat{x}_2 + 2.2\times10^{-27}x_2y - 0.004061\hat{x}_2^2 - 3\times10^{-9}\hat{x}_2y - 3\times10^{-12}y^2,$$

$$\Pi_{121}^{42}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -0.109591x_2^2 - 0.227151x_2\hat{x}_2 + 3.3\times10^{-9}x_2y - 0.004862\hat{x}_2^2 - 1.4\times10^{-7}\hat{x}_2y - 6.8\times10^{-10}y^2,$$

$$\Pi_{121}^{13}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 8\times10^{-18}x_2^2 - 0.126049x_2\hat{x}_2 + 5\times10^{-27}x_2y - 0.005706\hat{x}_2^2 - 4.7\times10^{-9}\hat{x}_2y - 4.9\times10^{-11}y^2,$$

$$\Pi_{121}^{23}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 2.8\times10^{-20}x_2^2 - 0.031930x_2\hat{x}_2 + 2.2\times10^{-27}x_2y - 0.004061\hat{x}_2^2 - 3\times10^{-9}\hat{x}_2y - 2.9\times10^{-12}y^2,$$

$$\Pi_{121}^{33}(\tilde{\boldsymbol{x}}) = 88.11266024029439\hat{x}_2^2,$$

$$\Pi_{121}^{43}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -2\times 10^{-19}x_2^2 + 0.00336x_2\hat{x}_2 - 5\times 10^{-26}x_2y - 0.027861\hat{x}_2^2$$
$$- 2\times 10^{-25}\hat{x}_2y - 6.4\times 10^{-12}y^2,$$
$$\Pi_{121}^{14}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.010687x_2^2 + 0.060349x_2\hat{x}_2 + 5.3\times 10^{-9}x_2y - 0.011182\hat{x}_2^2$$
$$- 2.3\times 10^{-7}\hat{x}_2y - 6.7\times 10^{-13}y^2,$$
$$\Pi_{121}^{24}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -0.109591x_2^2 - 0.227151x_2\hat{x}_2 + 3.3\times 10^{-9}x_2y - 0.004862\hat{x}_2^2$$
$$- 1.4\times 10^{-7}\hat{x}_2y - 6.8\times 10^{-10}y^2,$$
$$\Pi_{121}^{34}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -2\times 10^{-19}x_2^2 + 0.00336x_2\hat{x}_2 - 5\times 10^{-26}x_2y - 0.027861\hat{x}_2^2$$
$$- 2\times 10^{-25}\hat{x}_2y - 6.4\times 10^{-12}y^2,$$
$$\Pi_{121}^{44}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 91.6350536x_2^2 + 0.0402900x_2\hat{x}_2 + 92.3860823\hat{x}_2^2,$$
$$\Pi_{122}^{11}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 91.911527x_2^2 - 0.146250x_2\hat{x}_2 - 7.1\times 10^{-9}x_2y + 92.393616\hat{x}_2^2$$
$$- 2.5\times 10^{-8}\hat{x}_2y + 66.700773y^2,$$
$$\Pi_{122}^{21}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.050696x_2^2 + 0.026177x_2\hat{x}_2 - 4.7\times 10^{-10}x_2y + 0.026532\hat{x}_2^2$$
$$- 1.3\times 10^{-8}\hat{x}_2y + 41.346242y^2,$$
$$\Pi_{122}^{31}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -5\times 10^{-18}x_2^2 - 0.126347x_2\hat{x}_2 - 1\times 10^{-26}x_2y - 0.005735\hat{x}_2^2$$
$$- 1.1\times 10^{-9}\hat{x}_2y - 3\times 10^{-13}y^2,$$
$$\Pi_{122}^{41}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.010757x_2^2 + 0.060627x_2\hat{x}_2 - 3.5\times 10^{-9}x_2y - 0.011226\hat{x}_2^2$$
$$+ 1.1\times 10^{-7}\hat{x}_2y - 6.3\times 10^{-13}y^2,$$
$$\Pi_{122}^{12}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.050696x_2^2 + 0.026177x_2\hat{x}_2 - 4.7\times 10^{-10}x_2y + 0.026532\hat{x}_2^2$$
$$- 1.3\times 10^{-8}\hat{x}_2y + 41.346242y^2,$$
$$\Pi_{122}^{22}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 92.336443x_2^2 - 0.076838x_2\hat{x}_2 + 2.1\times 10^{-9}x_2y + 92.474182\hat{x}_2^2$$
$$- 6.2\times 10^{-9}\hat{x}_2y + 25.629608y^2,$$
$$\Pi_{122}^{32}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -4\times 10^{-20}x_2^2 - 0.03208x_2\hat{x}_2 + 1\times 10^{-25}x_2y - 0.004079\hat{x}_2^2$$
$$- 6.5\times 10^{-10}\hat{x}_2y + 2.2\times 10^{-12}y^2,$$
$$\Pi_{122}^{42}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -0.109626x_2^2 - 0.227207x_2\hat{x}_2 - 2.1\times 10^{-9}x_2y - 0.004857\hat{x}_2^2$$
$$+ 6.9\times 10^{-8}\hat{x}_2y - 4.6\times 10^{-13}y^2,$$
$$\Pi_{122}^{13}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -5\times 10^{-18}x_2^2 - 0.126347x_2\hat{x}_2 - 1.4\times 10^{-26}x_2y - 0.005735\hat{x}_2^2$$
$$- 1\times 10^{-9}\hat{x}_2y - 3\times 10^{-13}y^2,$$

$$\Pi_{122}^{23}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -4\times10^{-20}x_2^2 - 0.03208x_2\hat{x}_2 + 1\times10^{-25}x_2y - 0.004079\hat{x}_2^2 - 6.5\times10^{-10}\hat{x}_2y + 2.2\times10^{-12}y^2,$$

$$\Pi_{122}^{33}(\tilde{\boldsymbol{x}}) = 88.11230558468914\hat{x}_2^2,$$

$$\Pi_{122}^{43}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 2\times10^{-20}x_2^2 + 0.003391x_2\hat{x}_2 - 5\times10^{-25}x_2y - 0.027993\hat{x}_2^2 + 2\times10^{-25}\hat{x}_2y - 1.8\times10^{-12}y^2,$$

$$\Pi_{122}^{14}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.010757x_2^2 + 0.060627x_2\hat{x}_2 - 3.5\times10^{-9}x_2y - 0.011226\hat{x}_2^2 + 1.1\times10^{-7}\hat{x}_2y - 6.3\times10^{-13}y^2,$$

$$\Pi_{122}^{24}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -0.109626x_2^2 - 0.227207x_2\hat{x}_2 - 2.1\times10^{-9}x_2y - 0.004857\hat{x}_2^2 + 6.9\times10^{-8}\hat{x}_2y - 4.6\times10^{-13}y^2,$$

$$\Pi_{122}^{34}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 2\times10^{-20}x_2^2 + 0.003391x_2\hat{x}_2 - 5\times10^{-25}x_2y - 0.027993\hat{x}_2^2 + 2\times10^{-25}\hat{x}_2y - 1.8\times10^{-12}y^2,$$

$$\Pi_{122}^{44}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 91.6350452x_2^2 + 0.0402912x_2\hat{x}_2 + 92.3860386\hat{x}_2^2,$$

$$\Pi_{211}^{11}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 91.911283x_2^2 - 0.146212x_2\hat{x}_2 - 4.9\times10^{-9}x_2y + 92.393641\hat{x}_2^2 - 9.0\times10^{-9}\hat{x}_2y + 66.821425y^2,$$

$$\Pi_{211}^{21}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.050466x_2^2 + 0.026066x_2\hat{x}_2 - 2.4\times10^{-9}x_2y + 0.0264196\hat{x}_2^2 - 1.4\times10^{-8}\hat{x}_2y + 41.285281y^2,$$

$$\Pi_{211}^{31}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -3\times10^{-18}x_2^2 - 0.126049x_2\hat{x}_2 - 3\times10^{-27}x_2y - 0.005706\hat{x}_2^2 + 6\times10^{-9}\hat{x}_2y - 6\times10^{-11}y^2,$$

$$\Pi_{211}^{41}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.0106879x_2^2 + 0.060349x_2\hat{x}_2 - 5.5\times10^{-9}x_2y - 0.011182\hat{x}_2^2 - 2.2\times10^{-7}\hat{x}_2y - 1.5\times10^{-13}y^2,$$

$$\Pi_{211}^{12}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.050466x_2^2 + 0.026066x_2\hat{x}_2 - 2.4\times10^{-9}x_2y + 0.0264196\hat{x}_2^2 - 1.4\times10^{-8}\hat{x}_2y + 41.285281y^2,$$

$$\Pi_{211}^{22}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 92.336490x_2^2 - 0.076820x_2\hat{x}_2 - 1.2\times10^{-9}x_2y + 92.474211\hat{x}_2^2 - 1.3\times10^{-8}\hat{x}_2y + 25.507946y^2,$$

$$\Pi_{211}^{32}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 2\times10^{-19}x_2^2 - 0.031929x_2\hat{x}_2 + 1\times10^{-26}x_2y - 0.004061\hat{x}_2^2 + 4\times10^{-9}\hat{x}_2y + 7\times10^{-13}y^2,$$

$$\Pi_{211}^{42}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -0.109591x_2^2 - 0.227151x_2\hat{x}_2 - 3.4\times10^{-9}x_2y - 0.004863\hat{x}_2^2 - 1.4\times10^{-7}\hat{x}_2y - 3.2\times10^{-10}y^2,$$

$$\Pi_{211}^{13}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -3\times10^{-18}x_2^2 - 0.126049x_2\hat{x}_2 - 3\times10^{-27}x_2y - 0.005706\hat{x}_2^2 + 6\times10^{-9}\hat{x}_2y - 6\times10^{-11}y^2,$$

$$\Pi_{211}^{23}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 2\times10^{-19}x_2^2 - 0.031929x_2\hat{x}_2 + 1\times10^{-26}x_2y - 0.004061\hat{x}_2^2 + 4\times10^{-9}\hat{x}_2y + 7\times10^{-13}y^2,$$

$$\Pi_{211}^{33}(\tilde{\boldsymbol{x}}) = 88.1126600024803\hat{x}_2^2,$$

$$\Pi_{211}^{43}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -1\times10^{-19}x_2^2 + 0.003360x_2\hat{x}_2 - 2\times10^{-25}x_2y - 0.027862\hat{x}_2^2 - 2\times10^{-25}\hat{x}_2y - 3\times10^{-11}y^2,$$

$$\Pi_{211}^{14}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.0106879x_2^2 + 0.060349x_2\hat{x}_2 - 5.5\times10^{-9}x_2y - 0.011182\hat{x}_2^2 - 2.2\times10^{-7}\hat{x}_2y - 1.5\times10^{-13}y^2,$$

$$\Pi_{211}^{24}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -0.109591x_2^2 - 0.227151x_2\hat{x}_2 - 3.4\times10^{-9}x_2y - 0.004863\hat{x}_2^2 - 1.4\times10^{-7}\hat{x}_2y - 3.2\times10^{-10}y^2,$$

$$\Pi_{211}^{34}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -1\times10^{-19}x_2^2 + 0.003360x_2\hat{x}_2 - 2\times10^{-25}x_2y - 0.027862\hat{x}_2^2 - 2\times10^{-25}\hat{x}_2y - 3\times10^{-11}y^2,$$

$$\Pi_{211}^{44}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 91.6350524x_2^2 + 0.0402902x_2\hat{x}_2 + 92.3860800\hat{x}_2^2,$$

$$\Pi_{212}^{11}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 91.911527x_2^2 - 0.146250x_2\hat{x}_2 + 3.7\times10^{-9}x_2y + 92.393616\hat{x}_2^2 - 7.9\times10^{-9}\hat{x}_2y + 66.700773y^2,$$

$$\Pi_{212}^{21}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.050696x_2^2 + 0.026177x_2\hat{x}_2 - 3.0\times10^{-9}x_2y + 0.026531\hat{x}_2^2 - 1.2\times10^{-9}\hat{x}_2y + 41.346242y^2,$$

$$\Pi_{212}^{31}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -8\times10^{-18}x_2^2 - 0.126347x_2\hat{x}_2 - 3\times10^{-26}x_2y - 0.005735\hat{x}_2^2 + 2\times10^{-9}\hat{x}_2y - 2\times10^{-12}y^2,$$

$$\Pi^{41}_{212}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.010757x_2^2 + 0.060627x_2\hat{x}_2 + 8.7\times10^{-9}x_2y - 0.011226\hat{x}_2^2 + 1.1\times10^{-7}\hat{x}_2y + 8.0\times10^{-13}y^2,$$

$$\Pi^{12}_{212}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.050696x_2^2 + 0.026177x_2\hat{x}_2 - 3.0\times10^{-9}x_2y + 0.026531\hat{x}_2^2 - 1.2\times10^{-9}\hat{x}_2y + 41.346242y^2,$$

$$\Pi^{22}_{212}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 92.336444x_2^2 - 0.076839x_2\hat{x}_2 - 5.1\times10^{-9}x_2y + 92.474182\hat{x}_2^2 + 1.5\times10^{-9}\hat{x}_2y + 25.629608y^2,$$

$$\Pi^{32}_{212}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 3\times10^{-20}x_2^2 - 0.032080x_2\hat{x}_2 + 5\times10^{-26}x_2y - 0.004080\hat{x}_2^2 + 2\times10^{-9}\hat{x}_2y - 2\times10^{-14}y^2,$$

$$\Pi^{42}_{212}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -0.109626x_2^2 - 0.227207x_2\hat{x}_2 + 5.4\times10^{-9}x_2y - 0.004858\hat{x}_2^2 + 6.6\times10^{-8}\hat{x}_2y - 5.6\times10^{-12}y^2,$$

$$\Pi^{13}_{212}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -8\times10^{-18}x_2^2 - 0.126347x_2\hat{x}_2 - 3\times10^{-26}x_2y - 0.005735\hat{x}_2^2 + 2\times10^{-9}\hat{x}_2y - 2\times10^{-12}y^2,$$

$$\Pi^{23}_{212}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 3\times10^{-20}x_2^2 - 0.032080x_2\hat{x}_2 + 5\times10^{-26}x_2y - 0.004080\hat{x}_2^2 + 2\times10^{-9}\hat{x}_2y - 2\times10^{-14}y^2,$$

$$\Pi^{33}_{212}(\tilde{\boldsymbol{x}}) = 88.11230527390046\hat{x}_2^2,$$

$$\Pi^{43}_{212}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 3\times10^{-19}x_2^2 + 0.003391x_2\hat{x}_2 - 9\times10^{-26}x_2y - 0.027994\hat{x}_2^2 - 3\times10^{-25}\hat{x}_2y - 8\times10^{-13}y^2,$$

$$\Pi^{14}_{212}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.010757x_2^2 + 0.060627x_2\hat{x}_2 + 8.7\times10^{-9}x_2y - 0.011226\hat{x}_2^2 + 1.1\times10^{-7}\hat{x}_2y + 8.0\times10^{-13}y^2,$$

$$\Pi^{24}_{212}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -0.109626x_2^2 - 0.227207x_2\hat{x}_2 + 5.4\times10^{-9}x_2y - 0.004858\hat{x}_2^2 + 6.6\times10^{-8}\hat{x}_2y - 5.6\times10^{-12}y^2,$$

$$\Pi^{34}_{212}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 3\times10^{-19}x_2^2 + 0.003391x_2\hat{x}_2 - 9\times10^{-26}x_2y - 0.027994\hat{x}_2^2 - 3\times10^{-25}\hat{x}_2y - 8\times10^{-13}y^2,$$

$$\Pi^{44}_{212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 91.6350437x_2^2 + 0.0402915x_2\hat{x}_2 + 92.3860391\hat{x}_2^2,$$

$$\Pi^{11}_{221}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 90.633450x_2^2 - 0.442078x_2\hat{x}_2 - 8.9\times10^{-9}x_2y + 92.115985\hat{x}_2^2 - 5.0\times10^{-8}\hat{x}_2y + 66.477090y^2,$$

$$\Pi_{221}^{21}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.149400x_2^2 + 0.076135x_2\hat{x}_2 + 4.1\times10^{-11}x_2y + 0.077626\hat{x}_2^2 + 3.0\times10^{-7}\hat{x}_2y + 41.072603y^2,$$

$$\Pi_{221}^{31}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 3\times10^{-17}x_2^2 - 0.337092x_2\hat{x}_2 + 2\times10^{-26}x_2y - 0.030375\hat{x}_2^2 + 5\times10^{-7}\hat{x}_2y - 3\times10^{-10}y^2,$$

$$\Pi_{221}^{41}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.030138x_2^2 + 0.179682x_2\hat{x}_2 - 2\times10^{-9}x_2y - 0.032945hatx_2^2 + 4\times10^{-7}\hat{x}_2y + 3\times10^{-11}y^2,$$

$$\Pi_{221}^{12}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.149400x_2^2 + 0.076135x_2\hat{x}_2 + 4.1\times10^{-11}x_2y + 0.077626\hat{x}_2^2 + 3.0\times10^{-7}\hat{x}_2y + 41.072603y^2,$$

$$\Pi_{221}^{22}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 91.943092x_2^2 - 0.236815x_2\hat{x}_2 + 3.5\times10^{-9}x_2y + 92.337995\hat{x}_2^2 + 3.9\times10^{-7}\hat{x}_2y + 25.376574y^2,$$

$$\Pi_{221}^{32}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 7\times10^{-9}x_2^2 - 0.083848x_2\hat{x}_2 + 4.6\times10^{-27}x_2y - 0.005374\hat{x}_2^2 + 3\times10^{-7}\hat{x}_2y + 1\times10^{-12}y^2,$$

$$\Pi_{221}^{42}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -0.283324x_2^2 - 0.735999x_2\hat{x}_2 - 9\times10^{-10}x_2y + 0.037574\hat{x}_2^2 + 3\times10^{-7}\hat{x}_2y - 2\times10^{-10}y^2,$$

$$\Pi_{221}^{13}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 3\times10^{-17}x_2^2 - 0.337092x_2\hat{x}_2 + 2\times10^{-26}x_2y - 0.030375\hat{x}_2^2 + 5\times10^{-7}\hat{x}_2y - 3\times10^{-10}y^2,$$

$$\Pi_{221}^{23}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 7\times10^{-9}x_2^2 - 0.083848x_2\hat{x}_2 + 5\times10^{-27}x_2y - 0.005374\hat{x}_2^2 + 3\times10^{-7}\hat{x}_2y + 1.3\times10^{-12}y^2,$$

$$\Pi_{221}^{33}(\tilde{\boldsymbol{x}}) = 80.63678499777244\hat{x}_2^2,$$

$$\Pi_{221}^{43}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -6\times10^{-19}x_2^2 + 0.009411x_2\hat{x}_2 - 3\times10^{-25}x_2y - 0.078422\hat{x}_2^2 - 2\times10^{-24}\hat{x}_2y - 2\times10^{-11}y^2,$$

$$\Pi_{221}^{14}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.030138x_2^2 + 0.179682x_2\hat{x}_2 - 2\times10^{-9}x_2y - 0.032945hatx_2^2 + 4\times10^{-7}\hat{x}_2y + 3\times10^{-11}y^2,$$

$$\Pi_{221}^{24}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -0.283324x_2^2 - 0.735999x_2\hat{x}_2 - 9\times10^{-10}x_2y + 0.037574\hat{x}_2^2 + 3\times10^{-7}\hat{x}_2y - 2\times10^{-10}y^2,$$

$$\Pi_{221}^{34}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -6\times10^{-19}x_2^2 + 0.009411x_2\hat{x}_2 - 3\times10^{-25}x_2y - 0.078422\hat{x}_2^2 - 2\times10^{-24}\hat{x}_2y - 2\times10^{-11}y^2,$$

$$\Pi_{221}^{44}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 89.7319487x_2^2 + 0.2475572x_2\hat{x}_2 + 92.0435201\hat{x}_2^2,$$

$$\Pi_{222}^{11}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 90.634224x_2^2 - 0.442203x_2\hat{x}_2 + 2.8\times10^{-9}x_2y + 92.115917\hat{x}_2^2 + 1.6\times10^{-8}\hat{x}_2y + 66.358397y^2,$$

$$\Pi_{222}^{21}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.150078x_2^2 + 0.076462x_2\hat{x}_2 + 2.0\times10^{-8}x_2y + 0.077955\hat{x}_2^2 - 1.7\times10^{-7}\hat{x}_2y + 41.134062y^2,$$

$$\Pi_{222}^{31}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -1\times10^{-17}x_2^2 - 0.337858x_2\hat{x}_2 - 3\times10^{-26}x_2y - 0.030430\hat{x}_2^2 + 9\times10^{-7}\hat{x}_2y - 3\times10^{-13}y^2,$$

$$\Pi_{222}^{41}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.030336x_2^2 + 0.180508x_2\hat{x}_2 + 4.1\times10^{-9}x_2y - 0.033086\hat{x}_2^2 + 3.2\times10^{-7}\hat{x}_2y + 1.3\times10^{-12}y^2,$$

$$\Pi_{222}^{12}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.150078x_2^2 + 0.076462x_2\hat{x}_2 + 2.0\times10^{-8}x_2y + 0.077955\hat{x}_2^2 - 1.7\times10^{-7}\hat{x}_2y + 41.134062y^2,$$

$$\Pi_{222}^{22}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 91.942971x_2^2 - 0.236877x_2\hat{x}_2 + 1.4\times10^{-8}x_2y + 92.337922\hat{x}_2^2 - 2.2\times10^{-7}\hat{x}_2y + 25.498102y^2,$$

$$\Pi_{222}^{32}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -5\times10^{-19}x_2^2 - 0.084252x_2\hat{x}_2 + 1\times10^{-25}x_2y - 0.005399\hat{x}_2^2 + 6\times10^{-7}\hat{x}_2y - 3\times10^{-13}y^2,$$

$$\Pi_{222}^{42}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -0.283385x_2^2 - 0.736156x_2\hat{x}_2 + 2.6\times10^{-9}x_2y + 0.037614\hat{x}_2^2 + 2.0\times10^{-7}\hat{x}_2y - 4.4\times10^{-12}y^2,$$

$$\Pi^{13}_{222}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -1\times 10^{-17}x_2^2 - 0.337858x_2\hat{x}_2 - 3\times 10^{-26}x_2y - 0.030430\hat{x}_2^2 + 9\times 10^{-7}\hat{x}_2y - 3\times 10^{-13}y^2,$$

$$\Pi^{23}_{222}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -5\times 10^{-19}x_2^2 - 0.084252x_2\hat{x}_2 + 1\times 10^{-25}x_2y - 0.005399\hat{x}_2^2 + 6\times 10^{-7}\hat{x}_2y - 3\times 10^{-13}y^2,$$

$$\Pi^{33}_{222}(\tilde{\boldsymbol{x}}) = 80.63590073574424\hat{x}_2^2,$$

$$\Pi^{43}_{222}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -2\times 10^{-19}x_2^2 + 0.009471x_2\hat{x}_2 - 2\times 10^{-25}x_2y - 0.078797\hat{x}_2^2 + 5\times 10^{-25}\hat{x}_2y - 2\times 10^{-12}y^2,$$

$$\Pi^{14}_{222}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = 0.030336x_2^2 + 0.180508x_2\hat{x}_2 + 4.1\times 10^{-9}x_2y - 0.033086\hat{x}_2^2 + 3.2\times 10^{-7}\hat{x}_2y + 1.3\times 10^{-12}y^2,$$

$$\Pi^{24}_{222}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -0.283385x_2^2 - 0.736156x_2\hat{x}_2 + 2.6\times 10^{-9}x_2y + 0.037614\hat{x}_2^2 + 2.0\times 10^{-7}\hat{x}_2y - 4.4\times 10^{-12}y^2,$$

$$\Pi^{34}_{222}(\boldsymbol{x},\boldsymbol{\eta},\tilde{\boldsymbol{x}}) = -2\times 10^{-19}x_2^2 + 0.009471x_2\hat{x}_2 - 2\times 10^{-25}x_2y - 0.078797\hat{x}_2^2 + 5\times 10^{-25}\hat{x}_2y - 2\times 10^{-12}y^2,$$

$$\Pi^{44}_{222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 89.7319671x_2^2 + 0.24760438x_2\hat{x}_2 + 92.0433883\hat{x}_2^2$$

6.5 Appendix C

The feasible solutions of design example class III with unmeasurable premise variable in Chapter 5.2.1, i.e., λ_{ijksl} and $\mathbf{\Pi}_{ijkl}(\boldsymbol{x}, \tilde{\boldsymbol{x}})$, are as follows.

$$\lambda_{11111} = 596.2603127541606 \qquad \lambda_{11112} = 5.570827578441173$$

$$\lambda_{11121} = 1189.823767639005 \qquad \lambda_{11122} = 5664.161341526748$$

$$\lambda_{11211} = 596.2603078287702 \qquad \lambda_{11212} = 12.04256622767515$$

$$\lambda_{11221} = 674.5375115904262 \qquad \lambda_{11222} = 5664.161341526748$$

$$\lambda_{12111} = 596.2603056345661 \qquad \lambda_{12112} = 12.04256622767515$$

$$\lambda_{12121} = 674.5374970261983 \qquad \lambda_{12122} = 5664.161341526748$$

$$\lambda_{12211} = 596.2603073465334 \qquad \lambda_{12212} = 5.575579507489558$$

$$\lambda_{12221} = 1185.526889552348 \qquad \lambda_{12222} = 5664.161341526748$$

$$\lambda_{21111} = 596.2603076447876 \qquad \lambda_{21112} = 5.24219049367808$$

$$\lambda_{21121} = 1190.058515305776 \qquad \lambda_{21122} = 5664.161341526748$$

$$\lambda_{21211} = 596.2603064810264 \qquad \lambda_{21212} = 11.30521562563347$$

$$\lambda_{21221} = 674.1881197420444 \qquad \lambda_{21222} = 5664.161341526748$$

$$\lambda_{22111} = 596.2603064642117 \qquad \lambda_{22112} = 11.30521562563347$$

$$\lambda_{22121} = 674.1881201309479 \qquad \lambda_{22122} = 5664.161341526748$$

$$\lambda_{22211} = 596.2603079310484 \qquad \lambda_{22212} = 5.247353586972825$$

$$\lambda_{22221} = 1185.761516243532 \qquad \lambda_{22222} = 5664.161341526748$$

$$\Pi_{1111}^{11}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 26849.296x_2^4 + 517.616x_2^3\tilde{x}_1 + 988.703x_2^3\tilde{x}_2 + 28136.278x_2^2\tilde{x}_1^2 - 721.144x_2^2\tilde{x}_1\tilde{x}_2 + 20284.644x_2^2\tilde{x}_2^2 + 447.985x_2\tilde{x}_1^3 + 798.011x_2\tilde{x}_1^2\tilde{x}_2 - 440.769x_2\tilde{x}_1\tilde{x}_2^2 + 411.139x_2\tilde{x}_2^3 + 32871.788\tilde{x}_1^4 - 1345.143\tilde{x}_1^3\tilde{x}_2 + 20975.689\tilde{x}_1^2\tilde{x}_2^2 - 1723.100\tilde{x}_1\tilde{x}_2^3 + 23205.780\tilde{x}_2^4$$

$$\Pi_{1111}^{21}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -577.934x_2^4 + 979.045x_2^3\tilde{x}_1 - 573.646x_2^3\tilde{x}_2 - 463.476x_2^2\tilde{x}_1^2 + 2209.726x_2^2\tilde{x}_1\tilde{x}_2 - 1409.302x_2^2\tilde{x}_2^2 + 533.975x_2\tilde{x}_1^3 - 401.274x_2\tilde{x}_1^2\tilde{x}_2 + 561.409x_2\tilde{x}_1\tilde{x}_2^2 - 443.589x_2\tilde{x}_2^3 - 772.992\tilde{x}_1^4 + 2206.002\tilde{x}_1^3\tilde{x}_2 - 2440.470\tilde{x}_1^2\tilde{x}_2^2 + 1903.148\tilde{x}_1\tilde{x}_2^3 - 1165.006\tilde{x}_2^4$$

$$\Pi_{1111}^{31}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -4538.987x_2^4 - 754.564x_2^3\tilde{x}_1 - 769.902x_2^3\tilde{x}_2 - 8412.219x_2^2\tilde{x}_1^2 - 170.221x_2^2\tilde{x}_1\tilde{x}_2 - 1010.408x_2^2\tilde{x}_2^2 - 326.750x_2\tilde{x}_1^3 - 636.735x_2\tilde{x}_1^2\tilde{x}_2 + 111.016x_2\tilde{x}_1\tilde{x}_2^2 - 213.941x_2\tilde{x}_2^3 - 5390.247\tilde{x}_1^4 + 324.872\tilde{x}_1^3\tilde{x}_2 - 1280.408\tilde{x}_1^2\tilde{x}_2^2 + 876.570\tilde{x}_1\tilde{x}_2^3 - 574.232\tilde{x}_2^4$$

$$\Pi_{1111}^{41}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 620.027x_2^4 + 7647.864x_2^3\tilde{x}_1 - 467.841x_2^3\tilde{x}_2 + 1385.561x_2^2\tilde{x}_1^2 - 916.740x_2^2\tilde{x}_1\tilde{x}_2 + 749.330x_2^2\tilde{x}_2^2 + 6625.841x_2\tilde{x}_1^3 - 320.946x_2\tilde{x}_1^2\tilde{x}_2 + 1106.182x_2\tilde{x}_1\tilde{x}_2^2 - 636.478x_2\tilde{x}_2^3 + 1042.212\tilde{x}_1^4 - 1668.384\tilde{x}_1^3\tilde{x}_2 + 2040.665\tilde{x}_1^2\tilde{x}_2^2 - 1494.641\tilde{x}_1\tilde{x}_2^3 + 894.558\tilde{x}_2^4$$

$$\Pi_{1111}^{12}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -577.934x_2^4 + 979.045x_2^3\tilde{x}_1 - 573.646x_2^3\tilde{x}_2 - 463.476x_2^2\tilde{x}_1^2 + 2209.726x_2^2\tilde{x}_1\tilde{x}_2 - 1409.302x_2^2\tilde{x}_2^2 + 533.975x_2\tilde{x}_1^3 - 401.274x_2\tilde{x}_1^2\tilde{x}_2 + 561.409x_2\tilde{x}_1\tilde{x}_2^2 - 443.589x_2\tilde{x}_2^3 - 772.992\tilde{x}_1^4 + 2206.002\tilde{x}_1^3\tilde{x}_2 - 2440.470\tilde{x}_1^2\tilde{x}_2^2 + 1903.148\tilde{x}_1\tilde{x}_2^3 - 1165.006\tilde{x}_2^4$$

$$\Pi_{1111}^{22}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 25023.369x_2^4 - 603.039x_2^3\tilde{x}_1 + 1470.896x_2^3\tilde{x}_2 + 20426.711x_2^2\tilde{x}_1^2 - 1559.955x_2^2\tilde{x}_1\tilde{x}_2 + 21181.245x_2^2\tilde{x}_2^2 - 326.093x_2\tilde{x}_1^3 + 600.638x_2\tilde{x}_1^2\tilde{x}_2 - 640.908x_2\tilde{x}_1\tilde{x}_2^2 + 541.160x_2\tilde{x}_2^3 + 23139.162\tilde{x}_1^4 - 1652.013\tilde{x}_1^3\tilde{x}_2 + 21574.331\tilde{x}_1^2\tilde{x}_2^2 - 2119.680\tilde{x}_1\tilde{x}_2^3 + 23506.621\tilde{x}_2^4$$

$$\Pi_{1111}^{32}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -1951.710x_2^4 - 705.861x_2^3\tilde{x}_1 - 483.210x_2^3\tilde{x}_2 - 64.944x_2^2\tilde{x}_1^2 - 1579.561x_2^2\tilde{x}_1\tilde{x}_2 + 166.200x_2^2\tilde{x}_2^2 - 376.038x_2\tilde{x}_1^3 + 115.564x_2\tilde{x}_1^2\tilde{x}_2 - 278.587x_2\tilde{x}_1\tilde{x}_2^2 + 45.819x_2\tilde{x}_2^3 + 186.198\tilde{x}_1^4 - 1094.090\tilde{x}_1^3\tilde{x}_2 + 1372.421\tilde{x}_1^2\tilde{x}_2^2 - 1032.584\tilde{x}_1\tilde{x}_2^3 + 1076.508\tilde{x}_2^4$$

$$\Pi_{1111}^{42}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 1179.572x_2^4 - 423.610x_2^3\tilde{x}_1 + 3748.952x_2^3\tilde{x}_2 - 483.183x_2^2\tilde{x}_1^2 + 884.966x_2^2\tilde{x}_1\tilde{x}_2 - 670.012x_2^2\tilde{x}_2^2 - 158.830x_2\tilde{x}_1^3 + 1526.672x_2\tilde{x}_1^2\tilde{x}_2 - 1193.956x_2\tilde{x}_1\tilde{x}_2^2 + 1091.152x_2\tilde{x}_2^3 - 787.003\tilde{x}_1^4 + 1405.513\tilde{x}_1^3\tilde{x}_2 - 2015.892\tilde{x}_1^2\tilde{x}_2^2 + 1663.527\tilde{x}_1\tilde{x}_2^3 - 1024.988\tilde{x}_2^4$$

$$\Pi^{13}_{1111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -4538.987x_2^4 - 754.564x_2^3\tilde{x}_1 - 769.902x_2^3\tilde{x}_2 - 8412.219x_2^2\tilde{x}_1^2 - 170.221x_2^2\tilde{x}_1\tilde{x}_2 - 1010.408x_2^2\tilde{x}_2^2 - 326.750x_2\tilde{x}_1^3 - 636.735x_2\tilde{x}_1^2\tilde{x}_2 + 111.016x_2\tilde{x}_1\tilde{x}_2^2 - 213.941x_2\tilde{x}_2^3 - 5390.247\tilde{x}_1^4 + 324.872\tilde{x}_1^3\tilde{x}_2 - 1280.408\tilde{x}_1^2\tilde{x}_2^2 + 876.570\tilde{x}_1\tilde{x}_2^3 - 574.232\tilde{x}_2^4$$

$$\Pi^{23}_{1111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -1951.710x_2^4 - 705.861x_2^3\tilde{x}_1 - 483.210x_2^3\tilde{x}_2 - 64.944x_2^2\tilde{x}_1^2 - 1579.561x_2^2\tilde{x}_1\tilde{x}_2 + 166.200x_2^2\tilde{x}_2^2 - 376.038x_2\tilde{x}_1^3 + 115.564x_2\tilde{x}_1^2\tilde{x}_2 - 278.587x_2\tilde{x}_1\tilde{x}_2^2 + 45.819x_2\tilde{x}_2^3 + 186.198\tilde{x}_1^4 - 1094.090\tilde{x}_1^3\tilde{x}_2 + 1372.421\tilde{x}_1^2\tilde{x}_2^2 - 1032.584\tilde{x}_1\tilde{x}_2^3 + 1076.508\tilde{x}_2^4$$

$$\Pi^{33}_{1111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 41602.321x_2^4 + 490.159x_2^3\tilde{x}_1 + 1578.084x_2^3\tilde{x}_2 + 36231.938x_2^2\tilde{x}_1^2 + 762.588x_2^2\tilde{x}_1\tilde{x}_2 + 30999.639x_2^2\tilde{x}_2^2 + 359.727x_2\tilde{x}_1^3 + 524.685x_2\tilde{x}_1^2\tilde{x}_2 + 148.197x_2\tilde{x}_1\tilde{x}_2^2 - 227.463x_2\tilde{x}_2^3 + 34611.834\tilde{x}_1^4 + 315.232\tilde{x}_1^3\tilde{x}_2 + 27738.642\tilde{x}_1^2\tilde{x}_2^2 - 368.268\tilde{x}_1\tilde{x}_2^3 + 35978.075\tilde{x}_2^4$$

$$\Pi^{43}_{1111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -1527.688x_2^4 - 6850.347x_2^3\tilde{x}_1 - 2438.756x_2^3\tilde{x}_2 - 961.891x_2^2\tilde{x}_1^2 + 536.145x_2^2\tilde{x}_1\tilde{x}_2 - 700.763x_2^2\tilde{x}_2^2 - 5334.003x_2\tilde{x}_1^3 - 265.818x_2\tilde{x}_1^2\tilde{x}_2 - 1007.577x_2\tilde{x}_1\tilde{x}_2^2 - 28.441x_2\tilde{x}_2^3 - 352.271\tilde{x}_1^4 + 692.819\tilde{x}_1^3\tilde{x}_2 - 1231.579\tilde{x}_1^2\tilde{x}_2^2 + 827.153\tilde{x}_1\tilde{x}_2^3 - 1049.553\tilde{x}_2^4$$

$$\Pi^{14}_{1111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 620.027x_2^4 + 7647.864x_2^3\tilde{x}_1 - 467.841x_2^3\tilde{x}_2 + 1385.561x_2^2\tilde{x}_1^2 - 916.740x_2^2\tilde{x}_1\tilde{x}_2 + 749.330x_2^2\tilde{x}_2^2 + 6625.841x_2\tilde{x}_1^3 - 320.946x_2\tilde{x}_1^2\tilde{x}_2 + 1106.182x_2\tilde{x}_1\tilde{x}_2^2 - 636.478x_2\tilde{x}_2^3 + 1042.212\tilde{x}_1^4 - 1668.384\tilde{x}_1^3\tilde{x}_2 + 2040.665\tilde{x}_1^2\tilde{x}_2^2 - 1494.641\tilde{x}_1\tilde{x}_2^3 + 894.558\tilde{x}_2^4$$

$$\Pi^{24}_{1111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 1179.572x_2^4 - 423.610x_2^3\tilde{x}_1 + 3748.952x_2^3\tilde{x}_2 - 483.183x_2^2\tilde{x}_1^2 + 884.966x_2^2\tilde{x}_1\tilde{x}_2 - 670.012x_2^2\tilde{x}_2^2 - 158.830x_2\tilde{x}_1^3 + 1526.672x_2\tilde{x}_1^2\tilde{x}_2 - 1193.956x_2\tilde{x}_1\tilde{x}_2^2 + 1091.152x_2\tilde{x}_2^3 - 787.003\tilde{x}_1^4 + 1405.513\tilde{x}_1^3\tilde{x}_2 - 2015.892\tilde{x}_1^2\tilde{x}_2^2 + 1663.527\tilde{x}_1\tilde{x}_2^3 - 1024.988\tilde{x}_2^4$$

$$\Pi^{34}_{1111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -1527.688x_2^4 - 6850.347x_2^3\tilde{x}_1 - 2438.756x_2^3\tilde{x}_2 - 961.891x_2^2\tilde{x}_1^2 + 536.145x_2^2\tilde{x}_1\tilde{x}_2 - 700.763x_2^2\tilde{x}_2^2 - 5334.003x_2\tilde{x}_1^3 - 265.818x_2\tilde{x}_1^2\tilde{x}_2 - 1007.577x_2\tilde{x}_1\tilde{x}_2^2 - 28.441x_2\tilde{x}_2^3 - 352.271\tilde{x}_1^4 + 692.819\tilde{x}_1^3\tilde{x}_2 - 1231.579\tilde{x}_1^2\tilde{x}_2^2 + 827.153\tilde{x}_1\tilde{x}_2^3 - 1049.553\tilde{x}_2^4$$

$$\Pi^{44}_{1111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 38171.276x_2^4 + 1700.503x_2^3\tilde{x}_1 - 2962.117x_2^3\tilde{x}_2 + 28627.545x_2^2\tilde{x}_1^2 - 1692.052x_2^2\tilde{x}_1\tilde{x}_2 + 23260.000x_2^2\tilde{x}_2^2 + 1125.381x_2\tilde{x}_1^3 - 2456.913x_2\tilde{x}_1^2\tilde{x}_2 + 2148.451x_2\tilde{x}_1\tilde{x}_2^2 - 1511.851x_2\tilde{x}_2^3 + 26190.463\tilde{x}_1^4 - 1513.536\tilde{x}_1^3\tilde{x}_2 + 21762.774\tilde{x}_1^2\tilde{x}_2^2 - 2208.324\tilde{x}_1\tilde{x}_2^3 + 23691.397\tilde{x}_2^4$$

$$\Pi^{11}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 3160.740x_2^4 + 393.679x_2^3\tilde{x}_1 + 250.775x_2^3\tilde{x}_2 + 4418.959x_2^2\tilde{x}_1^2 + 129.511x_2^2\tilde{x}_1\tilde{x}_2 + 2274.192x_2^2\tilde{x}_2^2 + 337.840x_2\tilde{x}_1^3 + 268.620x_2\tilde{x}_1^2\tilde{x}_2 + 79.610x_2\tilde{x}_1\tilde{x}_2^2 + 43.19x_2\tilde{x}_2^3 + 5026.870\tilde{x}_1^4 - 55.922\tilde{x}_1^3\tilde{x}_2 + 2477.934\tilde{x}_1^2\tilde{x}_2^2 - 142.081\tilde{x}_1\tilde{x}_2^3 + 2340.985\tilde{x}_2^4$$

$$\Pi^{21}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 221.621x_2^4 + 187.608x_2^3\tilde{x}_1 + 133.232x_2^3\tilde{x}_2 + 291.688x_2^2\tilde{x}_1^2 + 531.876x_2^2\tilde{x}_1\tilde{x}_2 - 131.568x_2^2\tilde{x}_2^2 + 174.620x_2\tilde{x}_1^3 + 133.000x_2\tilde{x}_1^2\tilde{x}_2 + 26.388x_2\tilde{x}_1\tilde{x}_2^2 + 6.565x_2\tilde{x}_2^3 - 16.88\tilde{x}_1^4 + 592.886\tilde{x}_1^3\tilde{x}_2 - 384.892\tilde{x}_1^2\tilde{x}_2^2 + 233.099\tilde{x}_1\tilde{x}_2^3 - 124.799\tilde{x}_2^4$$

$$\Pi^{31}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -2431.643x_2^4 - 1276.298x_2^3\tilde{x}_1 - 646.610x_2^3\tilde{x}_2 - 5449.455x_2^2\tilde{x}_1^2 - 424.465x_2^2\tilde{x}_1\tilde{x}_2 - 851.529x_2^2\tilde{x}_2^2 - 1161.802x_2\tilde{x}_1^3 - 676.764x_2\tilde{x}_1^2\tilde{x}_2 - 215.264x_2\tilde{x}_1\tilde{x}_2^2 - 116.868x_2\tilde{x}_2^3 - 4287.820\tilde{x}_1^4 - 89.111\tilde{x}_1^3\tilde{x}_2 - 1150.871\tilde{x}_1^2\tilde{x}_2^2 + 141.959\tilde{x}_1\tilde{x}_2^3 - 189.592\tilde{x}_2^4$$

$$\Pi^{41}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 342.727x_2^4 + 1371.502x_2^3\tilde{x}_1 + 165.627x_2^3\tilde{x}_2 + 554.290x_2^2\tilde{x}_1^2 - 248.257x_2^2\tilde{x}_1\tilde{x}_2 + 321.301x_2^2\tilde{x}_2^2 + 1620.902x_2\tilde{x}_1^3 + 130.901x_2\tilde{x}_1^2\tilde{x}_2 + 376.588x_2\tilde{x}_1\tilde{x}_2^2 - 44.474x_2\tilde{x}_2^3 + 226.879\tilde{x}_1^4 - 435.631\tilde{x}_1^3\tilde{x}_2 + 471.669\tilde{x}_1^2\tilde{x}_2^2 - 212.123\tilde{x}_1\tilde{x}_2^3 + 137.888\tilde{x}_2^4$$

$$\Pi^{12}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 221.621x_2^4 + 187.608x_2^3\tilde{x}_1 + 133.232x_2^3\tilde{x}_2 + 291.688x_2^2\tilde{x}_1^2 + 531.876x_2^2\tilde{x}_1\tilde{x}_2 - 131.568x_2^2\tilde{x}_2^2 + 174.620x_2\tilde{x}_1^3 + 133.000x_2\tilde{x}_1^2\tilde{x}_2 + 26.388x_2\tilde{x}_1\tilde{x}_2^2 + 6.565x_2\tilde{x}_2^3 - 16.880\tilde{x}_1^4 + 592.886\tilde{x}_1^3\tilde{x}_2 - 384.892\tilde{x}_1^2\tilde{x}_2^2 + 233.099\tilde{x}_1\tilde{x}_2^3 - 124.799\tilde{x}_2^4$$

$$\Pi^{22}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2486.662x_2^4 + 90.670x_2^3\tilde{x}_1 + 50.926x_2^3\tilde{x}_2 + 2112.275x_2^2\tilde{x}_1^2 + 29.377x_2^2\tilde{x}_1\tilde{x}_2 + 2149.521x_2^2\tilde{x}_2^2 + 7.927x_2\tilde{x}_1^3 + 79.223x_2\tilde{x}_1^2\tilde{x}_2 + 20.576x_2\tilde{x}_1\tilde{x}_2^2 - 40.193x_2\tilde{x}_2^3 + 2327.454\tilde{x}_1^4 - 122.622\tilde{x}_1^3\tilde{x}_2 + 2250.652\tilde{x}_1^2\tilde{x}_2^2 - 286.685\tilde{x}_1\tilde{x}_2^3 + 2394.936\tilde{x}_2^4$$

$$\Pi^{32}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -718.506x_2^4 - 614.571x_2^3\tilde{x}_1 - 351.091x_2^3\tilde{x}_2 - 671.404x_2^2\tilde{x}_1^2 - 1033.132x_2^2\tilde{x}_1\tilde{x}_2 + 191.970x_2^2\tilde{x}_2^2 - 392.822x_2\tilde{x}_1^3 - 358.238x_2\tilde{x}_1^2\tilde{x}_2 - 97.750x_2\tilde{x}_1\tilde{x}_2^2 - 14.500x_2\tilde{x}_2^3 - 8.718\tilde{x}_1^4 - 939.250\tilde{x}_1^3\tilde{x}_2 + 531.537\tilde{x}_1^2\tilde{x}_2^2 - 334.607\tilde{x}_1\tilde{x}_2^3 + 241.475\tilde{x}_2^4$$

$$\Pi^{42}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -96.466x_2^4 + 253.217x_2^3\tilde{x}_1 + 548.119x_2^3\tilde{x}_2 - 17.994x_2^2\tilde{x}_1^2 + 287.284x_2^2\tilde{x}_1\tilde{x}_2 - 293.620x_2^2\tilde{x}_2^2 + 45.799x_2\tilde{x}_1^3 + 481.990x_2\tilde{x}_1^2\tilde{x}_2 - 197.761x_2\tilde{x}_1\tilde{x}_2^2 + 214.187x_2\tilde{x}_2^3 - 95.681\tilde{x}_1^4 + 184.333\tilde{x}_1^3\tilde{x}_2 - 334.180\tilde{x}_1^2\tilde{x}_2^2 + 290.048\tilde{x}_1\tilde{x}_2^3 - 180.051\tilde{x}_2^4$$

$$\Pi^{13}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -2431.643x_2^4 - 1276.298x_2^3\tilde{x}_1 - 646.610x_2^3\tilde{x}_2 - 5449.455x_2^2\tilde{x}_1^2 - 424.465x_2^2\tilde{x}_1\tilde{x}_2 - 851.529x_2^2\tilde{x}_2^2 - 1161.802x_2\tilde{x}_1^3 - 676.764x_2\tilde{x}_1^2\tilde{x}_2 - 215.264x_2\tilde{x}_1\tilde{x}_2^2 - 116.868x_2\tilde{x}_2^3 - 4287.820\tilde{x}_1^4 - 89.11\tilde{x}_1^3\tilde{x}_2 - 1150.871\tilde{x}_1^2\tilde{x}_2^2 + 141.959\tilde{x}_1\tilde{x}_2^3 - 189.592\tilde{x}_2^4$$

$$\Pi^{23}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -718.506x_2^4 - 614.571x_2^3\tilde{x}_1 - 351.091x_2^3\tilde{x}_2 - 671.404x_2^2\tilde{x}_1^2 - 1033.132x_2^2\tilde{x}_1\tilde{x}_2 + 191.970x_2^2\tilde{x}_2^2 - 392.822x_2\tilde{x}_1^3 - 358.238x_2\tilde{x}_1^2\tilde{x}_2 - 97.750x_2\tilde{x}_1\tilde{x}_2^2 - 14.500x_2\tilde{x}_2^3 - 8.718\tilde{x}_1^4 - 939.250\tilde{x}_1^3\tilde{x}_2 + 531.537\tilde{x}_1^2\tilde{x}_2^2 - 334.607\tilde{x}_1\tilde{x}_2^3 + 241.475\tilde{x}_2^4$$

$$\Pi^{33}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 10256.738x_2^4 + 4585.563x_2^3\tilde{x}_1 + 1606.160x_2^3\tilde{x}_2 + 15392.544x_2^2\tilde{x}_1^2 + 1307.607x_2^2\tilde{x}_1\tilde{x}_2 + 4771.824x_2^2\tilde{x}_2^2 + 3602.189x_2\tilde{x}_1^3 + 1670.815x_2\tilde{x}_1^2\tilde{x}_2 + 653.274x_2\tilde{x}_1\tilde{x}_2^2 + 237.275x_2\tilde{x}_2^3 + 11214.409\tilde{x}_1^4 + 489.399\tilde{x}_1^3\tilde{x}_2 + 4728.657\tilde{x}_1^2\tilde{x}_2^2 - 21.890\tilde{x}_1\tilde{x}_2^3 + 3180.534\tilde{x}_2^4$$

$$\Pi^{43}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -963.2265x_2^4 - 2945.723x_2^3\tilde{x}_1 - 575.654x_2^3\tilde{x}_2 - 1563.166x_2^2\tilde{x}_1^2 + 137.016x_2^2\tilde{x}_1\tilde{x}_2 - 664.250x_2^2\tilde{x}_2^2 - 2825.628x_2\tilde{x}_1^3 - 342.558x_2\tilde{x}_1^2\tilde{x}_2 - 667.354x_2\tilde{x}_1\tilde{x}_2^2 - 26.897x_2\tilde{x}_2^3 - 447.027\tilde{x}_1^4 + 564.605\tilde{x}_1^3\tilde{x}_2 - 728.005\tilde{x}_1^2\tilde{x}_2^2 + 253.537\tilde{x}_1\tilde{x}_2^3 - 273.196\tilde{x}_2^4$$

$$\Pi^{14}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 342.727x_2^4 + 1371.502x_2^3\tilde{x}_1 + 165.627x_2^3\tilde{x}_2 + 554.290x_2^2\tilde{x}_1^2 - 248.257x_2^2\tilde{x}_1\tilde{x}_2 + 321.301x_2^2\tilde{x}_2^2 + 1620.902x_2\tilde{x}_1^3 + 130.901x_2\tilde{x}_1^2\tilde{x}_2 + 376.588x_2\tilde{x}_1\tilde{x}_2^2 - 44.474x_2\tilde{x}_2^3 + 226.879\tilde{x}_1^4 - 435.631\tilde{x}_1^3\tilde{x}_2 + 471.669\tilde{x}_1^2\tilde{x}_2^2 - 212.123\tilde{x}_1\tilde{x}_2^3 + 137.888\tilde{x}_2^4$$

$$\Pi^{24}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -96.466x_2^4 + 253.217x_2^3\tilde{x}_1 + 548.119x_2^3\tilde{x}_2 - 17.994x_2^2\tilde{x}_1^2 + 287.284x_2^2\tilde{x}_1\tilde{x}_2 - 293.620x_2^2\tilde{x}_2^2 + 45.799x_2\tilde{x}_1^3 + 481.990x_2\tilde{x}_1^2\tilde{x}_2 - 197.761x_2\tilde{x}_1\tilde{x}_2^2 + 214.187x_2\tilde{x}_2^3 - 95.681\tilde{x}_1^4 + 184.333\tilde{x}_1^3\tilde{x}_2 - 334.180\tilde{x}_1^2\tilde{x}_2^2 + 290.048\tilde{x}_1\tilde{x}_2^3 - 180.051\tilde{x}_2^4$$

$$\Pi^{34}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -963.226x_2^4 - 2945.723x_2^3\tilde{x}_1 - 575.654x_2^3\tilde{x}_2 - 1563.166x_2^2\tilde{x}_1^2 + 137.016x_2^2\tilde{x}_1\tilde{x}_2 - 664.250x_2^2\tilde{x}_2^2 - 2825.628x_2\tilde{x}_1^3 - 342.558x_2\tilde{x}_1^2\tilde{x}_2 - 667.354x_2\tilde{x}_1\tilde{x}_2^2 - 26.897x_2\tilde{x}_2^3 - 447.027\tilde{x}_1^4 + 564.605\tilde{x}_1^3\tilde{x}_2 - 728.005\tilde{x}_1^2\tilde{x}_2^2 + 253.537\tilde{x}_1\tilde{x}_2^3 - 273.196\tilde{x}_2^4$$

$$\Pi^{44}_{1112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 3779.638x_2^4 + 498.278x_2^3\tilde{x}_1 - 874.676x_2^3\tilde{x}_2 + 3753.922x_2^2\tilde{x}_1^2 - 41.345x_2^2\tilde{x}_1\tilde{x}_2 + 2919.574x_2^2\tilde{x}_2^2 + 265.545x_2\tilde{x}_1^3 - 718.696x_2\tilde{x}_1^2\tilde{x}_2 + 536.647x_2\tilde{x}_1\tilde{x}_2^2 - 362.589x_2\tilde{x}_2^3 + 2650.774\tilde{x}_1^4 - 169.482\tilde{x}_1^3\tilde{x}_2 + 2431.452\tilde{x}_1^2\tilde{x}_2^2 - 326.179\tilde{x}_1\tilde{x}_2^3 + 2498.921\tilde{x}_2^4$$

$$\begin{aligned}\Pi^{11}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) =&27017.846x_2^4 + 944.690x_2^3\tilde{x}_1 + 1106.265x_2^3\tilde{x}_2 + 29330.597x_2^2\tilde{x}_1^2 - 339.651x_2^2\tilde{x}_1\tilde{x}_2+\\&20390.553x_2^2\tilde{x}_2^2 + 1169.444x_2\tilde{x}_1^3 + 915.976x_2\tilde{x}_1^2\tilde{x}_2 - 351.149x_2\tilde{x}_1\tilde{x}_2^2 + 359.97x_2\tilde{x}_2^3+\\&41152.86\tilde{x}_1^4 - 1358.984\tilde{x}_1^3\tilde{x}_2 + 20690.535\tilde{x}_1^2\tilde{x}_2^2 - 1480.09\tilde{x}_1\tilde{x}_2^3 + 23099.876\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{21}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) =&-119.612x_2^4 + 1057.889x_2^3\tilde{x}_1 - 342.352x_2^3\tilde{x}_2 - 137.807x_2^2\tilde{x}_1^2 + 2363.184x_2^2\tilde{x}_1\tilde{x}_2-\\&1202.373x_2^2\tilde{x}_2^2 + 633.419x_2\tilde{x}_1^3 - 294.382x_2\tilde{x}_1^2\tilde{x}_2 + 474.633x_2\tilde{x}_1\tilde{x}_2^2 - 361.268x_2\tilde{x}_2^3-\\&811.488\tilde{x}_1^4 + 2487.178\tilde{x}_1^3\tilde{x}_2 - 2258.447\tilde{x}_1^2\tilde{x}_2^2 + 1755.633\tilde{x}_1\tilde{x}_2^3 - 1062.748\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{31}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) =&-8170.111x_2^4 - 3235.576x_2^3\tilde{x}_1 - 1604.575x_2^3\tilde{x}_2 - 15658.489x_2^2\tilde{x}_1^2 - 361.595x_2^2\tilde{x}_1\tilde{x}_2-\\&1025.591x_2^2\tilde{x}_2^2 - 2691.535x_2\tilde{x}_1^3 - 1472.607x_2\tilde{x}_1^2\tilde{x}_2 + 245.81x_2\tilde{x}_1\tilde{x}_2^2 - 394.652x_2\tilde{x}_2^3-\\&9445.379\tilde{x}_1^4 + 200.262\tilde{x}_1^3\tilde{x}_2 - 1599.096\tilde{x}_1^2\tilde{x}_2^2 + 881.357\tilde{x}_1\tilde{x}_2^3 - 601.765\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{41}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) =&952.197x_2^4 + 7559.002x_2^3\tilde{x}_1 + 76.709x_2^3\tilde{x}_2 + 1690.752x_2^2\tilde{x}_1^2 - 955.092x_2^2\tilde{x}_1\tilde{x}_2+\\&777.416x_2^2\tilde{x}_2^2 + 7962.697x_2\tilde{x}_1^3 - 49.504x_2\tilde{x}_1^2\tilde{x}_2 + 1328.497x_2\tilde{x}_1\tilde{x}_2^2 - 556.973x_2\tilde{x}_2^3+\\&1581.804\tilde{x}_1^4 - 1863.474\tilde{x}_1^3\tilde{x}_2 + 1904.823\tilde{x}_1^2\tilde{x}_2^2 - 1408.792\tilde{x}_1\tilde{x}_2^3 + 849.132\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{12}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) =&-119.612x_2^4 + 1057.889x_2^3\tilde{x}_1 - 342.352x_2^3\tilde{x}_2 - 137.807x_2^2\tilde{x}_1^2 + 2363.184x_2^2\tilde{x}_1\tilde{x}_2-\\&1202.373x_2^2\tilde{x}_2^2 + 633.419x_2\tilde{x}_1^3 - 294.382x_2\tilde{x}_1^2\tilde{x}_2 + 474.633x_2\tilde{x}_1\tilde{x}_2^2 - 361.268x_2\tilde{x}_2^3-\\&811.488\tilde{x}_1^4 + 2487.178\tilde{x}_1^3\tilde{x}_2 - 2258.447\tilde{x}_1^2\tilde{x}_2^2 + 1755.633\tilde{x}_1\tilde{x}_2^3 - 1062.748\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{22}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) =&25687.469x_2^4 - 369.78x_2^3\tilde{x}_1 + 1799.173x_2^3\tilde{x}_2 + 20439.058x_2^2\tilde{x}_1^2 - 1240.982x_2^2\tilde{x}_1\tilde{x}_2+\\&21777.206x_2^2\tilde{x}_2^2 - 268.126x_2\tilde{x}_1^3 + 526.251x_2\tilde{x}_1^2\tilde{x}_2 - 518.388x_2\tilde{x}_1\tilde{x}_2^2 + 537.23x_2\tilde{x}_2^3+\\&23144.124\tilde{x}_1^4 - 1416.337\tilde{x}_1^3\tilde{x}_2 + 21380.306\tilde{x}_1^2\tilde{x}_2^2 - 1961.506\tilde{x}_1\tilde{x}_2^3 + 23607.702\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{32}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) =&-1820.329x_2^4 - 1540.43x_2^3\tilde{x}_1 - 204.917x_2^3\tilde{x}_2 - 440.366x_2^2\tilde{x}_1^2 - 1951.157x_2^2\tilde{x}_1\tilde{x}_2+\\&629.057x_2^2\tilde{x}_2^2 - 898.643x_2\tilde{x}_1^3 + 97.447x_2\tilde{x}_1^2\tilde{x}_2 - 519.264x_2\tilde{x}_1\tilde{x}_2^2 + 188.547x_2\tilde{x}_2^3+\\&136.759\tilde{x}_1^4 - 1646.533\tilde{x}_1^3\tilde{x}_2 + 1586.272\tilde{x}_1^2\tilde{x}_2^2 - 1058.486\tilde{x}_1\tilde{x}_2^3 + 1584.483\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{42}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) =&1178.702x_2^4 + 122.186x_2^3\tilde{x}_1 + 4581.035x_2^3\tilde{x}_2 - 456.178x_2^2\tilde{x}_1^2 + 908.59x_2^2\tilde{x}_1\tilde{x}_2-\\&942.795x_2^2\tilde{x}_2^2 - 12.194x_2\tilde{x}_1^3 + 1714.404x_2\tilde{x}_1^2\tilde{x}_2 - 1056.883x_2\tilde{x}_1\tilde{x}_2^2 + 1544.697x_2\tilde{x}_2^3-\\&805.851\tilde{x}_1^4 + 1259.571\tilde{x}_1^3\tilde{x}_2 - 1911.826\tilde{x}_1^2\tilde{x}_2^2 + 1586.733\tilde{x}_1\tilde{x}_2^3 - 1121.686\tilde{x}_2^4\end{aligned}$$

$$\Pi^{13}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) = -8170.111x_2^4 - 3235.576x_2^3\tilde{x}_1 - 1604.575x_2^3\tilde{x}_2 - 15658.489x_2^2\tilde{x}_1^2 - 361.595x_2^2\tilde{x}_1\tilde{x}_2 - 1025.591x_2^2\tilde{x}_2^2 - 2691.535x_2\tilde{x}_1^3 - 1472.607x_2\tilde{x}_1^2\tilde{x}_2 + 245.81x_2\tilde{x}_1\tilde{x}_2^2 - 394.652x_2\tilde{x}_2^3 - 9445.379\tilde{x}_1^4 + 200.262\tilde{x}_1^3\tilde{x}_2 - 1599.096\tilde{x}_1^2\tilde{x}_2^2 + 881.357\tilde{x}_1\tilde{x}_2^3 - 601.765\tilde{x}_2^4$$

$$\Pi^{23}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) = -1820.329x_2^4 - 1540.43x_2^3\tilde{x}_1 - 204.917x_2^3\tilde{x}_2 - 440.366x_2^2\tilde{x}_1^2 - 1951.157x_2^2\tilde{x}_1\tilde{x}_2 + 629.057x_2^2\tilde{x}_2^2 - 898.643x_2\tilde{x}_1^3 + 97.447x_2\tilde{x}_1^2\tilde{x}_2 - 519.264x_2\tilde{x}_1\tilde{x}_2^2 + 188.547x_2\tilde{x}_2^3 + 136.759\tilde{x}_1^4 - 1646.533\tilde{x}_1^3\tilde{x}_2 + 1586.272\tilde{x}_1^2\tilde{x}_2^2 - 1058.486\tilde{x}_1\tilde{x}_2^3 + 1584.483\tilde{x}_2^4$$

$$\Pi^{33}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) = 71146.788x_2^4 + 16898.321x_2^3\tilde{x}_1 + 4048.398x_2^3\tilde{x}_2 + 68779.91x_2^2\tilde{x}_1^2 + 903.644x_2^2\tilde{x}_1\tilde{x}_2 + 31735.879x_2^2\tilde{x}_2^2 + 8389.315x_2\tilde{x}_1^3 + 4337.756x_2\tilde{x}_1^2\tilde{x}_2 - 879.488x_2\tilde{x}_1\tilde{x}_2^2 - 611.688x_2\tilde{x}_2^3 + 59978.708\tilde{x}_1^4 - 583.177\tilde{x}_1^3\tilde{x}_2 + 28837.328\tilde{x}_1^2\tilde{x}_2^2 + 533.634\tilde{x}_1\tilde{x}_2^3 + 37677.988\tilde{x}_2^4$$

$$\Pi^{43}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) = -2941.975x_2^4 - 10159.505x_2^3\tilde{x}_1 - 2208.48x_2^3\tilde{x}_2 - 3987.198x_2^2\tilde{x}_1^2 - 182.562x_2^2\tilde{x}_1\tilde{x}_2 - 618.366x_2^2\tilde{x}_2^2 - 9537.852x_2\tilde{x}_1^3 - 610.944x_2\tilde{x}_1^2\tilde{x}_2 - 742.938x_2\tilde{x}_1\tilde{x}_2^2 + 67.587x_2\tilde{x}_2^3 - 1604.401\tilde{x}_1^4 + 700.16\tilde{x}_1^3\tilde{x}_2 - 1269.343\tilde{x}_1^2\tilde{x}_2^2 + 683.668\tilde{x}_1\tilde{x}_2^3 - 1488.294\tilde{x}_2^4$$

$$\Pi^{14}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) = 952.197x_2^4 + 7559.002x_2^3\tilde{x}_1 + 76.709x_2^3\tilde{x}_2 + 1690.752x_2^2\tilde{x}_1^2 - 955.092x_2^2\tilde{x}_1\tilde{x}_2 + 777.416x_2^2\tilde{x}_2^2 + 7962.697x_2\tilde{x}_1^3 - 49.504x_2\tilde{x}_1^2\tilde{x}_2 + 1328.497x_2\tilde{x}_1\tilde{x}_2^2 - 556.973x_2\tilde{x}_2^3 + 1581.804\tilde{x}_1^4 - 1863.474\tilde{x}_1^3\tilde{x}_2 + 1904.823\tilde{x}_1^2\tilde{x}_2^2 - 1408.792\tilde{x}_1\tilde{x}_2^3 + 849.132\tilde{x}_2^4$$

$$\Pi^{24}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) = 1178.702x_2^4 + 122.186x_2^3\tilde{x}_1 + 4581.035x_2^3\tilde{x}_2 - 456.178x_2^2\tilde{x}_1^2 + 908.59x_2^2\tilde{x}_1\tilde{x}_2 - 942.795x_2^2\tilde{x}_2^2 - 12.194x_2\tilde{x}_1^3 + 1714.404x_2\tilde{x}_1^2\tilde{x}_2 - 1056.883x_2\tilde{x}_1\tilde{x}_2^2 + 1544.697x_2\tilde{x}_2^3 - 805.851\tilde{x}_1^4 + 1259.571\tilde{x}_1^3\tilde{x}_2 - 1911.826\tilde{x}_1^2\tilde{x}_2^2 + 1586.733\tilde{x}_1\tilde{x}_2^3 - 1121.686\tilde{x}_2^4$$

$$\Pi^{34}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) = -2941.975x_2^4 - 10159.505x_2^3\tilde{x}_1 - 2208.48x_2^3\tilde{x}_2 - 3987.198x_2^2\tilde{x}_1^2 - 182.562x_2^2\tilde{x}_1\tilde{x}_2 - 618.366x_2^2\tilde{x}_2^2 - 9537.852x_2\tilde{x}_1^3 - 610.944x_2\tilde{x}_1^2\tilde{x}_2 - 742.938x_2\tilde{x}_1\tilde{x}_2^2 + 67.587x_2\tilde{x}_2^3 - 1604.401\tilde{x}_1^4 + 700.16\tilde{x}_1^3\tilde{x}_2 - 1269.343\tilde{x}_1^2\tilde{x}_2^2 + 683.668\tilde{x}_1\tilde{x}_2^3 - 1488.294\tilde{x}_2^4$$

$$\Pi^{44}_{1121}(\boldsymbol{x}.\tilde{\boldsymbol{x}}) = 38093.91x_2^4 + 1598.177x_2^3\tilde{x}_1 - 3959.58x_2^3\tilde{x}_2 + 28439.331x_2^2\tilde{x}_1^2 - 901.771x_2^2\tilde{x}_1\tilde{x}_2 + 24716.995x_2^2\tilde{x}_2^2 + 1235.809x_2\tilde{x}_1^3 - 2658.165x_2\tilde{x}_1^2\tilde{x}_2 + 2028.118x_2\tilde{x}_1\tilde{x}_2^2 - 2132.933x_2\tilde{x}_2^3 + 26568.986\tilde{x}_1^4 - 1234.178\tilde{x}_1^3\tilde{x}_2 + 21792.682\tilde{x}_1^2\tilde{x}_2^2 - 2048.265\tilde{x}_1\tilde{x}_2^3 + 24036.54\tilde{x}_2^4$$

$$\begin{aligned}
\Pi^{11}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&2596.379x_2^4 + 303.039x_2^3\tilde{x}_1 + 135.967x_2^3\tilde{x}_2 + 2792.1x_2^2\tilde{x}_1^2 + 143.523x_2^2\tilde{x}_1\tilde{x}_2+\\
&2022.071x_2^2\tilde{x}_2^2 + 293.581x_2\tilde{x}_1^3 + 158.415x_2\tilde{x}_1^2\tilde{x}_2 + 41.844x_2\tilde{x}_1\tilde{x}_2^2 + 16.884x_2\tilde{x}_2^3+\\
&2998.341\tilde{x}_1^4 + 48.946\tilde{x}_1^3\tilde{x}_2 + 2022.906\tilde{x}_1^2\tilde{x}_2^2 - 39.864\tilde{x}_1\tilde{x}_2^3 + 2261.421\tilde{x}_2^4\\
\Pi^{21}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&110.673x_2^4 + 130.594x_2^3\tilde{x}_1 + 75.544x_2^3\tilde{x}_2 + 166.584x_2^2\tilde{x}_1^2 + 202.626x_2^2\tilde{x}_1\tilde{x}_2-\\
&16.344x_2^2\tilde{x}_2^2 + 86.851x_2\tilde{x}_1^3 + 83.802x_2\tilde{x}_1^2\tilde{x}_2 + 27.165x_2\tilde{x}_1\tilde{x}_2^2 + 1.334x_2\tilde{x}_2^3+\\
&26.045\tilde{x}_1^4 + 146.307\tilde{x}_1^3\tilde{x}_2 - 73.307\tilde{x}_1^2\tilde{x}_2^2 + 65.718\tilde{x}_1\tilde{x}_2^3 - 32.843\tilde{x}_2^4\\
\Pi^{31}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&-975.297x_2^4 - 946.598x_2^3\tilde{x}_1 - 357.132x_2^3\tilde{x}_2 - 2160.121x_2^2\tilde{x}_1^2 - 393.368x_2^2\tilde{x}_1\tilde{x}_2-\\
&225.765x_2^2\tilde{x}_2^2 - 856.576x_2\tilde{x}_1^3 - 418.628x_2\tilde{x}_1^2\tilde{x}_2 - 117.332x_2\tilde{x}_1\tilde{x}_2^2 - 39.837x_2\tilde{x}_2^3-\\
&1129.663\tilde{x}_1^4 - 163.429\tilde{x}_1^3\tilde{x}_2 - 230.145\tilde{x}_1^2\tilde{x}_2^2 + 30.983\tilde{x}_1\tilde{x}_2^3 - 41.274\tilde{x}_2^4\\
\Pi^{41}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&183.347x_2^4 + 521.668x_2^3\tilde{x}_1 + 102.412x_2^3\tilde{x}_2 + 368.837x_2^2\tilde{x}_1^2 + 17.727x_2^2\tilde{x}_1\tilde{x}_2+\\
&91.686x_2^2\tilde{x}_2^2 + 506.941x_2\tilde{x}_1^3 + 101.227x_2\tilde{x}_1^2\tilde{x}_2 + 113.753x_2\tilde{x}_1\tilde{x}_2^2 - 8.872x_2\tilde{x}_2^3+\\
&116.842\tilde{x}_1^4 - 64.977\tilde{x}_1^3\tilde{x}_2 + 111.522\tilde{x}_1^2\tilde{x}_2^2 - 52.171\tilde{x}_1\tilde{x}_2^3 + 34.414\tilde{x}_2^4\\
\Pi^{12}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&110.673x_2^4 + 130.594x_2^3\tilde{x}_1 + 75.544x_2^3\tilde{x}_2 + 166.584x_2^2\tilde{x}_1^2 + 202.626x_2^2\tilde{x}_1\tilde{x}_2-\\
&16.344x_2^2\tilde{x}_2^2 + 86.851x_2\tilde{x}_1^3 + 83.802x_2\tilde{x}_1^2\tilde{x}_2 + 27.165x_2\tilde{x}_1\tilde{x}_2^2 + 1.334x_2\tilde{x}_2^3+\\
&26.045\tilde{x}_1^4 + 146.307\tilde{x}_1^3\tilde{x}_2 - 73.307\tilde{x}_1^2\tilde{x}_2^2 + 65.718\tilde{x}_1\tilde{x}_2^3 - 32.843\tilde{x}_2^4\\
\Pi^{22}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&2343.491x_2^4 + 56x_2^3\tilde{x}_1 + 52.499x_2^3\tilde{x}_2 + 2009.232x_2^2\tilde{x}_1^2 + 30.404x_2^2\tilde{x}_1\tilde{x}_2+\\
&1999.619x_2^2\tilde{x}_2^2 + 15.699x_2\tilde{x}_1^3 + 46.07x_2\tilde{x}_1^2\tilde{x}_2 + 11.192x_2\tilde{x}_1\tilde{x}_2^2 + 0.163x_2\tilde{x}_2^3+\\
&2267.429\tilde{x}_1^4 - 27.843\tilde{x}_1^3\tilde{x}_2 + 2006.954\tilde{x}_1^2\tilde{x}_2^2 - 70.913\tilde{x}_1\tilde{x}_2^3 + 2273.578\tilde{x}_2^4\\
\Pi^{32}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&-350.08x_2^4 - 383.155x_2^3\tilde{x}_1 - 209.885x_2^3\tilde{x}_2 - 443.275x_2^2\tilde{x}_1^2 - 414.401x_2^2\tilde{x}_1\tilde{x}_2-\\
&3.335x_2^2\tilde{x}_2^2 - 232.546x_2\tilde{x}_1^3 - 233.212x_2\tilde{x}_1^2\tilde{x}_2 - 60.833x_2\tilde{x}_1\tilde{x}_2^2 - 8.755x_2\tilde{x}_2^3-\\
&77.496\tilde{x}_1^4 - 251.711\tilde{x}_1^3\tilde{x}_2 + 82.154\tilde{x}_1^2\tilde{x}_2^2 - 80.533\tilde{x}_1\tilde{x}_2^3 + 56.778\tilde{x}_2^4\\
\Pi^{42}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&24.438x_2^4 + 133.066x_2^3\tilde{x}_1 + 192.564x_2^3\tilde{x}_2 + 59.399x_2^2\tilde{x}_1^2 + 123.301x_2^2\tilde{x}_1\tilde{x}_2-\\
&54.218x_2^2\tilde{x}_2^2 + 54.867x_2\tilde{x}_1^3 + 150.686x_2\tilde{x}_1^2\tilde{x}_2 - 29.659x_2\tilde{x}_1\tilde{x}_2^2 + 60.851x_2\tilde{x}_2^3-\\
&17.85\tilde{x}_1^4 + 63.659\tilde{x}_1^3\tilde{x}_2 - 79.044\tilde{x}_1^2\tilde{x}_2^2 + 72.297\tilde{x}_1\tilde{x}_2^3 - 43.429\tilde{x}_2^4
\end{aligned}$$

$$\Pi^{13}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -975.297x_2^4 - 946.598x_2^3\tilde{x}_1 - 357.132x_2^3\tilde{x}_2 - 2160.121x_2^2\tilde{x}_1^2 - 393.368x_2^2\tilde{x}_1\tilde{x}_2 -$$
$$225.765x_2^2\tilde{x}_2^2 - 856.576x_2\tilde{x}_1^3 - 418.628x_2\tilde{x}_1^2\tilde{x}_2 - 117.332x_2\tilde{x}_1\tilde{x}_2^2 - 39.837x_2\tilde{x}_2^3 -$$
$$1129.663\tilde{x}_1^4 - 163.429\tilde{x}_1^3\tilde{x}_2 - 230.145\tilde{x}_1^2\tilde{x}_2^2 + 30.983\tilde{x}_1\tilde{x}_2^3 - 41.274\tilde{x}_2^4$$

$$\Pi^{23}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -350.08x_2^4 - 383.155x_2^3\tilde{x}_1 - 209.885x_2^3\tilde{x}_2 - 443.275x_2^2\tilde{x}_1^2 - 414.401x_2^2\tilde{x}_1\tilde{x}_2 -$$
$$3.335x_2^2\tilde{x}_2^2 - 232.546x_2\tilde{x}_1^3 - 233.212x_2\tilde{x}_1^2\tilde{x}_2 - 60.833x_2\tilde{x}_1\tilde{x}_2^2 - 8.755x_2\tilde{x}_2^3 -$$
$$77.496\tilde{x}_1^4 - 251.711\tilde{x}_1^3\tilde{x}_2 + 82.154\tilde{x}_1^2\tilde{x}_2^2 - 80.533\tilde{x}_1\tilde{x}_2^3 + 56.778\tilde{x}_2^4$$

$$\Pi^{33}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 5645.645x_2^4 + 3263.526x_2^3\tilde{x}_1 + 943.734x_2^3\tilde{x}_2 + 8147.656x_2^2\tilde{x}_1^2 + 1104.153x_2^2\tilde{x}_1\tilde{x}_2 +$$
$$2674.779x_2^2\tilde{x}_2^2 + 2660.942x_2\tilde{x}_1^3 + 1120.413x_2\tilde{x}_1^2\tilde{x}_2 + 299.717x_2\tilde{x}_1\tilde{x}_2^2 + 69.534x_2\tilde{x}_2^3 +$$
$$5319.224\tilde{x}_1^4 + 474.997\tilde{x}_1^3\tilde{x}_2 + 2530.818\tilde{x}_1^2\tilde{x}_2^2 + 31.553\tilde{x}_1\tilde{x}_2^3 + 2529.078\tilde{x}_2^4$$

$$\Pi^{43}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -553.247x_2^4 - 1231.282x_2^3\tilde{x}_1 - 307.72x_2^3\tilde{x}_2 - 1090.139x_2^2\tilde{x}_1^2 - 181.257x_2^2\tilde{x}_1\tilde{x}_2 -$$
$$195.063x_2^2\tilde{x}_2^2 - 1049.604x_2\tilde{x}_1^3 - 250.744x_2\tilde{x}_1^2\tilde{x}_2 - 191.878x_2\tilde{x}_1\tilde{x}_2^2 - 14.7x_2\tilde{x}_2^3 -$$
$$291.142\tilde{x}_1^4 + 35.478\tilde{x}_1^3\tilde{x}_2 - 183.749\tilde{x}_1^2\tilde{x}_2^2 + 46.896\tilde{x}_1\tilde{x}_2^3 - 63.475\tilde{x}_2^4$$

$$\Pi^{14}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 183.347x_2^4 + 521.668x_2^3\tilde{x}_1 + 102.412x_2^3\tilde{x}_2 + 368.837x_2^2\tilde{x}_1^2 + 17.727x_2^2\tilde{x}_1\tilde{x}_2 +$$
$$91.686x_2^2\tilde{x}_2^2 + 506.941x_2\tilde{x}_1^3 + 101.227x_2\tilde{x}_1^2\tilde{x}_2 + 113.753x_2\tilde{x}_1\tilde{x}_2^2 - 8.872x_2\tilde{x}_2^3 +$$
$$116.842\tilde{x}_1^4 - 64.977\tilde{x}_1^3\tilde{x}_2 + 111.522\tilde{x}_1^2\tilde{x}_2^2 - 52.171\tilde{x}_1\tilde{x}_2^3 + 34.414\tilde{x}_2^4$$

$$\Pi^{24}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 24.438x_2^4 + 133.066x_2^3\tilde{x}_1 + 192.564x_2^3\tilde{x}_2 + 59.399x_2^2\tilde{x}_1^2 + 123.301x_2^2\tilde{x}_1\tilde{x}_2 -$$
$$54.218x_2^2\tilde{x}_2^2 + 54.867x_2\tilde{x}_1^3 + 150.686x_2\tilde{x}_1^2\tilde{x}_2 - 29.659x_2\tilde{x}_1\tilde{x}_2^2 + 60.851x_2\tilde{x}_2^3 -$$
$$17.85\tilde{x}_1^4 + 63.659\tilde{x}_1^3\tilde{x}_2 - 79.044\tilde{x}_1^2\tilde{x}_2^2 + 72.297\tilde{x}_1\tilde{x}_2^3 - 43.429\tilde{x}_2^4$$

$$\Pi^{34}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -553.247x_2^4 - 1231.282x_2^3\tilde{x}_1 - 307.72x_2^3\tilde{x}_2 - 1090.139x_2^2\tilde{x}_1^2 - 181.257x_2^2\tilde{x}_1\tilde{x}_2 -$$
$$195.063x_2^2\tilde{x}_2^2 - 1049.604x_2\tilde{x}_1^3 - 250.744x_2\tilde{x}_1^2\tilde{x}_2 - 191.878x_2\tilde{x}_1\tilde{x}_2^2 - 14.7x_2\tilde{x}_2^3 -$$
$$291.142\tilde{x}_1^4 + 35.478\tilde{x}_1^3\tilde{x}_2 - 183.749\tilde{x}_1^2\tilde{x}_2^2 + 46.896\tilde{x}_1\tilde{x}_2^3 - 63.475\tilde{x}_2^4$$

$$\Pi^{44}_{1122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2736.57x_2^4 + 263.014x_2^3\tilde{x}_1 - 180.952x_2^3\tilde{x}_2 + 2535.761x_2^2\tilde{x}_1^2 + 55.907x_2^2\tilde{x}_1\tilde{x}_2 +$$
$$2190.304x_2^2\tilde{x}_2^2 + 157.572x_2\tilde{x}_1^3 - 126.977x_2\tilde{x}_1^2\tilde{x}_2 + 138.93x_2\tilde{x}_1\tilde{x}_2^2 - 92.726x_2\tilde{x}_2^3 +$$
$$2405.098\tilde{x}_1^4 - 31.346\tilde{x}_1^3\tilde{x}_2 + 2057.305\tilde{x}_1^2\tilde{x}_2^2 - 81.756\tilde{x}_1\tilde{x}_2^3 + 2299.553\tilde{x}_2^4$$

$$\Pi^{11}_{1211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 36164.768x_2^4 + 743.678x_2^3\tilde{x}_1 + 2710.283x_2^3\tilde{x}_2 + 46744.657x_2^2\tilde{x}_1^2 - 2799.523x_2^2\tilde{x}_1\tilde{x}_2 + 22998.747x_2^2\tilde{x}_2^2 + 373.255x_2\tilde{x}_1^3 + 2088.335x_2\tilde{x}_1^2\tilde{x}_2 - 892.601x_2\tilde{x}_1\tilde{x}_2^2 + 1128.96x_2\tilde{x}_2^3 + 57416.362\tilde{x}_1^4 - 5593.966\tilde{x}_1^3\tilde{x}_2 + 26298.871\tilde{x}_1^2\tilde{x}_2^2 - 5167.327\tilde{x}_1\tilde{x}_2^3 + 25549.047\tilde{x}_2^4$$

$$\Pi^{21}_{1211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -1577.196x_2^4 + 2304.062x_2^3\tilde{x}_1 - 1403.838x_2^3\tilde{x}_2 - 2131.458x_2^2\tilde{x}_1^2 + 6665.962x_2^2\tilde{x}_1\tilde{x}_2 - 4439.655x_2^2\tilde{x}_2^2 + 1451.095x_2\tilde{x}_1^3 - 1012.214x_2\tilde{x}_1^2\tilde{x}_2 + 1259.928x_2\tilde{x}_1\tilde{x}_2^2 - 1130.804x_2\tilde{x}_2^3 - 3526.859\tilde{x}_1^4 + 7972.516\tilde{x}_1^3\tilde{x}_2 - 8402.823\tilde{x}_1^2\tilde{x}_2^2 + 5957.033\tilde{x}_1\tilde{x}_2^3 - 3859.838\tilde{x}_2^4$$

$$\Pi^{31}_{1211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -16549.918x_2^4 - 1267.953x_2^3\tilde{x}_1 - 2289.631x_2^3\tilde{x}_2 - 27981.871x_2^2\tilde{x}_1^2 + 497.064x_2^2\tilde{x}_1\tilde{x}_2 - 4822.432x_2^2\tilde{x}_2^2 - 354.218x_2\tilde{x}_1^3 - 1764.573x_2\tilde{x}_1^2\tilde{x}_2 + 275.32x_2\tilde{x}_1\tilde{x}_2^2 - 834.909x_2\tilde{x}_2^3 - 25064.797\tilde{x}_1^4 + 2363.661\tilde{x}_1^3\tilde{x}_2 - 7045.477\tilde{x}_1^2\tilde{x}_2^2 + 3348.394\tilde{x}_1\tilde{x}_2^3 - 2813.533\tilde{x}_2^4$$

$$\Pi^{41}_{1211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2115.435x_2^4 + 23449.118x_2^3\tilde{x}_1 - 1601.932x_2^3\tilde{x}_2 + 3984.747x_2^2\tilde{x}_1^2 - 3885.405x_2^2\tilde{x}_1\tilde{x}_2 + 3202.173x_2^2\tilde{x}_2^2 + 20537.371x_2\tilde{x}_1^3 - 1867.309x_2\tilde{x}_1^2\tilde{x}_2 + 3866.452x_2\tilde{x}_1\tilde{x}_2^2 - 1858.355x_2\tilde{x}_2^3 + 3780.079\tilde{x}_1^4 - 6501.948\tilde{x}_1^3\tilde{x}_2 + 7403.322\tilde{x}_1^2\tilde{x}_2^2 - 4919.265\tilde{x}_1\tilde{x}_2^3 + 3197.538\tilde{x}_2^4$$

$$\Pi^{12}_{1211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -1577.196x_2^4 + 2304.062x_2^3\tilde{x}_1 - 1403.838x_2^3\tilde{x}_2 - 2131.458x_2^2\tilde{x}_1^2 + 6665.962x_2^2\tilde{x}_1\tilde{x}_2 - 4439.655x_2^2\tilde{x}_2^2 + 1451.095x_2\tilde{x}_1^3 - 1012.214x_2\tilde{x}_1^2\tilde{x}_2 + 1259.928x_2\tilde{x}_1\tilde{x}_2^2 - 1130.804x_2\tilde{x}_2^3 - 3526.859\tilde{x}_1^4 + 7972.516\tilde{x}_1^3\tilde{x}_2 - 8402.823\tilde{x}_1^2\tilde{x}_2^2 + 5957.033\tilde{x}_1\tilde{x}_2^3 - 3859.838\tilde{x}_2^4$$

$$\Pi^{22}_{1211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 29125.003x_2^4 - 1478.658x_2^3\tilde{x}_1 + 3149.247x_2^3\tilde{x}_2 + 22709.278x_2^2\tilde{x}_1^2 - 4191.067x_2^2\tilde{x}_1\tilde{x}_2 + 24927.262x_2^2\tilde{x}_2^2 - 966.729x_2\tilde{x}_1^3 + 1527.297x_2\tilde{x}_1^2\tilde{x}_2 - 1618.543x_2\tilde{x}_1\tilde{x}_2^2 + 1178.122x_2\tilde{x}_2^3 + 25158.544\tilde{x}_1^4 - 5109.681\tilde{x}_1^3\tilde{x}_2 + 26875.78\tilde{x}_1^2\tilde{x}_2^2 - 6945.855\tilde{x}_1\tilde{x}_2^3 + 26809.096\tilde{x}_2^4$$

$$\Pi^{32}_{1211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -4528.514x_2^4 - 1834.077x_2^3\tilde{x}_1 - 623.588x_2^3\tilde{x}_2 + 698.021x_2^2\tilde{x}_1^2 - 5215.64x_2^2\tilde{x}_1\tilde{x}_2 + 1671.798x_2^2\tilde{x}_2^2 - 1177.3x_2\tilde{x}_1^3 + 393.009x_2\tilde{x}_1^2\tilde{x}_2 - 843.299x_2\tilde{x}_1\tilde{x}_2^2 + 349.448x_2\tilde{x}_2^3 + 1867.509\tilde{x}_1^4 - 5047.882\tilde{x}_1^3\tilde{x}_2 + 5808.269\tilde{x}_1^2\tilde{x}_2^2 - 4257.51\tilde{x}_1\tilde{x}_2^3 + 4698.937\tilde{x}_2^4$$

$$\Pi^{42}_{1211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 1254.879x_2^4 - 1465.655x_2^3\tilde{x}_1 + 11052.764x_2^3\tilde{x}_2 - 2113.02x_2^2\tilde{x}_1^2 + 3071.433x_2^2\tilde{x}_1\tilde{x}_2 - 3405.255x_2^2\tilde{x}_2^2 - 1117.176x_2\tilde{x}_1^3 + 5123.321x_2\tilde{x}_1^2\tilde{x}_2 - 3922.566x_2\tilde{x}_1\tilde{x}_2^2 + 3394.477x_2\tilde{x}_2^3 - 2852.528\tilde{x}_1^4 + 4659.563\tilde{x}_1^3\tilde{x}_2 - 6848.7\tilde{x}_1^2\tilde{x}_2^2 + 5761.39\tilde{x}_1\tilde{x}_2^3 - 3835.028\tilde{x}_2^4$$

$$\Pi_{1211}^{13}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -16549.918x_2^4 - 1267.953x_2^3\tilde{x}_1 - 2289.631x_2^3\tilde{x}_2 - 27981.871x_2^2\tilde{x}_1^2 + 497.064x_2^2\tilde{x}_1\tilde{x}_2 - 4822.432x_2^2\tilde{x}_2^2 - 354.218x_2\tilde{x}_1^3 - 1764.573x_2\tilde{x}_1^2\tilde{x}_2 + 275.32x_2\tilde{x}_1\tilde{x}_2^2 - 834.909x_2\tilde{x}_2^3 - 25064.797\tilde{x}_1^4 + 2363.661\tilde{x}_1^3\tilde{x}_2 - 7045.477\tilde{x}_1^2\tilde{x}_2^2 + 3348.394\tilde{x}_1\tilde{x}_2^3 - 2813.533\tilde{x}_2^4$$

$$\Pi_{1211}^{23}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -4528.514x_2^4 - 1834.077x_2^3\tilde{x}_1 - 623.588x_2^3\tilde{x}_2 + 698.021x_2^2\tilde{x}_1^2 - 5215.64x_2^2\tilde{x}_1\tilde{x}_2 + 1671.798x_2^2\tilde{x}_2^2 - 1177.3x_2\tilde{x}_1^3 + 393.009x_2\tilde{x}_1^2\tilde{x}_2 - 843.299x_2\tilde{x}_1\tilde{x}_2^2 + 349.448x_2\tilde{x}_2^3 + 1867.509\tilde{x}_1^4 - 5047.882\tilde{x}_1^3\tilde{x}_2 + 5808.269\tilde{x}_1^2\tilde{x}_2^2 - 4257.51\tilde{x}_1\tilde{x}_2^3 + 4698.937\tilde{x}_2^4$$

$$\Pi_{1211}^{33}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 83094.32x_2^4 + 748.841x_2^3\tilde{x}_1 + 3388.531x_2^3\tilde{x}_2 + 73260.076x_2^2\tilde{x}_1^2 + 1163.269x_2^2\tilde{x}_1\tilde{x}_2 + 56867.135x_2^2\tilde{x}_2^2 + 615.305x_2\tilde{x}_1^3 + 1590.774x_2\tilde{x}_1^2\tilde{x}_2 + 272.705x_2\tilde{x}_1\tilde{x}_2^2 - 341.091x_2\tilde{x}_2^3 + 68366.337\tilde{x}_1^4 - 353.895\tilde{x}_1^3\tilde{x}_2 + 50124.498\tilde{x}_1^2\tilde{x}_2^2 - 1752.337\tilde{x}_1\tilde{x}_2^3 + 66253.906\tilde{x}_2^4$$

$$\Pi_{1211}^{43}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -3203.273x_2^4 - 21952.886x_2^3\tilde{x}_1 - 5642.251x_2^3\tilde{x}_2 - 2877.881x_2^2\tilde{x}_1^2 + 2792.872x_2^2\tilde{x}_1\tilde{x}_2 - 2781.478x_2^2\tilde{x}_2^2 - 17489.582x_2\tilde{x}_1^3 - 136.887x_2\tilde{x}_1^2\tilde{x}_2 - 4394.829x_2\tilde{x}_1\tilde{x}_2^2 + 158.738x_2\tilde{x}_2^3 - 1936.471\tilde{x}_1^4 + 3758.853\tilde{x}_1^3\tilde{x}_2 - 5314.138\tilde{x}_1^2\tilde{x}_2^2 + 3513.898\tilde{x}_1\tilde{x}_2^3 - 4459.707\tilde{x}_2^4$$

$$\Pi_{1211}^{14}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2115.435x_2^4 + 23449.118x_2^3\tilde{x}_1 - 1601.932x_2^3\tilde{x}_2 + 3984.747x_2^2\tilde{x}_1^2 - 3885.405x_2^2\tilde{x}_1\tilde{x}_2 + 3202.173x_2^2\tilde{x}_2^2 + 20537.371x_2\tilde{x}_1^3 - 1867.309x_2\tilde{x}_1^2\tilde{x}_2 + 3866.452x_2\tilde{x}_1\tilde{x}_2^2 - 1858.355x_2\tilde{x}_2^3 + 3780.079\tilde{x}_1^4 - 6501.948\tilde{x}_1^3\tilde{x}_2 + 7403.322\tilde{x}_1^2\tilde{x}_2^2 - 4919.265\tilde{x}_1\tilde{x}_2^3 + 3197.538\tilde{x}_2^4$$

$$\Pi_{1211}^{24}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 1254.879x_2^4 - 1465.655x_2^3\tilde{x}_1 + 11052.764x_2^3\tilde{x}_2 - 2113.02x_2^2\tilde{x}_1^2 + 3071.433x_2^2\tilde{x}_1\tilde{x}_2 - 3405.255x_2^2\tilde{x}_2^2 - 1117.176x_2\tilde{x}_1^3 + 5123.321x_2\tilde{x}_1^2\tilde{x}_2 - 3922.566x_2\tilde{x}_1\tilde{x}_2^2 + 3394.477x_2\tilde{x}_2^3 - 2852.528\tilde{x}_1^4 + 4659.563\tilde{x}_1^3\tilde{x}_2 - 6848.7\tilde{x}_1^2\tilde{x}_2^2 + 5761.39\tilde{x}_1\tilde{x}_2^3 - 3835.028\tilde{x}_2^4$$

$$\Pi_{1211}^{34}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -3203.273x_2^4 - 21952.886x_2^3\tilde{x}_1 - 5642.251x_2^3\tilde{x}_2 - 2877.881x_2^2\tilde{x}_1^2 + 2792.872x_2^2\tilde{x}_1\tilde{x}_2 - 2781.478x_2^2\tilde{x}_2^2 - 17489.582x_2\tilde{x}_1^3 - 136.887x_2\tilde{x}_1^2\tilde{x}_2 - 4394.829x_2\tilde{x}_1\tilde{x}_2^2 + 158.738x_2\tilde{x}_2^3 - 1936.471\tilde{x}_1^4 + 3758.853\tilde{x}_1^3\tilde{x}_2 - 5314.138\tilde{x}_1^2\tilde{x}_2^2 + 3513.898\tilde{x}_1\tilde{x}_2^3 - 4459.707\tilde{x}_2^4$$

$$\Pi_{1211}^{44}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 77060.613x_2^4 + 5411.043x_2^3\tilde{x}_1 - 13081.751x_2^3\tilde{x}_2 + 49224.305x_2^2\tilde{x}_1^2 - 6082.514x_2^2\tilde{x}_1\tilde{x}_2 + 33342.35x_2^2\tilde{x}_2^2 + 3678.188x_2\tilde{x}_1^3 - 8724.398x_2\tilde{x}_1^2\tilde{x}_2 + 7202.424x_2\tilde{x}_1\tilde{x}_2^2 - 5209.47x_2\tilde{x}_2^3 + 33657.501\tilde{x}_1^4 - 5274.73\tilde{x}_1^3\tilde{x}_2 + 28167.379\tilde{x}_1^2\tilde{x}_2^2 - 7024.639\tilde{x}_1\tilde{x}_2^3 + 27433.61\tilde{x}_2^4$$

$$\begin{aligned}\Pi^{11}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&2596.38x_2^4 + 303.04x_2^3\tilde{x}_1 + 135.967x_2^3\tilde{x}_2 + 2792.101x_2^2\tilde{x}_1^2 + 143.523x_2^2\tilde{x}_1\tilde{x}_2+\\ &2022.071x_2^2\tilde{x}_2^2 + 293.581x_2\tilde{x}_1^3 + 158.415x_2\tilde{x}_1^2\tilde{x}_2 + 41.844x_2\tilde{x}_1\tilde{x}_2^2 + 16.884x_2\tilde{x}_2^3+\\ &2998.34\tilde{x}_1^4 + 48.947\tilde{x}_1^3\tilde{x}_2 + 2022.906\tilde{x}_1^2\tilde{x}_2^2 - 39.864\tilde{x}_1\tilde{x}_2^3 + 2261.421\tilde{x}_2^4\\ \Pi^{21}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&110.673x_2^4 + 130.595x_2^3\tilde{x}_1 + 75.544x_2^3\tilde{x}_2 + 166.584x_2^2\tilde{x}_1^2 + 202.626x_2^2\tilde{x}_1\tilde{x}_2-\\ &16.344x_2^2\tilde{x}_2^2 + 86.851x_2\tilde{x}_1^3 + 83.801x_2\tilde{x}_1^2\tilde{x}_2 + 27.165x_2\tilde{x}_1\tilde{x}_2^2 + 1.335x_2\tilde{x}_2^3\\ &+ 26.046\tilde{x}_1^4 + 146.308\tilde{x}_1^3\tilde{x}_2 - 73.307\tilde{x}_1^2\tilde{x}_2^2 + 65.718\tilde{x}_1\tilde{x}_2^3 - 32.843\tilde{x}_2^4\\ \Pi^{31}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&- 975.297x_2^4 - 946.599x_2^3\tilde{x}_1 - 357.132x_2^3\tilde{x}_2 - 2160.121x_2^2\tilde{x}_1^2 - 393.369x_2^2\tilde{x}_1\tilde{x}_2-\\ &225.765x_2^2\tilde{x}_2^2 - 856.577x_2\tilde{x}_1^3 - 418.628x_2\tilde{x}_1^2\tilde{x}_2 - 117.332x_2\tilde{x}_1\tilde{x}_2^2 - 39.837x_2\tilde{x}_2^3-\\ &1129.664\tilde{x}_1^4 - 163.43\tilde{x}_1^3\tilde{x}_2 - 230.145\tilde{x}_1^2\tilde{x}_2^2 + 30.983\tilde{x}_1\tilde{x}_2^3 - 41.274\tilde{x}_2^4\\ \Pi^{41}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&183.347x_2^4 + 521.668x_2^3\tilde{x}_1 + 102.412x_2^3\tilde{x}_2 + 368.837x_2^2\tilde{x}_1^2 + 17.727x_2^2\tilde{x}_1\tilde{x}_2+\\ &91.686x_2^2\tilde{x}_2^2 + 506.941x_2\tilde{x}_1^3 + 101.227x_2\tilde{x}_1^2\tilde{x}_2 + 113.753x_2\tilde{x}_1\tilde{x}_2^2 - 8.872x_2\tilde{x}_2^3+\\ &116.843\tilde{x}_1^4 - 64.977\tilde{x}_1^3\tilde{x}_2 + 111.522\tilde{x}_1^2\tilde{x}_2^2 - 52.171\tilde{x}_1\tilde{x}_2^3 + 34.414\tilde{x}_2^4\\ \Pi^{12}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&110.673x_2^4 + 130.595x_2^3\tilde{x}_1 + 75.544x_2^3\tilde{x}_2 + 166.584x_2^2\tilde{x}_1^2 + 202.626x_2^2\tilde{x}_1\tilde{x}_2-\\ &16.344x_2^2\tilde{x}_2^2 + 86.851x_2\tilde{x}_1^3 + 83.801x_2\tilde{x}_1^2\tilde{x}_2 + 27.165x_2\tilde{x}_1\tilde{x}_2^2 + 1.335x_2\tilde{x}_2^3+\\ &26.046\tilde{x}_1^4 + 146.308\tilde{x}_1^3\tilde{x}_2 - 73.307\tilde{x}_1^2\tilde{x}_2^2 + 65.718\tilde{x}_1\tilde{x}_2^3 - 32.843\tilde{x}_2^4\\ \Pi^{22}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&2343.491x_2^4 + 56.001x_2^3\tilde{x}_1 + 52.499x_2^3\tilde{x}_2 + 2009.231x_2^2\tilde{x}_1^2 + 30.404x_2^2\tilde{x}_1\tilde{x}_2+\\ &1999.618x_2^2\tilde{x}_2^2 + 15.698x_2\tilde{x}_1^3 + 46.07x_2\tilde{x}_1^2\tilde{x}_2 + 11.192x_2\tilde{x}_1\tilde{x}_2^2 + 0.162x_2\tilde{x}_2^3+\\ &2267.429\tilde{x}_1^4 - 27.843\tilde{x}_1^3\tilde{x}_2 + 2006.953\tilde{x}_1^2\tilde{x}_2^2 - 70.913\tilde{x}_1\tilde{x}_2^3 + 2273.578\tilde{x}_2^4\\ \Pi^{32}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&- 350.081x_2^4 - 383.154x_2^3\tilde{x}_1 - 209.884x_2^3\tilde{x}_2 - 443.275x_2^2\tilde{x}_1^2 - 414.401x_2^2\tilde{x}_1\tilde{x}_2-\\ &3.335x_2^2\tilde{x}_2^2 - 232.546x_2\tilde{x}_1^3 - 233.211x_2\tilde{x}_1^2\tilde{x}_2 - 60.833x_2\tilde{x}_1\tilde{x}_2^2 - 8.755x_2\tilde{x}_2^3-\\ &77.497\tilde{x}_1^4 - 251.711\tilde{x}_1^3\tilde{x}_2 + 82.154\tilde{x}_1^2\tilde{x}_2^2 - 80.533\tilde{x}_1\tilde{x}_2^3 + 56.778\tilde{x}_2^4\\ \Pi^{42}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =&24.438x_2^4 + 133.066x_2^3\tilde{x}_1 + 192.564x_2^3\tilde{x}_2 + 59.399x_2^2\tilde{x}_1^2 + 123.301x_2^2\tilde{x}_1\tilde{x}_2-\\ &54.219x_2^2\tilde{x}_2^2 + 54.867x_2\tilde{x}_1^3 + 150.686x_2\tilde{x}_1^2\tilde{x}_2 - 29.659x_2\tilde{x}_1\tilde{x}_2^2 + 60.851x_2\tilde{x}_2^3-\\ &17.851\tilde{x}_1^4 + 63.658\tilde{x}_1^3\tilde{x}_2 - 79.044\tilde{x}_1^2\tilde{x}_2^2 + 72.297\tilde{x}_1\tilde{x}_2^3 - 43.429\tilde{x}_2^4\end{aligned}$$

$$\Pi^{13}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -975.297x_2^4 - 946.599x_2^3\tilde{x}_1 - 357.132x_2^3\tilde{x}_2 - 2160.121x_2^2\tilde{x}_1^2 - 393.369x_2^2\tilde{x}_1\tilde{x}_2 - 225.765x_2^2\tilde{x}_2^2 - 856.577x_2\tilde{x}_1^3 - 418.628x_2\tilde{x}_1^2\tilde{x}_2 - 117.332x_2\tilde{x}_1\tilde{x}_2^2 - 39.837x_2\tilde{x}_2^3 - 1129.664\tilde{x}_1^4 - 163.43\tilde{x}_1^3\tilde{x}_2 - 230.145\tilde{x}_1^2\tilde{x}_2^2 + 30.983\tilde{x}_1\tilde{x}_2^3 - 41.274\tilde{x}_2^4$$

$$\Pi^{23}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -350.081x_2^4 - 383.154x_2^3\tilde{x}_1 - 209.884x_2^3\tilde{x}_2 - 443.275x_2^2\tilde{x}_1^2 - 414.401x_2^2\tilde{x}_1\tilde{x}_2 - 3.335x_2^2\tilde{x}_2^2 - 232.546x_2\tilde{x}_1^3 - 233.211x_2\tilde{x}_1^2\tilde{x}_2 - 60.833x_2\tilde{x}_1\tilde{x}_2^2 - 8.755x_2\tilde{x}_2^3 - 77.497\tilde{x}_1^4 - 251.711\tilde{x}_1^3\tilde{x}_2 + 82.154\tilde{x}_1^2\tilde{x}_2^2 - 80.533\tilde{x}_1\tilde{x}_2^3 + 56.778\tilde{x}_2^4$$

$$\Pi^{33}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 5645.648x_2^4 + 3263.528x_2^3\tilde{x}_1 + 943.734x_2^3\tilde{x}_2 + 8147.659x_2^2\tilde{x}_1^2 + 1104.154x_2^2\tilde{x}_1\tilde{x}_2 + 2674.776x_2^2\tilde{x}_2^2 + 2660.943x_2\tilde{x}_1^3 + 1120.413x_2\tilde{x}_1^2\tilde{x}_2 + 299.716x_2\tilde{x}_1\tilde{x}_2^2 + 69.534x_2\tilde{x}_2^3 + 5319.225\tilde{x}_1^4 + 474.998\tilde{x}_1^3\tilde{x}_2 + 2530.815\tilde{x}_1^2\tilde{x}_2^2 + 31.552\tilde{x}_1\tilde{x}_2^3 + 2529.079\tilde{x}_2^4$$

$$\Pi^{43}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -553.247x_2^4 - 1231.283x_2^3\tilde{x}_1 - 307.72x_2^3\tilde{x}_2 - 1090.14x_2^2\tilde{x}_1^2 - 181.257x_2^2\tilde{x}_1\tilde{x}_2 - 195.063x_2^2\tilde{x}_2^2 - 1049.604x_2\tilde{x}_1^3 - 250.745x_2\tilde{x}_1^2\tilde{x}_2 - 191.878x_2\tilde{x}_1\tilde{x}_2^2 - 14.7x_2\tilde{x}_2^3 - 291.142\tilde{x}_1^4 + 35.479\tilde{x}_1^3\tilde{x}_2 - 183.749\tilde{x}_1^2\tilde{x}_2^2 + 46.896\tilde{x}_1\tilde{x}_2^3 - 63.475\tilde{x}_2^4$$

$$\Pi^{14}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 183.347x_2^4 + 521.668x_2^3\tilde{x}_1 + 102.412x_2^3\tilde{x}_2 + 368.837x_2^2\tilde{x}_1^2 + 17.727x_2^2\tilde{x}_1\tilde{x}_2 + 91.686x_2^2\tilde{x}_2^2 + 506.941x_2\tilde{x}_1^3 + 101.227x_2\tilde{x}_1^2\tilde{x}_2 + 113.753x_2\tilde{x}_1\tilde{x}_2^2 - 8.872x_2\tilde{x}_2^3 + 116.843\tilde{x}_1^4 - 64.977\tilde{x}_1^3\tilde{x}_2 + 111.522\tilde{x}_1^2\tilde{x}_2^2 - 52.171\tilde{x}_1\tilde{x}_2^3 + 34.414\tilde{x}_2^4$$

$$\Pi^{24}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 24.438x_2^4 + 133.066x_2^3\tilde{x}_1 + 192.564x_2^3\tilde{x}_2 + 59.399x_2^2\tilde{x}_1^2 + 123.301x_2^2\tilde{x}_1\tilde{x}_2 - 54.219x_2^2\tilde{x}_2^2 + 54.867x_2\tilde{x}_1^3 + 150.686x_2\tilde{x}_1^2\tilde{x}_2 - 29.659x_2\tilde{x}_1\tilde{x}_2^2 + 60.851x_2\tilde{x}_2^3 - 17.851\tilde{x}_1^4 + 63.658\tilde{x}_1^3\tilde{x}_2 - 79.044\tilde{x}_1^2\tilde{x}_2^2 + 72.297\tilde{x}_1\tilde{x}_2^3 - 43.429\tilde{x}_2^4$$

$$\Pi^{34}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -553.247x_2^4 - 1231.283x_2^3\tilde{x}_1 - 307.72x_2^3\tilde{x}_2 - 1090.14x_2^2\tilde{x}_1^2 - 181.257x_2^2\tilde{x}_1\tilde{x}_2 - 195.063x_2^2\tilde{x}_2^2 - 1049.604x_2\tilde{x}_1^3 - 250.745x_2\tilde{x}_1^2\tilde{x}_2 - 191.878x_2\tilde{x}_1\tilde{x}_2^2 - 14.7x_2\tilde{x}_2^3 - 291.142\tilde{x}_1^4 + 35.479\tilde{x}_1^3\tilde{x}_2 - 183.749\tilde{x}_1^2\tilde{x}_2^2 + 46.896\tilde{x}_1\tilde{x}_2^3 - 63.475\tilde{x}_2^4$$

$$\Pi^{44}_{1212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2736.571x_2^4 + 263.014x_2^3\tilde{x}_1 - 180.952x_2^3\tilde{x}_2 + 2535.76x_2^2\tilde{x}_1^2 + 55.907x_2^2\tilde{x}_1\tilde{x}_2 + 2190.304x_2^2\tilde{x}_2^2 + 157.572x_2\tilde{x}_1^3 - 126.977x_2\tilde{x}_1^2\tilde{x}_2 + 138.93x_2\tilde{x}_1\tilde{x}_2^2 - 92.726x_2\tilde{x}_2^3 + 2405.099\tilde{x}_1^4 - 31.346\tilde{x}_1^3\tilde{x}_2 + 2057.304\tilde{x}_1^2\tilde{x}_2^2 - 81.757\tilde{x}_1\tilde{x}_2^3 + 2299.553\tilde{x}_2^4$$

$$\Pi^{11}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 35633.591x_2^4 + 1327.163x_2^3\tilde{x}_1 + 2940.569x_2^3\tilde{x}_2 + 48358.448x_2^2\tilde{x}_1^2 - 1561.031x_2^2\tilde{x}_1\tilde{x}_2 + 23188.752x_2^2\tilde{x}_2^2 + 1906.638x_2\tilde{x}_1^3 + 2327.052x_2\tilde{x}_1^2\tilde{x}_2 - 713.088x_2\tilde{x}_1\tilde{x}_2^2 + 1036.408x_2\tilde{x}_2^3 + 80339.637\tilde{x}_1^4 - 5161.085\tilde{x}_1^3\tilde{x}_2 + 26240.293\tilde{x}_1^2\tilde{x}_2^2 - 4419.908\tilde{x}_1\tilde{x}_2^3 + 25210.366\tilde{x}_2^4$$

$$\Pi^{21}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -568.77x_2^4 + 2440.15x_2^3\tilde{x}_1 - 1035.495x_2^3\tilde{x}_2 - 1108.092x_2^2\tilde{x}_1^2 + 6996.77x_2^2\tilde{x}_1\tilde{x}_2 - 3821.181x_2^2\tilde{x}_2^2 + 1748.561x_2\tilde{x}_1^3 - 849.41x_2\tilde{x}_1^2\tilde{x}_2 + 1027.157x_2\tilde{x}_1\tilde{x}_2^2 - 944.032x_2\tilde{x}_2^3 - 3439.913\tilde{x}_1^4 + 8969.638\tilde{x}_1^3\tilde{x}_2 - 7827.662\tilde{x}_1^2\tilde{x}_2^2 + 5541.362\tilde{x}_1\tilde{x}_2^3 - 3615.807\tilde{x}_2^4$$

$$\Pi^{31}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -23237.27x_2^4 - 4760.671x_2^3\tilde{x}_1 - 4114.92x_2^3\tilde{x}_2 - 39648.976x_2^2\tilde{x}_1^2 + 48.308x_2^2\tilde{x}_1\tilde{x}_2 - 4617.983x_2^2\tilde{x}_2^2 - 3764.596x_2\tilde{x}_1^3 - 3457.938x_2\tilde{x}_1^2\tilde{x}_2 + 546.56x_2\tilde{x}_1\tilde{x}_2^2 - 1287.328x_2\tilde{x}_2^3 - 34437.445\tilde{x}_1^4 + 1857.739\tilde{x}_1^3\tilde{x}_2 - 7524.846\tilde{x}_1^2\tilde{x}_2^2 + 3067.06\tilde{x}_1\tilde{x}_2^3 - 2834.512\tilde{x}_2^4$$

$$\Pi^{41}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2176.123x_2^4 + 22113.483x_2^3\tilde{x}_1 - 154.158x_2^3\tilde{x}_2 + 3716.317x_2^2\tilde{x}_1^2 - 4056.246x_2^2\tilde{x}_1\tilde{x}_2 + 2998.188x_2^2\tilde{x}_2^2 + 23463.785x_2\tilde{x}_1^3 - 893.197x_2\tilde{x}_1^2\tilde{x}_2 + 4533.915x_2\tilde{x}_1\tilde{x}_2^2 - 1708.663x_2\tilde{x}_2^3 + 4986.325\tilde{x}_1^4 - 7258.212\tilde{x}_1^3\tilde{x}_2 + 6910.525\tilde{x}_1^2\tilde{x}_2^2 - 4669.862\tilde{x}_1\tilde{x}_2^3 + 3113.51\tilde{x}_2^4$$

$$\Pi^{12}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -568.77x_2^4 + 2440.15x_2^3\tilde{x}_1 - 1035.495x_2^3\tilde{x}_2 - 1108.092x_2^2\tilde{x}_1^2 + 6996.77x_2^2\tilde{x}_1\tilde{x}_2 - 3821.181x_2^2\tilde{x}_2^2 + 1748.561x_2\tilde{x}_1^3 - 849.41x_2\tilde{x}_1^2\tilde{x}_2 + 1027.157x_2\tilde{x}_1\tilde{x}_2^2 - 944.032x_2\tilde{x}_2^3 - 3439.913\tilde{x}_1^4 + 8969.638\tilde{x}_1^3\tilde{x}_2 - 7827.662\tilde{x}_1^2\tilde{x}_2^2 + 5541.362\tilde{x}_1\tilde{x}_2^3 - 3615.807\tilde{x}_2^4$$

$$\Pi^{22}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 31097.044x_2^4 - 1068.416x_2^3\tilde{x}_1 + 3958.736x_2^3\tilde{x}_2 + 22809.916x_2^2\tilde{x}_1^2 - 3507.446x_2^2\tilde{x}_1\tilde{x}_2 + 26829.344x_2^2\tilde{x}_2^2 - 829.613x_2\tilde{x}_1^3 + 1318.039x_2\tilde{x}_1^2\tilde{x}_2 - 1372.674x_2\tilde{x}_1\tilde{x}_2^2 + 1081.737x_2\tilde{x}_2^3 + 25106.69\tilde{x}_1^4 - 4322.646\tilde{x}_1^3\tilde{x}_2 + 26279.579\tilde{x}_1^2\tilde{x}_2^2 - 6534.807\tilde{x}_1\tilde{x}_2^3 + 27245.202\tilde{x}_2^4$$

$$\Pi^{32}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -4427.717x_2^4 - 3704.229x_2^3\tilde{x}_1 + 158.3x_2^3\tilde{x}_2 - 152.97x_2^2\tilde{x}_1^2 - 5897.885x_2^2\tilde{x}_1\tilde{x}_2 + 2625.809x_2^2\tilde{x}_2^2 - 2276.997x_2\tilde{x}_1^3 + 541.156x_2\tilde{x}_1^2\tilde{x}_2 - 1452.065x_2\tilde{x}_1\tilde{x}_2^2 + 711.151x_2\tilde{x}_2^3 + 1555.842\tilde{x}_1^4 - 6479.242\tilde{x}_1^3\tilde{x}_2 + 5970.709\tilde{x}_1^2\tilde{x}_2^2 - 4083.793\tilde{x}_1\tilde{x}_2^3 + 6055.722\tilde{x}_2^4$$

$$\Pi^{42}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 1102.963x_2^4 - 36.018x_2^3\tilde{x}_1 + 13771.063x_2^3\tilde{x}_2 - 2161.643x_2^2\tilde{x}_1^2 + 2859.247x_2^2\tilde{x}_1\tilde{x}_2 - 4522.464x_2^2\tilde{x}_2^2 - 503.532x_2\tilde{x}_1^3 + 5592.906x_2\tilde{x}_1^2\tilde{x}_2 - 3505.046x_2\tilde{x}_1\tilde{x}_2^2 + 4903.105x_2\tilde{x}_2^3 - 2892.203\tilde{x}_1^4 + 4073.264\tilde{x}_1^3\tilde{x}_2 - 6502.713\tilde{x}_1^2\tilde{x}_2^2 + 5528.901\tilde{x}_1\tilde{x}_2^3 - 4308.065\tilde{x}_2^4$$

$$\Pi^{13}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -23237.27x_2^4 - 4760.671x_2^3\tilde{x}_1 - 4114.92x_2^3\tilde{x}_2 - 39648.976x_2^2\tilde{x}_1^2 + 48.308x_2^2\tilde{x}_1\tilde{x}_2 - 4617.983x_2^2\tilde{x}_2^2 - 3764.596x_2\tilde{x}_1^3 - 3457.938x_2\tilde{x}_1^2\tilde{x}_2 + 546.56x_2\tilde{x}_1\tilde{x}_2^2 - 1287.328x_2\tilde{x}_2^3 - 34437.445\tilde{x}_1^4 + 1857.739\tilde{x}_1^3\tilde{x}_2 - 7524.846\tilde{x}_1^2\tilde{x}_2^2 + 3067.06\tilde{x}_1\tilde{x}_2^3 - 2834.512\tilde{x}_2^4$$

$$\Pi^{23}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -4427.717x_2^4 - 3704.229x_2^3\tilde{x}_1 + 158.3x_2^3\tilde{x}_2 - 152.97x_2^2\tilde{x}_1^2 - 5897.885x_2^2\tilde{x}_1\tilde{x}_2 + 2625.809x_2^2\tilde{x}_2^2 - 2276.997x_2\tilde{x}_1^3 + 541.156x_2\tilde{x}_1^2\tilde{x}_2 - 1452.065x_2\tilde{x}_1\tilde{x}_2^2 + 711.151x_2\tilde{x}_2^3 + 1555.842\tilde{x}_1^4 - 6479.242\tilde{x}_1^3\tilde{x}_2 + 5970.709\tilde{x}_1^2\tilde{x}_2^2 - 4083.793\tilde{x}_1\tilde{x}_2^3 + 6055.722\tilde{x}_2^4$$

$$\Pi^{33}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 147718.122x_2^4 + 25970.923x_2^3\tilde{x}_1 + 8244.505x_2^3\tilde{x}_2 + 129666.302x_2^2\tilde{x}_1^2 + 586.987x_2^2\tilde{x}_1\tilde{x}_2 + 59300.038x_2^2\tilde{x}_2^2 + 9254.258x_2\tilde{x}_1^3 + 8563.91x_2\tilde{x}_1^2\tilde{x}_2 - 1333.465x_2\tilde{x}_1\tilde{x}_2^2 - 869.436x_2\tilde{x}_2^3 + 126564.946\tilde{x}_1^4 - 2540.704\tilde{x}_1^3\tilde{x}_2 + 53020.876\tilde{x}_1^2\tilde{x}_2^2 + 244.424\tilde{x}_1\tilde{x}_2^3 + 70386.289\tilde{x}_2^4$$

$$\Pi^{43}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -4657.123x_2^4 - 26080.174x_2^3\tilde{x}_1 - 5462.885x_2^3\tilde{x}_2 - 6386.657x_2^2\tilde{x}_1^2 + 1145.345x_2^2\tilde{x}_1\tilde{x}_2 - 2311.023x_2^2\tilde{x}_2^2 - 24825.221x_2\tilde{x}_1^3 - 950.7431x_2\tilde{x}_1^2\tilde{x}_2 - 3426.741x_2\tilde{x}_1\tilde{x}_2^2 + 359.683x_2\tilde{x}_2^3 - 3642.807\tilde{x}_1^4 + 4091.266\tilde{x}_1^3\tilde{x}_2 - 5114.115\tilde{x}_1^2\tilde{x}_2^2 + 2935.402\tilde{x}_1\tilde{x}_2^3 - 5684.86\tilde{x}_2^4$$

$$\Pi^{14}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2176.123x_2^4 + 22113.483x_2^3\tilde{x}_1 - 154.158x_2^3\tilde{x}_2 + 3716.317x_2^2\tilde{x}_1^2 - 4056.246x_2^2\tilde{x}_1\tilde{x}_2 + 2998.188x_2^2\tilde{x}_2^2 + 23463.785x_2\tilde{x}_1^3 - 893.197x_2\tilde{x}_1^2\tilde{x}_2 + 4533.915x_2\tilde{x}_1\tilde{x}_2^2 - 1708.663x_2\tilde{x}_2^3 + 4986.325\tilde{x}_1^4 - 7258.212\tilde{x}_1^3\tilde{x}_2 + 6910.525\tilde{x}_1^2\tilde{x}_2^2 - 4669.862\tilde{x}_1\tilde{x}_2^3 + 3113.51\tilde{x}_2^4$$

$$\Pi^{24}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 1102.963x_2^4 - 36.018x_2^3\tilde{x}_1 + 13771.063x_2^3\tilde{x}_2 - 2161.643x_2^2\tilde{x}_1^2 + 2859.247x_2^2\tilde{x}_1\tilde{x}_2 - 4522.464x_2^2\tilde{x}_2^2 - 503.532x_2\tilde{x}_1^3 + 5592.906x_2\tilde{x}_1^2\tilde{x}_2 - 3505.046x_2\tilde{x}_1\tilde{x}_2^2 + 4903.105x_2\tilde{x}_2^3 - 2892.203\tilde{x}_1^4 + 4073.264\tilde{x}_1^3\tilde{x}_2 - 6502.713\tilde{x}_1^2\tilde{x}_2^2 + 5528.901\tilde{x}_1\tilde{x}_2^3 - 4308.065\tilde{x}_2^4$$

$$\Pi^{34}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -4657.123x_2^4 - 26080.174x_2^3\tilde{x}_1 - 5462.885x_2^3\tilde{x}_2 - 6386.657x_2^2\tilde{x}_1^2 + 1145.345x_2^2\tilde{x}_1\tilde{x}_2 - 2311.023x_2^2\tilde{x}_2^2 - 24825.221x_2\tilde{x}_1^3 - 950.743x_2\tilde{x}_1^2\tilde{x}_2 - 3426.741x_2\tilde{x}_1\tilde{x}_2^2 + 359.683x_2\tilde{x}_2^3 - 3642.807\tilde{x}_1^4 + 4091.266\tilde{x}_1^3\tilde{x}_2 - 5114.115\tilde{x}_1^2\tilde{x}_2^2 + 2935.402\tilde{x}_1\tilde{x}_2^3 - 5684.86\tilde{x}_2^4$$

$$\Pi^{44}_{1221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 76911.02x_2^4 + 3746.502x_2^3\tilde{x}_1 - 16615.895x_2^3\tilde{x}_2 + 47529.002x_2^2\tilde{x}_1^2 - 3763.796x_2^2\tilde{x}_1\tilde{x}_2 + 38231.399x_2^2\tilde{x}_2^2 + 3262.505x_2\tilde{x}_1^3 - 9301.352x_2\tilde{x}_1^2\tilde{x}_2 + 6593.794x_2\tilde{x}_1\tilde{x}_2^2 - 7397.34x_2\tilde{x}_2^3 + 34029.726\tilde{x}_1^4 - 4257.866\tilde{x}_1^3\tilde{x}_2 + 28280.972\tilde{x}_1^2\tilde{x}_2^2 - 6617.901\tilde{x}_1\tilde{x}_2^3 + 28746.497\tilde{x}_2^4$$

$$\Pi^{11}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 3160.551x_2^4 + 393.636x_2^3\tilde{x}_1 + 250.744x_2^3\tilde{x}_2 + 4417.932x_2^2\tilde{x}_1^2 + 129.53x_2^2\tilde{x}_1\tilde{x}_2 + 2274.163x_2^2\tilde{x}_2^2 + 337.77x_2\tilde{x}_1^3 + 268.556x_2\tilde{x}_1^2\tilde{x}_2 + 79.617x_2\tilde{x}_1\tilde{x}_2^2 + 43.192x_2\tilde{x}_2^3 + 5024.397\tilde{x}_1^4 - 56\tilde{x}_1^3\tilde{x}_2 + 2477.779\tilde{x}_1^2\tilde{x}_2^2 - 142.108\tilde{x}_1\tilde{x}_2^3 + 2340.998\tilde{x}_2^4$$

$$\Pi^{21}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 221.527x_2^4 + 187.563x_2^3\tilde{x}_1 + 133.174x_2^3\tilde{x}_2 + 291.539x_2^2\tilde{x}_1^2 + 531.857x_2^2\tilde{x}_1\tilde{x}_2 - 131.587x_2^2\tilde{x}_2^2 + 174.531x_2\tilde{x}_1^3 + 132.962x_2\tilde{x}_1^2\tilde{x}_2 + 26.386x_2\tilde{x}_1\tilde{x}_2^2 + 6.552x_2\tilde{x}_2^3 - 16.944\tilde{x}_1^4 + 592.638\tilde{x}_1^3\tilde{x}_2 - 384.807\tilde{x}_1^2\tilde{x}_2^2 + 233.125\tilde{x}_1\tilde{x}_2^3 - 124.816\tilde{x}_2^4$$

$$\Pi^{31}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -2430.681x_2^4 - 1275.766x_2^3\tilde{x}_1 - 646.244x_2^3\tilde{x}_2 - 5446.379x_2^2\tilde{x}_1^2 - 424.212x_2^2\tilde{x}_1\tilde{x}_2 - 851.337x_2^2\tilde{x}_2^2 - 1161.242x_2\tilde{x}_1^3 - 676.27x_2\tilde{x}_1^2\tilde{x}_2 - 215.151x_2\tilde{x}_1\tilde{x}_2^2 - 116.851x_2\tilde{x}_2^3 - 4282.856\tilde{x}_1^4 - 88.965\tilde{x}_1^3\tilde{x}_2 - 1150.401\tilde{x}_1^2\tilde{x}_2^2 + 142.033\tilde{x}_1\tilde{x}_2^3 - 189.612\tilde{x}_2^4$$

$$\Pi^{41}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 342.613x_2^4 + 1371.336x_2^3\tilde{x}_1 + 165.59x_2^3\tilde{x}_2 + 554.128x_2^2\tilde{x}_1^2 - 248.281x_2^2\tilde{x}_1\tilde{x}_2 + 321.248x_2^2\tilde{x}_2^2 + 1620.064x_2\tilde{x}_1^3 + 130.84x_2\tilde{x}_1^2\tilde{x}_2 + 376.584x_2\tilde{x}_1\tilde{x}_2^2 - 44.489x_2\tilde{x}_2^3 + 226.831\tilde{x}_1^4 - 435.455\tilde{x}_1^3\tilde{x}_2 + 471.539\tilde{x}_1^2\tilde{x}_2^2 - 212.149\tilde{x}_1\tilde{x}_2^3 + 137.897\tilde{x}_2^4$$

$$\Pi^{12}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 221.527x_2^4 + 187.563x_2^3\tilde{x}_1 + 133.174x_2^3\tilde{x}_2 + 291.539x_2^2\tilde{x}_1^2 + 531.857x_2^2\tilde{x}_1\tilde{x}_2 - 131.587x_2^2\tilde{x}_2^2 + 174.531x_2\tilde{x}_1^3 + 132.962x_2\tilde{x}_1^2\tilde{x}_2 + 26.386x_2\tilde{x}_1\tilde{x}_2^2 + 6.552x_2\tilde{x}_2^3 - 16.944\tilde{x}_1^4 + 592.638\tilde{x}_1^3\tilde{x}_2 - 384.807\tilde{x}_1^2\tilde{x}_2^2 + 233.125\tilde{x}_1\tilde{x}_2^3 - 124.816\tilde{x}_2^4$$

$$\Pi^{22}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2486.668x_2^4 + 90.63x_2^3\tilde{x}_1 + 50.927x_2^3\tilde{x}_2 + 2112.292x_2^2\tilde{x}_1^2 + 29.299x_2^2\tilde{x}_1\tilde{x}_2 + 2149.592x_2^2\tilde{x}_2^2 + 7.933x_2\tilde{x}_1^3 + 79.209x_2\tilde{x}_1^2\tilde{x}_2 + 20.55x_2\tilde{x}_1\tilde{x}_2^2 - 40.173x_2\tilde{x}_2^3 + 2327.472\tilde{x}_1^4 - 122.655\tilde{x}_1^3\tilde{x}_2 + 2250.677\tilde{x}_1^2\tilde{x}_2^2 - 286.705\tilde{x}_1\tilde{x}_2^3 + 2394.959\tilde{x}_2^4$$

$$\Pi^{32}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -718.124x_2^4 - 614.294x_2^3\tilde{x}_1 - 350.834x_2^3\tilde{x}_2 - 670.958x_2^2\tilde{x}_1^2 - 1032.841x_2^2\tilde{x}_1\tilde{x}_2 + 192.008x_2^2\tilde{x}_2^2 - 392.583x_2\tilde{x}_1^3 - 358.01x_2\tilde{x}_1^2\tilde{x}_2 - 97.724x_2\tilde{x}_1\tilde{x}_2^2 - 14.483x_2\tilde{x}_2^3 - 8.677\tilde{x}_1^4 - 938.716\tilde{x}_1^3\tilde{x}_2 + 531.389\tilde{x}_1^2\tilde{x}_2^2 - 334.642\tilde{x}_1\tilde{x}_2^3 + 241.51\tilde{x}_2^4$$

$$\Pi^{42}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -96.526x_2^4 + 253.118x_2^3\tilde{x}_1 + 548.16x_2^3\tilde{x}_2 - 18.065x_2^2\tilde{x}_1^2 + 287.243x_2^2\tilde{x}_1\tilde{x}_2 - 293.661x_2^2\tilde{x}_2^2 + 45.797x_2\tilde{x}_1^3 + 481.938x_2\tilde{x}_1^2\tilde{x}_2 - 197.75x_2\tilde{x}_1\tilde{x}_2^2 + 214.209x_2\tilde{x}_2^3 - 95.685\tilde{x}_1^4 + 184.364\tilde{x}_1^3\tilde{x}_2 - 334.209\tilde{x}_1^2\tilde{x}_2^2 + 290.054\tilde{x}_1\tilde{x}_2^3 - 180.064\tilde{x}_2^4$$

$$\Pi^{13}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -2430.681x_2^4 - 1275.766x_2^3\tilde{x}_1 - 646.244x_2^3\tilde{x}_2 - 5446.379x_2^2\tilde{x}_1^2 - 424.212x_2^2\tilde{x}_1\tilde{x}_2 - 851.337x_2^2\tilde{x}_2^2 - 1161.242x_2\tilde{x}_1^3 - 676.272x_2\tilde{x}_1^2\tilde{x}_2 - 215.151x_2\tilde{x}_1\tilde{x}_2^2 - 116.851x_2\tilde{x}_2^3 - 4282.856\tilde{x}_1^4 - 88.965\tilde{x}_1^3\tilde{x}_2 - 1150.401\tilde{x}_1^2\tilde{x}_2^2 + 142.033\tilde{x}_1\tilde{x}_2^3 - 189.612\tilde{x}_2^4$$

$$\Pi^{23}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -718.124x_2^4 - 614.294x_2^3\tilde{x}_1 - 350.834x_2^3\tilde{x}_2 - 670.958x_2^2\tilde{x}_1^2 - 1032.841x_2^2\tilde{x}_1\tilde{x}_2 + 192.008x_2^2\tilde{x}_2^2 - 392.583x_2\tilde{x}_1^3 - 358.01x_2\tilde{x}_1^2\tilde{x}_2 - 97.724x_2\tilde{x}_1\tilde{x}_2^2 - 14.483x_2\tilde{x}_2^3 - 8.677\tilde{x}_1^4 - 938.716\tilde{x}_1^3\tilde{x}_2 + 531.389\tilde{x}_1^2\tilde{x}_2^2 - 334.642\tilde{x}_1\tilde{x}_2^3 + 241.51\tilde{x}_2^4$$

$$\Pi^{33}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 10252.886x_2^4 + 4583.339x_2^3\tilde{x}_1 + 1604.134x_2^3\tilde{x}_2 + 15384.702x_2^2\tilde{x}_1^2 + 1306.182x_2^2\tilde{x}_1\tilde{x}_2 + 4771.089x_2^2\tilde{x}_2^2 + 3600.37x_2\tilde{x}_1^3 + 1669.186x_2\tilde{x}_1^2\tilde{x}_2 + 652.698x_2\tilde{x}_1\tilde{x}_2^2 + 236.947x_2\tilde{x}_2^3 + 11204.794\tilde{x}_1^4 + 488.789\tilde{x}_1^3\tilde{x}_2 + 4727.471\tilde{x}_1^2\tilde{x}_2^2 - 22.105\tilde{x}_1\tilde{x}_2^3 + 3180.618\tilde{x}_2^4$$

$$\Pi^{43}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -962.723x_2^4 - 2944.668x_2^3\tilde{x}_1 - 575.4x_2^3\tilde{x}_2 - 1562.436x_2^2\tilde{x}_1^2 + 137.255x_2^2\tilde{x}_1\tilde{x}_2 - 663.993x_2^2\tilde{x}_2^2 - 2823.862x_2\tilde{x}_1^3 - 342.318x_2\tilde{x}_1^2\tilde{x}_2 - 667.22x_2\tilde{x}_1\tilde{x}_2^2 - 26.834x_2\tilde{x}_2^3 - 446.858\tilde{x}_1^4 + 564.287\tilde{x}_1^3\tilde{x}_2 - 727.686\tilde{x}_1^2\tilde{x}_2^2 + 253.571\tilde{x}_1\tilde{x}_2^3 - 273.204\tilde{x}_2^4$$

$$\Pi^{14}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 342.613x_2^4 + 1371.336x_2^3\tilde{x}_1 + 165.59x_2^3\tilde{x}_2 + 554.128x_2^2\tilde{x}_1^2 - 248.281x_2^2\tilde{x}_1\tilde{x}_2 + 321.248x_2^2\tilde{x}_2^2 + 1620.064x_2\tilde{x}_1^3 + 130.84x_2\tilde{x}_1^2\tilde{x}_2 + 376.584x_2\tilde{x}_1\tilde{x}_2^2 - 44.489x_2\tilde{x}_2^3 + 226.831\tilde{x}_1^4 - 435.455\tilde{x}_1^3\tilde{x}_2 + 471.539\tilde{x}_1^2\tilde{x}_2^2 - 212.149\tilde{x}_1\tilde{x}_2^3 + 137.897\tilde{x}_2^4$$

$$\Pi^{24}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -96.526x_2^4 + 253.118x_2^3\tilde{x}_1 + 548.16x_2^3\tilde{x}_2 - 18.065x_2^2\tilde{x}_1^2 + 287.243x_2^2\tilde{x}_1\tilde{x}_2 - 293.661x_2^2\tilde{x}_2^2 + 45.797x_2\tilde{x}_1^3 + 481.938x_2\tilde{x}_1^2\tilde{x}_2 - 197.75x_2\tilde{x}_1\tilde{x}_2^2 + 214.209x_2\tilde{x}_2^3 - 95.685\tilde{x}_1^4 + 184.364\tilde{x}_1^3\tilde{x}_2 - 334.209\tilde{x}_1^2\tilde{x}_2^2 + 290.054\tilde{x}_1\tilde{x}_2^3 - 180.064\tilde{x}_2^4$$

$$\Pi^{34}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -962.723x_2^4 - 2944.668x_2^3\tilde{x}_1 - 575.4x_2^3\tilde{x}_2 - 1562.436x_2^2\tilde{x}_1^2 + 137.255x_2^2\tilde{x}_1\tilde{x}_2 - 663.993x_2^2\tilde{x}_2^2 - 2823.862x_2\tilde{x}_1^3 - 342.318x_2\tilde{x}_1^2\tilde{x}_2 - 667.22x_2\tilde{x}_1\tilde{x}_2^2 - 26.834x_2\tilde{x}_2^3 - 446.858\tilde{x}_1^4 + 564.287\tilde{x}_1^3\tilde{x}_2 - 727.686\tilde{x}_1^2\tilde{x}_2^2 + 253.571\tilde{x}_1\tilde{x}_2^3 - 273.204\tilde{x}_2^4$$

$$\Pi^{44}_{1222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 3779.733x_2^4 + 498.072x_2^3\tilde{x}_1 - 874.824x_2^3\tilde{x}_2 + 3753.733x_2^2\tilde{x}_1^2 - 41.385x_2^2\tilde{x}_1\tilde{x}_2 + 2919.647x_2^2\tilde{x}_2^2 + 265.487x_2\tilde{x}_1^3 - 718.652x_2\tilde{x}_1^2\tilde{x}_2 + 536.548x_2\tilde{x}_1\tilde{x}_2^2 - 362.645x_2\tilde{x}_2^3 + 2650.816\tilde{x}_1^4 - 169.489\tilde{x}_1^3\tilde{x}_2 + 2431.497\tilde{x}_1^2\tilde{x}_2^2 - 326.187\tilde{x}_1\tilde{x}_2^3 + 2498.937\tilde{x}_2^4$$

$$\Pi_{2111}^{11}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = 26853.102x_2^4 + 516.654x_2^3\tilde{x}_1 + 989.506x_2^3\tilde{x}_2 + 28141.422x_2^2\tilde{x}_1^2 - 723.344x_2^2\tilde{x}_1\tilde{x}_2 + 20285.323x_2^2\tilde{x}_2^2 + 455.434x_2\tilde{x}_1^3 + 797.522x_2\tilde{x}_1^2\tilde{x}_2 - 441.521x_2\tilde{x}_1\tilde{x}_2^2 + 411.567x_2\tilde{x}_2^3 + 32895.917\tilde{x}_1^4 - 1345.164\tilde{x}_1^3\tilde{x}_2 + 20975.333\tilde{x}_1^2\tilde{x}_2^2 - 1723.523\tilde{x}_1\tilde{x}_2^3 + 23205.97\tilde{x}_2^4$$

$$\Pi_{2111}^{21}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = -581.494x_2^4 + 979.808x_2^3\tilde{x}_1 - 575.775x_2^3\tilde{x}_2 - 465.652x_2^2\tilde{x}_1^2 + 2210.687x_2^2\tilde{x}_1\tilde{x}_2 - 1410.288x_2^2\tilde{x}_2^2 + 533.674x_2\tilde{x}_1^3 - 401.717x_2\tilde{x}_1^2\tilde{x}_2 + 561.774x_2\tilde{x}_1\tilde{x}_2^2 - 444.031x_2\tilde{x}_2^3 - 773.223\tilde{x}_1^4 + 2206.349\tilde{x}_1^3\tilde{x}_2 - 2440.599\tilde{x}_1^2\tilde{x}_2^2 + 1903.384\tilde{x}_1\tilde{x}_2^3 - 1165.096\tilde{x}_2^4$$

$$\Pi_{2111}^{31}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = -4529.835x_2^4 - 754.91x_2^3\tilde{x}_1 - 766.546x_2^3\tilde{x}_2 - 8385.12x_2^2\tilde{x}_1^2 - 171.519x_2^2\tilde{x}_1\tilde{x}_2 - 1007.345x_2^2\tilde{x}_2^2 - 323.815x_2\tilde{x}_1^3 - 635.451x_2\tilde{x}_1^2\tilde{x}_2 + 110.07x_2\tilde{x}_1\tilde{x}_2^2 - 213.241x_2\tilde{x}_2^3 - 5319.932\tilde{x}_1^4 + 326.525\tilde{x}_1^3\tilde{x}_2 - 1278.135\tilde{x}_1^2\tilde{x}_2^2 + 875.614\tilde{x}_1\tilde{x}_2^3 - 573.254\tilde{x}_2^4$$

$$\Pi_{2111}^{41}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = 614.825x_2^4 + 7653.45x_2^3\tilde{x}_1 - 470.902x_2^3\tilde{x}_2 + 1386.902x_2^2\tilde{x}_1^2 - 916.963x_2^2\tilde{x}_1\tilde{x}_2 + 748.451x_2^2\tilde{x}_2^2 + 6631.559x_2\tilde{x}_1^3 - 322.735x_2\tilde{x}_1^2\tilde{x}_2 + 1106.58x_2\tilde{x}_1\tilde{x}_2^2 - 636.97x_2\tilde{x}_2^3 + 1047.228\tilde{x}_1^4 - 1668.968\tilde{x}_1^3\tilde{x}_2 + 2040.094\tilde{x}_1^2\tilde{x}_2^2 - 1494.43\tilde{x}_1\tilde{x}_2^3 + 894.298\tilde{x}_2^4$$

$$\Pi_{2111}^{12}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = -581.494x_2^4 + 979.808x_2^3\tilde{x}_1 - 575.775x_2^3\tilde{x}_2 - 465.652x_2^2\tilde{x}_1^2 + 2210.687x_2^2\tilde{x}_1\tilde{x}_2 - 1410.288x_2^2\tilde{x}_2^2 + 533.674x_2\tilde{x}_1^3 - 401.717x_2\tilde{x}_1^2\tilde{x}_2 + 561.774x_2\tilde{x}_1\tilde{x}_2^2 - 444.031x_2\tilde{x}_2^3 - 773.223\tilde{x}_1^4 + 2206.349\tilde{x}_1^3\tilde{x}_2 - 2440.599\tilde{x}_1^2\tilde{x}_2^2 + 1903.384\tilde{x}_1\tilde{x}_2^3 - 1165.096\tilde{x}_2^4$$

$$\Pi_{2111}^{22}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = 25024.141x_2^4 - 605.112x_2^3\tilde{x}_1 + 1471.271x_2^3\tilde{x}_2 + 20427.779x_2^2\tilde{x}_1^2 - 1561.401x_2^2\tilde{x}_1\tilde{x}_2 + 21181.634x_2^2\tilde{x}_2^2 - 326.555x_2\tilde{x}_1^3 + 601.113x_2\tilde{x}_1^2\tilde{x}_2 - 641.372x_2\tilde{x}_1\tilde{x}_2^2 + 541.415x_2\tilde{x}_2^3 + 23139.709\tilde{x}_1^4 - 1652.476\tilde{x}_1^3\tilde{x}_2 + 21574.555\tilde{x}_1^2\tilde{x}_2^2 - 2119.667\tilde{x}_1\tilde{x}_2^3 + 23506.504\tilde{x}_2^4$$

$$\Pi_{2111}^{32}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = -1956.347x_2^4 - 702.874x_2^3\tilde{x}_1 - 486.438x_2^3\tilde{x}_2 - 66.238x_2^2\tilde{x}_1^2 - 1576.205x_2^2\tilde{x}_1\tilde{x}_2 + 162.914x_2^2\tilde{x}_2^2 - 375.726x_2\tilde{x}_1^3 + 114.757x_2\tilde{x}_1^2\tilde{x}_2 - 277.439x_2\tilde{x}_1\tilde{x}_2^2 + 44.908x_2\tilde{x}_2^3 + 186.868\tilde{x}_1^4 - 1091.235\tilde{x}_1^3\tilde{x}_2 + 1371.006\tilde{x}_1^2\tilde{x}_2^2 - 1030.952\tilde{x}_1\tilde{x}_2^3 + 1074.823\tilde{x}_2^4$$

$$\Pi_{2111}^{42}(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = 1178.627x_2^4 - 426.75x_2^3\tilde{x}_1 + 3748.7x_2^3\tilde{x}_2 - 483.226x_2^2\tilde{x}_1^2 + 883.811x_2^2\tilde{x}_1\tilde{x}_2 - 670.079x_2^2\tilde{x}_2^2 - 159.755x_2\tilde{x}_1^3 + 1527.184x_2\tilde{x}_1^2\tilde{x}_2 - 1194.556x_2\tilde{x}_1\tilde{x}_2^2 + 1091.344x_2\tilde{x}_2^3 - 787.607\tilde{x}_1^4 + 1405.644\tilde{x}_1^3\tilde{x}_2 - 2015.765\tilde{x}_1^2\tilde{x}_2^2 + 1663.066\tilde{x}_1\tilde{x}_2^3 - 1024.612\tilde{x}_2^4$$

$$\Pi^{13}_{2111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -4529.835x_2^4 - 754.91x_2^3\tilde{x}_1 - 766.546x_2^3\tilde{x}_2 - 8385.12x_2^2\tilde{x}_1^2 - 171.519x_2^2\tilde{x}_1\tilde{x}_2 - 1007.345x_2^2\tilde{x}_2^2 - 323.815x_2\tilde{x}_1^3 - 635.451x_2\tilde{x}_1^2\tilde{x}_2 + 110.07x_2\tilde{x}_1\tilde{x}_2^2 - 213.241x_2\tilde{x}_2^3 - 5319.932\tilde{x}_1^4 + 326.525\tilde{x}_1^3\tilde{x}_2 - 1278.135\tilde{x}_1^2\tilde{x}_2^2 + 875.614\tilde{x}_1\tilde{x}_2^3 - 573.254\tilde{x}_2^4$$

$$\Pi^{23}_{2111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -1956.347x_2^4 - 702.874x_2^3\tilde{x}_1 - 486.438x_2^3\tilde{x}_2 - 66.238x_2^2\tilde{x}_1^2 - 1576.205x_2^2\tilde{x}_1\tilde{x}_2 + 162.914x_2^2\tilde{x}_2^2 - 375.726x_2\tilde{x}_1^3 + 114.757x_2\tilde{x}_1^2\tilde{x}_2 - 277.439x_2\tilde{x}_1\tilde{x}_2^2 + 44.908x_2\tilde{x}_2^3 + 186.868\tilde{x}_1^4 - 1091.235\tilde{x}_1^3\tilde{x}_2 + 1371.006\tilde{x}_1^2\tilde{x}_2^2 - 1030.952\tilde{x}_1\tilde{x}_2^3 + 1074.823\tilde{x}_2^4$$

$$\Pi^{33}_{2111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 41588.275x_2^4 + 497.864x_2^3\tilde{x}_1 + 1576.525x_2^3\tilde{x}_2 + 36252.513x_2^2\tilde{x}_1^2 + 763.164x_2^2\tilde{x}_1\tilde{x}_2 + 30981.753x_2^2\tilde{x}_2^2 + 359.418x_2\tilde{x}_1^3 + 511.038x_2\tilde{x}_1^2\tilde{x}_2 + 147.67x_2\tilde{x}_1\tilde{x}_2^2 - 223.123x_2\tilde{x}_2^3 + 34680.027\tilde{x}_1^4 + 323.523\tilde{x}_1^3\tilde{x}_2 + 27725.356\tilde{x}_1^2\tilde{x}_2^2 - 370.013\tilde{x}_1\tilde{x}_2^3 + 35973.027\tilde{x}_2^4$$

$$\Pi^{43}_{2111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -1532.362x_2^4 - 6840.894x_2^3\tilde{x}_1 - 2442.265x_2^3\tilde{x}_2 - 960.73x_2^2\tilde{x}_1^2 + 537.419x_2^2\tilde{x}_1\tilde{x}_2 - 701.11x_2^2\tilde{x}_2^2 - 5314.424x_2\tilde{x}_1^3 - 266.359x_2\tilde{x}_1^2\tilde{x}_2 - 1005.326x_2\tilde{x}_1\tilde{x}_2^2 - 29.758x_2\tilde{x}_2^3 - 351.022\tilde{x}_1^4 + 690.549\tilde{x}_1^3\tilde{x}_2 - 1231.037\tilde{x}_1^2\tilde{x}_2^2 + 826.23\tilde{x}_1\tilde{x}_2^3 - 1048.423\tilde{x}_2^4$$

$$\Pi^{14}_{2111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 614.825x_2^4 + 7653.45x_2^3\tilde{x}_1 - 470.902x_2^3\tilde{x}_2 + 1386.902x_2^2\tilde{x}_1^2 - 916.963x_2^2\tilde{x}_1\tilde{x}_2 + 748.451x_2^2\tilde{x}_2^2 + 6631.559x_2\tilde{x}_1^3 - 322.735x_2\tilde{x}_1^2\tilde{x}_2 + 1106.58x_2\tilde{x}_1\tilde{x}_2^2 - 636.97x_2\tilde{x}_2^3 + 1047.228\tilde{x}_1^4 - 1668.968\tilde{x}_1^3\tilde{x}_2 + 2040.094\tilde{x}_1^2\tilde{x}_2^2 - 1494.43\tilde{x}_1\tilde{x}_2^3 + 894.298\tilde{x}_2^4$$

$$\Pi^{24}_{2111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 1178.627x_2^4 - 426.75x_2^3\tilde{x}_1 + 3748.7x_2^3\tilde{x}_2 - 483.226x_2^2\tilde{x}_1^2 + 883.811x_2^2\tilde{x}_1\tilde{x}_2 - 670.079x_2^2\tilde{x}_2^2 - 159.755x_2\tilde{x}_1^3 + 1527.184x_2\tilde{x}_1^2\tilde{x}_2 - 1194.556x_2\tilde{x}_1\tilde{x}_2^2 + 1091.344x_2\tilde{x}_2^3 - 787.607\tilde{x}_1^4 + 1405.644\tilde{x}_1^3\tilde{x}_2 - 2015.765\tilde{x}_1^2\tilde{x}_2^2 + 1663.066\tilde{x}_1\tilde{x}_2^3 - 1024.612\tilde{x}_2^4$$

$$\Pi^{34}_{2111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -1532.362x_2^4 - 6840.894x_2^3\tilde{x}_1 - 2442.265x_2^3\tilde{x}_2 - 960.73x_2^2\tilde{x}_1^2 + 537.419x_2^2\tilde{x}_1\tilde{x}_2 - 701.11x_2^2\tilde{x}_2^2 - 5314.424x_2\tilde{x}_1^3 - 266.359x_2\tilde{x}_1^2\tilde{x}_2 - 1005.326x_2\tilde{x}_1\tilde{x}_2^2 - 29.758x_2\tilde{x}_2^3 - 351.022\tilde{x}_1^4 + 690.549\tilde{x}_1^3\tilde{x}_2 - 1231.037\tilde{x}_1^2\tilde{x}_2^2 + 826.23\tilde{x}_1\tilde{x}_2^3 - 1048.423\tilde{x}_2^4$$

$$\Pi^{44}_{2111}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 38169.503x_2^4 + 1696.231x_2^3\tilde{x}_1 - 2963.809x_2^3\tilde{x}_2 + 28634.599x_2^2\tilde{x}_1^2 - 1694.989x_2^2\tilde{x}_1\tilde{x}_2 + 23259.592x_2^2\tilde{x}_2^2 + 1127.915x_2\tilde{x}_1^3 - 2457.533x_2\tilde{x}_1^2\tilde{x}_2 + 2147.973x_2\tilde{x}_1\tilde{x}_2^2 - 1511.901x_2\tilde{x}_2^3 + 26196.906\tilde{x}_1^4 - 1514.686\tilde{x}_1^3\tilde{x}_2 + 21762.915\tilde{x}_1^2\tilde{x}_2^2 - 2208.374\tilde{x}_1\tilde{x}_2^3 + 23691.17\tilde{x}_2^4$$

$$\Pi^{11}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 3157.264x_2^4 + 394.093x_2^3\tilde{x}_1 + 250.73x_2^3\tilde{x}_2 + 4407.181x_2^2\tilde{x}_1^2 + 131.114x_2^2\tilde{x}_1\tilde{x}_2 + 2273.094x_2^2\tilde{x}_2^2 + 337.219x_2\tilde{x}_1^3 + 268.549x_2\tilde{x}_1^2\tilde{x}_2 + 79.941x_2\tilde{x}_1\tilde{x}_2^2 + 43.098x_2\tilde{x}_2^3 + 5006.528\tilde{x}_1^4 - 54.333\tilde{x}_1^3\tilde{x}_2 + 2476.038\tilde{x}_1^2\tilde{x}_2^2 - 142.055\tilde{x}_1\tilde{x}_2^3 + 2340.897\tilde{x}_2^4$$

$$\Pi^{21}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 221.186x_2^4 + 187.858x_2^3\tilde{x}_1 + 133.313x_2^3\tilde{x}_2 + 291.768x_2^2\tilde{x}_1^2 + 531.429x_2^2\tilde{x}_1\tilde{x}_2 - 130.91x_2^2\tilde{x}_2^2 + 174.481x_2\tilde{x}_1^3 + 133.157x_2\tilde{x}_1^2\tilde{x}_2 + 26.428x_2\tilde{x}_1\tilde{x}_2^2 + 6.653x_2\tilde{x}_2^3 - 16.013\tilde{x}_1^4 + 590.496\tilde{x}_1^3\tilde{x}_2 - 383.156\tilde{x}_1^2\tilde{x}_2^2 + 232.953\tilde{x}_1\tilde{x}_2^3 - 124.72\tilde{x}_2^4$$

$$\Pi^{31}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -2411.759x_2^4 - 1271.176x_2^3\tilde{x}_1 - 645.551x_2^3\tilde{x}_2 - 5405.152x_2^2\tilde{x}_1^2 - 427.33x_2^2\tilde{x}_1\tilde{x}_2 - 846.756x_2^2\tilde{x}_2^2 - 1156.2x_2\tilde{x}_1^3 - 676.331x_2\tilde{x}_1^2\tilde{x}_2 - 215.184x_2\tilde{x}_1\tilde{x}_2^2 - 116.734x_2\tilde{x}_2^3 - 4247.151\tilde{x}_1^4 - 92.205\tilde{x}_1^3\tilde{x}_2 - 1143.006\tilde{x}_1^2\tilde{x}_2^2 + 141.949\tilde{x}_1\tilde{x}_2^3 - 189.306\tilde{x}_2^4$$

$$\Pi^{41}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 342.159x_2^4 + 1367.785x_2^3\tilde{x}_1 + 165.62x_2^3\tilde{x}_2 + 552.488x_2^2\tilde{x}_1^2 - 247.214x_2^2\tilde{x}_1\tilde{x}_2 + 320.46x_2^2\tilde{x}_2^2 + 1611.962x_2\tilde{x}_1^3 + 131.336x_2\tilde{x}_1^2\tilde{x}_2 + 376.211x_2\tilde{x}_1\tilde{x}_2^2 - 44.685x_2\tilde{x}_2^3 + 225.276\tilde{x}_1^4 - 433.317\tilde{x}_1^3\tilde{x}_2 + 469.87\tilde{x}_1^2\tilde{x}_2^2 - 212.003\tilde{x}_1\tilde{x}_2^3 + 137.802\tilde{x}_2^4$$

$$\Pi^{12}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 221.186x_2^4 + 187.858x_2^3\tilde{x}_1 + 133.313x_2^3\tilde{x}_2 + 291.768x_2^2\tilde{x}_1^2 + 531.429x_2^2\tilde{x}_1\tilde{x}_2 - 130.91x_2^2\tilde{x}_2^2 + 174.481x_2\tilde{x}_1^3 + 133.157x_2\tilde{x}_1^2\tilde{x}_2 + 26.428x_2\tilde{x}_1\tilde{x}_2^2 + 6.653x_2\tilde{x}_2^3 - 16.013\tilde{x}_1^4 + 590.496\tilde{x}_1^3\tilde{x}_2 - 383.156\tilde{x}_1^2\tilde{x}_2^2 + 232.953\tilde{x}_1\tilde{x}_2^3 - 124.72\tilde{x}_2^4$$

$$\Pi^{22}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2486.782x_2^4 + 90.892x_2^3\tilde{x}_1 + 51.144x_2^3\tilde{x}_2 + 2112.982x_2^2\tilde{x}_1^2 + 29.121x_2^2\tilde{x}_1\tilde{x}_2 + 2150.027x_2^2\tilde{x}_2^2 + 8.257x_2\tilde{x}_1^3 + 79.231x_2\tilde{x}_1^2\tilde{x}_2 + 20.767x_2\tilde{x}_1\tilde{x}_2^2 - 40.117x_2\tilde{x}_2^3 + 2327.663\tilde{x}_1^4 - 122.379\tilde{x}_1^3\tilde{x}_2 + 2250.327\tilde{x}_1^2\tilde{x}_2^2 - 286.263\tilde{x}_1\tilde{x}_2^3 + 2394.854\tilde{x}_2^4$$

$$\Pi^{32}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -714.237x_2^4 - 612.543x_2^3\tilde{x}_1 - 350.197x_2^3\tilde{x}_2 - 669.758x_2^2\tilde{x}_1^2 - 1029.082x_2^2\tilde{x}_1\tilde{x}_2 + 189.102x_2^2\tilde{x}_2^2 - 391.386x_2\tilde{x}_1^3 - 357.295x_2\tilde{x}_1^2\tilde{x}_2 - 98.562x_2\tilde{x}_1\tilde{x}_2^2 - 14.56x_2\tilde{x}_2^3 - 10.592\tilde{x}_1^4 - 931.814\tilde{x}_1^3\tilde{x}_2 + 526.883\tilde{x}_1^2\tilde{x}_2^2 - 334.159\tilde{x}_1\tilde{x}_2^3 + 241.15\tilde{x}_2^4$$

$$\Pi^{42}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -96.255x_2^4 + 252.757x_2^3\tilde{x}_1 + 548.367x_2^3\tilde{x}_2 - 18.074x_2^2\tilde{x}_1^2 + 287.363x_2^2\tilde{x}_1\tilde{x}_2 - 293.591x_2^2\tilde{x}_2^2 + 46.481x_2\tilde{x}_1^3 + 481.046x_2\tilde{x}_1^2\tilde{x}_2 - 197.004x_2\tilde{x}_1\tilde{x}_2^2 + 214.356x_2\tilde{x}_2^3 - 95.5610\tilde{x}_1^4 + 184.212\tilde{x}_1^3\tilde{x}_2 - 333.759\tilde{x}_1^2\tilde{x}_2^2 + 289.699\tilde{x}_1\tilde{x}_2^3 - 179.941\tilde{x}_2^4$$

$$\Pi^{13}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -2411.759x_2^4 - 1271.176x_2^3\tilde{x}_1 - 645.551x_2^3\tilde{x}_2 - 5405.152x_2^2\tilde{x}_1^2 - 427.33x_2^2\tilde{x}_1\tilde{x}_2 - 846.756x_2^2\tilde{x}_2^2 - 1156.2x_2\tilde{x}_1^3 - 676.331x_2\tilde{x}_1^2\tilde{x}_2 - 215.184x_2\tilde{x}_1\tilde{x}_2^2 - 116.734x_2\tilde{x}_2^3 - 4247.151\tilde{x}_1^4 - 92.205\tilde{x}_1^3\tilde{x}_2 - 1143.006\tilde{x}_1^2\tilde{x}_2^2 + 141.949\tilde{x}_1\tilde{x}_2^3 - 189.306\tilde{x}_2^4$$

$$\Pi^{23}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -714.237x_2^4 - 612.543x_2^3\tilde{x}_1 - 350.197x_2^3\tilde{x}_2 - 669.758x_2^2\tilde{x}_1^2 - 1029.082x_2^2\tilde{x}_1\tilde{x}_2 + 189.102x_2^2\tilde{x}_2^2 - 391.386x_2\tilde{x}_1^3 - 357.295x_2\tilde{x}_1^2\tilde{x}_2 - 98.562x_2\tilde{x}_1\tilde{x}_2^2 - 14.56x_2\tilde{x}_2^3 - 10.592\tilde{x}_1^4 - 931.814\tilde{x}_1^3\tilde{x}_2 + 526.883\tilde{x}_1^2\tilde{x}_2^2 - 334.159\tilde{x}_1\tilde{x}_2^3 + 241.15\tilde{x}_2^4$$

$$\Pi^{33}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 10157.233x_2^4 + 4543.378x_2^3\tilde{x}_1 + 1603.134x_2^3\tilde{x}_2 + 15243.306x_2^2\tilde{x}_1^2 + 1313.18x_2^2\tilde{x}_1\tilde{x}_2 + 4751.176x_2^2\tilde{x}_2^2 + 3568.581x_2\tilde{x}_1^3 + 1669.747x_2\tilde{x}_1^2\tilde{x}_2 + 650.152x_2\tilde{x}_1\tilde{x}_2^2 + 237.039x_2\tilde{x}_2^3 + 11081.026\tilde{x}_1^4 + 492.54\tilde{x}_1^3\tilde{x}_2 + 4705.431\tilde{x}_1^2\tilde{x}_2^2 - 22.199\tilde{x}_1\tilde{x}_2^3 + 3179.011\tilde{x}_2^4$$

$$\Pi^{43}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -956.889x_2^4 - 2925.629x_2^3\tilde{x}_1 - 574.357x_2^3\tilde{x}_2 - 1553.326x_2^2\tilde{x}_1^2 + 133.131x_2^2\tilde{x}_1\tilde{x}_2 - 660.026x_2^2\tilde{x}_2^2 - 2800.378x_2\tilde{x}_1^3 - 342.878x_2\tilde{x}_1^2\tilde{x}_2 - 664.946x_2\tilde{x}_1\tilde{x}_2^2 - 26.487x_2\tilde{x}_2^3 - 443.193\tilde{x}_1^4 + 558.482\tilde{x}_1^3\tilde{x}_2 - 722.878\tilde{x}_1^2\tilde{x}_2^2 + 253.138\tilde{x}_1\tilde{x}_2^3 - 272.806\tilde{x}_2^4$$

$$\Pi^{14}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 342.159x_2^4 + 1367.785x_2^3\tilde{x}_1 + 165.62x_2^3\tilde{x}_2 + 552.488x_2^2\tilde{x}_1^2 - 247.214x_2^2\tilde{x}_1\tilde{x}_2 + 320.46x_2^2\tilde{x}_2^2 + 1611.962x_2\tilde{x}_1^3 + 131.336x_2\tilde{x}_1^2\tilde{x}_2 + 376.211x_2\tilde{x}_1\tilde{x}_2^2 - 44.685x_2\tilde{x}_2^3 + 225.276\tilde{x}_1^4 - 433.317\tilde{x}_1^3\tilde{x}_2 + 469.87\tilde{x}_1^2\tilde{x}_2^2 - 212.003\tilde{x}_1\tilde{x}_2^3 + 137.802\tilde{x}_2^4$$

$$\Pi^{24}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -96.255x_2^4 + 252.757x_2^3\tilde{x}_1 + 548.367x_2^3\tilde{x}_2 - 18.074x_2^2\tilde{x}_1^2 + 287.363x_2^2\tilde{x}_1\tilde{x}_2 - 293.591x_2^2\tilde{x}_2^2 + 46.481x_2\tilde{x}_1^3 + 481.046x_2\tilde{x}_1^2\tilde{x}_2 - 197.004x_2\tilde{x}_1\tilde{x}_2^2 + 214.356x_2\tilde{x}_2^3 - 95.5610\tilde{x}_1^4 + 184.212\tilde{x}_1^3\tilde{x}_2 - 333.759\tilde{x}_1^2\tilde{x}_2^2 + 289.699\tilde{x}_1\tilde{x}_2^3 - 179.941\tilde{x}_2^4$$

$$\Pi^{34}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -956.889x_2^4 - 2925.629x_2^3\tilde{x}_1 - 574.357x_2^3\tilde{x}_2 - 1553.326x_2^2\tilde{x}_1^2 + 133.131x_2^2\tilde{x}_1\tilde{x}_2 - 660.026x_2^2\tilde{x}_2^2 - 2800.378x_2\tilde{x}_1^3 - 342.878x_2\tilde{x}_1^2\tilde{x}_2 - 664.946x_2\tilde{x}_1\tilde{x}_2^2 - 26.487x_2\tilde{x}_2^3 - 443.193\tilde{x}_1^4 + 558.482\tilde{x}_1^3\tilde{x}_2 - 722.878\tilde{x}_1^2\tilde{x}_2^2 + 253.138\tilde{x}_1\tilde{x}_2^3 - 272.806\tilde{x}_2^4$$

$$\Pi^{44}_{2112}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 3779.898x_2^4 + 496.384x_2^3\tilde{x}_1 - 874.671x_2^3\tilde{x}_2 + 3749.294x_2^2\tilde{x}_1^2 - 41.339x_2^2\tilde{x}_1\tilde{x}_2 + 2919.442x_2^2\tilde{x}_2^2 + 263.818x_2\tilde{x}_1^3 - 716.544x_2\tilde{x}_1^2\tilde{x}_2 + 534.928x_2\tilde{x}_1\tilde{x}_2^2 - 362.914x_2\tilde{x}_2^3 + 2650.823\tilde{x}_1^4 - 168.933\tilde{x}_1^3\tilde{x}_2 + 2431.35\tilde{x}_1^2\tilde{x}_2^2 - 325.892\tilde{x}_1\tilde{x}_2^3 + 2498.828\tilde{x}_2^4$$

$$\Pi^{11}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 27018.797x_2^4 + 948.573x_2^3\tilde{x}_1 + 1106.205x_2^3\tilde{x}_2 + 29335.504x_2^2\tilde{x}_1^2 - 339.04x_2^2\tilde{x}_1\tilde{x}_2 + 20390.112x_2^2\tilde{x}_2^2 + 1172.53x_2\tilde{x}_1^3 + 916.255x_2\tilde{x}_1^2\tilde{x}_2 - 351.407x_2\tilde{x}_1\tilde{x}_2^2 + 360.051x_2\tilde{x}_2^3 + 41147.973\tilde{x}_1^4 - 1358.083\tilde{x}_1^3\tilde{x}_2 + 20691.045\tilde{x}_1^2\tilde{x}_2^2 - 1479.939\tilde{x}_1\tilde{x}_2^3 + 23099.699\tilde{x}_2^4$$

$$\Pi^{21}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -119.907x_2^4 + 1057.969x_2^3\tilde{x}_1 - 342.786x_2^3\tilde{x}_2 - 137.072x_2^2\tilde{x}_1^2 + 2363.045x_2^2\tilde{x}_1\tilde{x}_2 - 1202.772x_2^2\tilde{x}_2^2 + 633.647x_2\tilde{x}_1^3 - 294.445x_2\tilde{x}_1^2\tilde{x}_2 + 474.652x_2\tilde{x}_1\tilde{x}_2^2 - 361.441x_2\tilde{x}_2^3 - 810.793\tilde{x}_1^4 + 2487.451\tilde{x}_1^3\tilde{x}_2 - 2258.5\tilde{x}_1^2\tilde{x}_2^2 + 1755.323\tilde{x}_1\tilde{x}_2^3 - 1062.617\tilde{x}_2^4$$

$$\Pi^{31}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -8180.553x_2^4 - 3256.031x_2^3\tilde{x}_1 - 1602.099x_2^3\tilde{x}_2 - 15684.67x_2^2\tilde{x}_1^2 - 363.35x_2^2\tilde{x}_1\tilde{x}_2 - 1024.01x_2^2\tilde{x}_2^2 - 2722.097x_2\tilde{x}_1^3 - 1473.898x_2\tilde{x}_1^2\tilde{x}_2 + 247.544x_2\tilde{x}_1\tilde{x}_2^2 - 394.272x_2\tilde{x}_2^3 - 9485.53\tilde{x}_1^4 + 194.599\tilde{x}_1^3\tilde{x}_2 - 1596.043\tilde{x}_1^2\tilde{x}_2^2 + 880.673\tilde{x}_1\tilde{x}_2^3 - 601.163\tilde{x}_2^4$$

$$\Pi^{41}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 954.609x_2^4 + 7560.401x_2^3\tilde{x}_1 + 76.7630x_2^3\tilde{x}_2 + 1696.826x_2^2\tilde{x}_1^2 - 954.815x_2^2\tilde{x}_1\tilde{x}_2 + 777.009x_2^2\tilde{x}_2^2 + 7965.905x_2\tilde{x}_1^3 - 48.649x_2\tilde{x}_1^2\tilde{x}_2 + 1328.135x_2\tilde{x}_1\tilde{x}_2^2 - 557.070x_2\tilde{x}_2^3 + 1583.411\tilde{x}_1^4 - 1863.571\tilde{x}_1^3\tilde{x}_2 + 1904.37\tilde{x}_1^2\tilde{x}_2^2 - 1408.384\tilde{x}_1\tilde{x}_2^3 + 848.904\tilde{x}_2^4$$

$$\Pi^{12}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -119.907x_2^4 + 1057.969x_2^3\tilde{x}_1 - 342.786x_2^3\tilde{x}_2 - 137.072x_2^2\tilde{x}_1^2 + 2363.045x_2^2\tilde{x}_1\tilde{x}_2 - 1202.772x_2^2\tilde{x}_2^2 + 633.647x_2\tilde{x}_1^3 - 294.445x_2\tilde{x}_1^2\tilde{x}_2 + 474.652x_2\tilde{x}_1\tilde{x}_2^2 - 361.441x_2\tilde{x}_2^3 - 810.793\tilde{x}_1^4 + 2487.451\tilde{x}_1^3\tilde{x}_2 - 2258.5\tilde{x}_1^2\tilde{x}_2^2 + 1755.323\tilde{x}_1\tilde{x}_2^3 - 1062.617\tilde{x}_2^4$$

$$\Pi^{22}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 25686.672x_2^4 - 370.246x_2^3\tilde{x}_1 + 1798.36x_2^3\tilde{x}_2 + 20439.065x_2^2\tilde{x}_1^2 - 1241.394x_2^2\tilde{x}_1\tilde{x}_2 + 21776.957x_2^2\tilde{x}_2^2 - 268.208x_2\tilde{x}_1^3 + 526.463x_2\tilde{x}_1^2\tilde{x}_2 - 518.699x_2\tilde{x}_1\tilde{x}_2^2 + 537.079x_2\tilde{x}_2^3 + 23144.054\tilde{x}_1^4 - 1416.056\tilde{x}_1^3\tilde{x}_2 + 21379.893\tilde{x}_1^2\tilde{x}_2^2 - 1961.317\tilde{x}_1\tilde{x}_2^3 + 23607.531\tilde{x}_2^4$$

$$\Pi^{32}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -1820.218x_2^4 - 1539.265x_2^3\tilde{x}_1 - 203.633x_2^3\tilde{x}_2 - 441.534x_2^2\tilde{x}_1^2 - 1951.146x_2^2\tilde{x}_1\tilde{x}_2 + 630.744x_2^2\tilde{x}_2^2 - 898.746x_2\tilde{x}_1^3 + 98.1500x_2\tilde{x}_1^2\tilde{x}_2 - 518.37x_2\tilde{x}_1\tilde{x}_2^2 + 188.886x_2\tilde{x}_2^3 + 133.339\tilde{x}_1^4 - 1645.647\tilde{x}_1^3\tilde{x}_2 + 1585.858\tilde{x}_1^2\tilde{x}_2^2 - 1057.763\tilde{x}_1\tilde{x}_2^3 + 1583.695\tilde{x}_2^4$$

$$\Pi^{42}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 1178.331x_2^4 + 122.232x_2^3\tilde{x}_1 + 4580.571x_2^3\tilde{x}_2 - 455.801x_2^2\tilde{x}_1^2 + 908.444x_2^2\tilde{x}_1\tilde{x}_2 - 943.587x_2^2\tilde{x}_2^2 - 11.485x_2\tilde{x}_1^3 + 1714.184x_2\tilde{x}_1^2\tilde{x}_2 - 1057.053x_2\tilde{x}_1\tilde{x}_2^2 + 1544.658x_2\tilde{x}_2^3 - 805.744\tilde{x}_1^4 + 1259.479\tilde{x}_1^3\tilde{x}_2 - 1911.433\tilde{x}_1^2\tilde{x}_2^2 + 1586.343\tilde{x}_1\tilde{x}_2^3 - 1121.598\tilde{x}_2^4$$

$$\begin{aligned}\Pi^{13}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -8180.553x_2^4 - 3256.031x_2^3\tilde{x}_1 - 1602.099x_2^3\tilde{x}_2 - 15684.67x_2^2\tilde{x}_1^2 - 363.35x_2^2\tilde{x}_1\tilde{x}_2 - \\ & 1024.01x_2^2\tilde{x}_2^2 - 2722.097x_2\tilde{x}_1^3 - 1473.898x_2\tilde{x}_1^2\tilde{x}_2 + 247.544x_2\tilde{x}_1\tilde{x}_2^2 - 394.272x_2\tilde{x}_2^3 - \\ & 9485.53\tilde{x}_1^4 + 194.599\tilde{x}_1^3\tilde{x}_2 - 1596.043\tilde{x}_1^2\tilde{x}_2^2 + 880.673\tilde{x}_1\tilde{x}_2^3 - 601.163\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{23}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -1820.218x_2^4 - 1539.265x_2^3\tilde{x}_1 - 203.633x_2^3\tilde{x}_2 - 441.534x_2^2\tilde{x}_1^2 - 1951.146x_2^2\tilde{x}_1\tilde{x}_2 + \\ & 630.744x_2^2\tilde{x}_2^2 - 898.746x_2\tilde{x}_1^3 + 98.1500x_2\tilde{x}_1^2\tilde{x}_2 - 518.37x_2\tilde{x}_1\tilde{x}_2^2 + 188.886x_2\tilde{x}_2^3 + \\ & 133.339\tilde{x}_1^4 - 1645.647\tilde{x}_1^3\tilde{x}_2 + 1585.858\tilde{x}_1^2\tilde{x}_2^2 - 1057.763\tilde{x}_1\tilde{x}_2^3 + 1583.695\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{33}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 71211.796x_2^4 + 17044.284x_2^3\tilde{x}_1 + 4035.496x_2^3\tilde{x}_2 + 68888.861x_2^2\tilde{x}_1^2 + 925.938x_2^2\tilde{x}_1\tilde{x}_2 + \\ & 31723.813x_2^2\tilde{x}_2^2 + 8408.841x_2\tilde{x}_1^3 + 4357.383x_2\tilde{x}_1^2\tilde{x}_2 - 876.636x_2\tilde{x}_1\tilde{x}_2^2 - 615.856x_2\tilde{x}_2^3 + \\ & 59928.367\tilde{x}_1^4 - 578.211\tilde{x}_1^3\tilde{x}_2 + 28856.832\tilde{x}_1^2\tilde{x}_2^2 + 531.501\tilde{x}_1\tilde{x}_2^3 + 37667.781\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{43}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -2957.622x_2^4 - 10176.475x_2^3\tilde{x}_1 - 2208.879x_2^3\tilde{x}_2 - 4011.199x_2^2\tilde{x}_1^2 - 180.224x_2^2\tilde{x}_1\tilde{x}_2 - \\ & 617.044x_2^2\tilde{x}_2^2 - 9549.712x_2\tilde{x}_1^3 - 613.317x_2\tilde{x}_1^2\tilde{x}_2 - 742.766x_2\tilde{x}_1\tilde{x}_2^2 + 68.035x_2\tilde{x}_2^3 - \\ & 1618.552\tilde{x}_1^4 + 698.788\tilde{x}_1^3\tilde{x}_2 - 1266.846\tilde{x}_1^2\tilde{x}_2^2 + 683.241\tilde{x}_1\tilde{x}_2^3 - 1487.309\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{14}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 954.609x_2^4 + 7560.401x_2^3\tilde{x}_1 + 76.7630x_2^3\tilde{x}_2 + 1696.826x_2^2\tilde{x}_1^2 - 954.815x_2^2\tilde{x}_1\tilde{x}_2 + \\ & 777.009x_2^2\tilde{x}_2^2 + 7965.905x_2\tilde{x}_1^3 - 48.649x_2\tilde{x}_1^2\tilde{x}_2 + 1328.135x_2\tilde{x}_1\tilde{x}_2^2 - 557.070x_2\tilde{x}_2^3 + \\ & 1583.411\tilde{x}_1^4 - 1863.571\tilde{x}_1^3\tilde{x}_2 + 1904.37\tilde{x}_1^2\tilde{x}_2^2 - 1408.384\tilde{x}_1\tilde{x}_2^3 + 848.904\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{24}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 1178.331x_2^4 + 122.232x_2^3\tilde{x}_1 + 4580.571x_2^3\tilde{x}_2 - 455.801x_2^2\tilde{x}_1^2 + 908.444x_2^2\tilde{x}_1\tilde{x}_2 - \\ & 943.587x_2^2\tilde{x}_2^2 - 11.485x_2\tilde{x}_1^3 + 1714.184x_2\tilde{x}_1^2\tilde{x}_2 - 1057.053x_2\tilde{x}_1\tilde{x}_2^2 + 1544.658x_2\tilde{x}_2^3 - \\ & 805.744\tilde{x}_1^4 + 1259.479\tilde{x}_1^3\tilde{x}_2 - 1911.433\tilde{x}_1^2\tilde{x}_2^2 + 1586.343\tilde{x}_1\tilde{x}_2^3 - 1121.598\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{34}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -2957.622x_2^4 - 10176.475x_2^3\tilde{x}_1 - 2208.879x_2^3\tilde{x}_2 - 4011.199x_2^2\tilde{x}_1^2 - 180.224x_2^2\tilde{x}_1\tilde{x}_2 - \\ & 617.044x_2^2\tilde{x}_2^2 - 9549.712x_2\tilde{x}_1^3 - 613.317x_2\tilde{x}_1^2\tilde{x}_2 - 742.766x_2\tilde{x}_1\tilde{x}_2^2 + 68.035x_2\tilde{x}_2^3 - \\ & 1618.552\tilde{x}_1^4 + 698.788\tilde{x}_1^3\tilde{x}_2 - 1266.846\tilde{x}_1^2\tilde{x}_2^2 + 683.241\tilde{x}_1\tilde{x}_2^3 - 1487.309\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{44}_{2121}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 38095.274x_2^4 + 1602.054x_2^3\tilde{x}_1 - 3959.662x_2^3\tilde{x}_2 + 28441.516x_2^2\tilde{x}_1^2 - 901.611x_2^2\tilde{x}_1\tilde{x}_2 + \\ & 24717.225x_2^2\tilde{x}_2^2 + 1239.7x_2\tilde{x}_1^3 - 2657.601x_2\tilde{x}_1^2\tilde{x}_2 + 2027.732x_2\tilde{x}_1\tilde{x}_2^2 - 2133.395x_2\tilde{x}_2^3 + \\ & 26568.832\tilde{x}_1^4 - 1233.835\tilde{x}_1^3\tilde{x}_2 + 21792.084\tilde{x}_1^2\tilde{x}_2^2 - 2047.95\tilde{x}_1\tilde{x}_2^3 + 24036.44\tilde{x}_2^4\end{aligned}$$

$$\Pi^{11}_{2122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2596.255x_2^4 + 304.116x_2^3\tilde{x}_1 + 136.691x_2^3\tilde{x}_2 + 2791.184x_2^2\tilde{x}_1^2 + 144.985x_2^2\tilde{x}_1\tilde{x}_2 + 2022.337x_2^2\tilde{x}_2^2 + 293.762x_2\tilde{x}_1^3 + 159.598x_2\tilde{x}_1^2\tilde{x}_2 + 42.278x_2\tilde{x}_1\tilde{x}_2^2 + 16.944x_2\tilde{x}_2^3 + 2991.941\tilde{x}_1^4 + 49.545\tilde{x}_1^3\tilde{x}_2 + 2023.559\tilde{x}_1^2\tilde{x}_2^2 - 39.821\tilde{x}_1\tilde{x}_2^3 + 2261.426\tilde{x}_2^4$$

$$\Pi^{21}_{2122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 111.023x_2^4 + 131.388x_2^3\tilde{x}_1 + 75.984x_2^3\tilde{x}_2 + 167.522x_2^2\tilde{x}_1^2 + 203.33x_2^2\tilde{x}_1\tilde{x}_2 - 16.139x_2^2\tilde{x}_2^2 + 87.438x_2\tilde{x}_1^3 + 84.38x_2\tilde{x}_1^2\tilde{x}_2 + 27.385x_2\tilde{x}_1\tilde{x}_2^2 + 1.376x_2\tilde{x}_2^3 + 26.335\tilde{x}_1^4 + 146.321\tilde{x}_1^3\tilde{x}_2 - 73.057\tilde{x}_1^2\tilde{x}_2^2 + 65.77\tilde{x}_1\tilde{x}_2^3 - 32.859\tilde{x}_2^4$$

$$\Pi^{31}_{2122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -970.118x_2^4 - 945.407x_2^3\tilde{x}_1 - 358.299x_2^3\tilde{x}_2 - 2148.59x_2^2\tilde{x}_1^2 - 396.363x_2^2\tilde{x}_1\tilde{x}_2 - 226.156x_2^2\tilde{x}_2^2 - 854.626x_2\tilde{x}_1^3 - 420.745x_2\tilde{x}_1^2\tilde{x}_2 - 118.191x_2\tilde{x}_1\tilde{x}_2^2 - 40.067x_2\tilde{x}_2^3 - 1122.044\tilde{x}_1^4 - 164.877\tilde{x}_1^3\tilde{x}_2 - 229.793\tilde{x}_1^2\tilde{x}_2^2 + 30.902\tilde{x}_1\tilde{x}_2^3 - 41.349\tilde{x}_2^4$$

$$\Pi^{41}_{2122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 183.688x_2^4 + 521.672x_2^3\tilde{x}_1 + 102.904x_2^3\tilde{x}_2 + 369.08x_2^2\tilde{x}_1^2 + 18.622x_2^2\tilde{x}_1\tilde{x}_2 + 91.859x_2^2\tilde{x}_2^2 + 505.407x_2\tilde{x}_1^3 + 101.948x_2\tilde{x}_1^2\tilde{x}_2 + 114.096x_2\tilde{x}_1\tilde{x}_2^2 - 8.855x_2\tilde{x}_2^3 + 116.481\tilde{x}_1^4 - 64.4120\tilde{x}_1^3\tilde{x}_2 + 111.531\tilde{x}_1^2\tilde{x}_2^2 - 52.16\tilde{x}_1\tilde{x}_2^3 + 34.443\tilde{x}_2^4$$

$$\Pi^{12}_{2122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 111.023x_2^4 + 131.388x_2^3\tilde{x}_1 + 75.984x_2^3\tilde{x}_2 + 167.522x_2^2\tilde{x}_1^2 + 203.33x_2^2\tilde{x}_1\tilde{x}_2 - 16.139x_2^2\tilde{x}_2^2 + 87.438x_2\tilde{x}_1^3 + 84.38x_2\tilde{x}_1^2\tilde{x}_2 + 27.385x_2\tilde{x}_1\tilde{x}_2^2 + 1.376x_2\tilde{x}_2^3 + 26.335\tilde{x}_1^4 + 146.321\tilde{x}_1^3\tilde{x}_2 - 73.057\tilde{x}_1^2\tilde{x}_2^2 + 65.77\tilde{x}_1\tilde{x}_2^3 - 32.859\tilde{x}_2^4$$

$$\Pi^{22}_{2122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2343.807x_2^4 + 56.451x_2^3\tilde{x}_1 + 52.824x_2^3\tilde{x}_2 + 2009.829x_2^2\tilde{x}_1^2 + 30.77x_2^2\tilde{x}_1\tilde{x}_2 + 1999.811x_2^2\tilde{x}_2^2 + 15.986x_2\tilde{x}_1^3 + 46.392x_2\tilde{x}_1^2\tilde{x}_2 + 11.339x_2\tilde{x}_1\tilde{x}_2^2 + 0.178x_2\tilde{x}_2^3 + 2267.573\tilde{x}_1^4 - 27.705\tilde{x}_1^3\tilde{x}_2 + 2007.028\tilde{x}_1^2\tilde{x}_2^2 - 70.923\tilde{x}_1\tilde{x}_2^3 + 2273.601\tilde{x}_2^4$$

$$\Pi^{32}_{2122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -349.429x_2^4 - 383.85x_2^3\tilde{x}_1 - 210.361x_2^3\tilde{x}_2 - 443.819x_2^2\tilde{x}_1^2 - 415.122x_2^2\tilde{x}_1\tilde{x}_2 - 4.16x_2^2\tilde{x}_2^2 - 232.987x_2\tilde{x}_1^3 - 233.81x_2\tilde{x}_1^2\tilde{x}_2 - 61.613x_2\tilde{x}_1\tilde{x}_2^2 - 8.8379x_2\tilde{x}_2^3 - 78.0070\tilde{x}_1^4 - 250.769\tilde{x}_1^3\tilde{x}_2 + 81.345\tilde{x}_1^2\tilde{x}_2^2 - 80.681\tilde{x}_1\tilde{x}_2^3 + 56.813\tilde{x}_2^4$$

$$\Pi^{42}_{2122}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 24.812x_2^4 + 133.53x_2^3\tilde{x}_1 + 192.965x_2^3\tilde{x}_2 + 59.979x_2^2\tilde{x}_1^2 + 123.796x_2^2\tilde{x}_1\tilde{x}_2 - 54.107x_2^2\tilde{x}_2^2 + 55.302x_2\tilde{x}_1^3 + 150.989x_2\tilde{x}_1^2\tilde{x}_2 - 29.491x_2\tilde{x}_1\tilde{x}_2^2 + 60.938x_2\tilde{x}_2^3 - 17.698\tilde{x}_1^4 + 63.776\tilde{x}_1^3\tilde{x}_2 - 78.977\tilde{x}_1^2\tilde{x}_2^2 + 72.345\tilde{x}_1\tilde{x}_2^3 - 43.445\tilde{x}_2^4$$

$$\Pi_{2122}^{13}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -970.118x_2^4 - 945.407x_2^3\tilde{x}_1 - 358.299x_2^3\tilde{x}_2 - 2148.59x_2^2\tilde{x}_1^2 - 396.363x_2^2\tilde{x}_1\tilde{x}_2 - 226.156x_2^2\tilde{x}_2^2 - 854.626x_2\tilde{x}_1^3 - 420.745x_2\tilde{x}_1^2\tilde{x}_2 - 118.191x_2\tilde{x}_1\tilde{x}_2^2 - 40.067x_2\tilde{x}_2^3 - 1122.044\tilde{x}_1^4 - 164.877\tilde{x}_1^3\tilde{x}_2 - 229.793\tilde{x}_1^2\tilde{x}_2^2 + 30.902\tilde{x}_1\tilde{x}_2^3 - 41.349\tilde{x}_2^4$$

$$\Pi_{2122}^{23}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -349.429x_2^4 - 383.85x_2^3\tilde{x}_1 - 210.361x_2^3\tilde{x}_2 - 443.819x_2^2\tilde{x}_1^2 - 415.122x_2^2\tilde{x}_1\tilde{x}_2 - 4.16x_2^2\tilde{x}_2^2 - 232.987x_2\tilde{x}_1^3 - 233.81x_2\tilde{x}_1^2\tilde{x}_2 - 61.613x_2\tilde{x}_1\tilde{x}_2^2 - 8.8379x_2\tilde{x}_2^3 - 78.0070\tilde{x}_1^4 - 250.769\tilde{x}_1^3\tilde{x}_2 + 81.345\tilde{x}_1^2\tilde{x}_2^2 - 80.681\tilde{x}_1\tilde{x}_2^3 + 56.813\tilde{x}_2^4$$

$$\Pi_{2122}^{33}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 5606.859x_2^4 + 3241.144x_2^3\tilde{x}_1 + 945.666x_2^3\tilde{x}_2 + 8088.73x_2^2\tilde{x}_1^2 + 1111.219x_2^2\tilde{x}_1\tilde{x}_2 + 2674.511x_2^2\tilde{x}_2^2 + 2641.082x_2\tilde{x}_1^3 + 1124.636x_2\tilde{x}_1^2\tilde{x}_2 + 301.66x_2\tilde{x}_1\tilde{x}_2^2 + 70.14x_2\tilde{x}_2^3 + 5270.637\tilde{x}_1^4 + 476.658\tilde{x}_1^3\tilde{x}_2 + 2530.093\tilde{x}_1^2\tilde{x}_2^2 + 31.559\tilde{x}_1\tilde{x}_2^3 + 2529.071\tilde{x}_2^4$$

$$\Pi_{2122}^{43}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -551.123x_2^4 - 1226.255x_2^3\tilde{x}_1 - 308.317x_2^3\tilde{x}_2 - 1085.896x_2^2\tilde{x}_1^2 - 183.629x_2^2\tilde{x}_1\tilde{x}_2 - 194.978x_2^2\tilde{x}_2^2 - 1042.718x_2\tilde{x}_1^3 - 252.033x_2\tilde{x}_1^2\tilde{x}_2 - 192.455x_2\tilde{x}_1\tilde{x}_2^2 - 14.823x_2\tilde{x}_2^3 - 289.252\tilde{x}_1^4 + 33.983\tilde{x}_1^3\tilde{x}_2 - 183.296\tilde{x}_1^2\tilde{x}_2^2 + 46.813\tilde{x}_1\tilde{x}_2^3 - 63.54\tilde{x}_2^4$$

$$\Pi_{2122}^{14}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 183.688x_2^4 + 521.672x_2^3\tilde{x}_1 + 102.904x_2^3\tilde{x}_2 + 369.08x_2^2\tilde{x}_1^2 + 18.622x_2^2\tilde{x}_1\tilde{x}_2 + 91.859x_2^2\tilde{x}_2^2 + 505.407x_2\tilde{x}_1^3 + 101.948x_2\tilde{x}_1^2\tilde{x}_2 + 114.096x_2\tilde{x}_1\tilde{x}_2^2 - 8.855x_2\tilde{x}_2^3 + 116.481\tilde{x}_1^4 - 64.4120\tilde{x}_1^3\tilde{x}_2 + 111.531\tilde{x}_1^2\tilde{x}_2^2 - 52.16\tilde{x}_1\tilde{x}_2^3 + 34.443\tilde{x}_2^4$$

$$\Pi_{2122}^{24}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 24.812x_2^4 + 133.53x_2^3\tilde{x}_1 + 192.965x_2^3\tilde{x}_2 + 59.979x_2^2\tilde{x}_1^2 + 123.796x_2^2\tilde{x}_1\tilde{x}_2 - 54.107x_2^2\tilde{x}_2^2 + 55.302x_2\tilde{x}_1^3 + 150.989x_2\tilde{x}_1^2\tilde{x}_2 - 29.491x_2\tilde{x}_1\tilde{x}_2^2 + 60.938x_2\tilde{x}_2^3 - 17.698\tilde{x}_1^4 + 63.776\tilde{x}_1^3\tilde{x}_2 - 78.977\tilde{x}_1^2\tilde{x}_2^2 + 72.345\tilde{x}_1\tilde{x}_2^3 - 43.445\tilde{x}_2^4$$

$$\Pi_{2122}^{34}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -551.123x_2^4 - 1226.255x_2^3\tilde{x}_1 - 308.317x_2^3\tilde{x}_2 - 1085.896x_2^2\tilde{x}_1^2 - 183.629x_2^2\tilde{x}_1\tilde{x}_2 - 194.978x_2^2\tilde{x}_2^2 - 1042.718x_2\tilde{x}_1^3 - 252.033x_2\tilde{x}_1^2\tilde{x}_2 - 192.455x_2\tilde{x}_1\tilde{x}_2^2 - 14.823x_2\tilde{x}_2^3 - 289.252\tilde{x}_1^4 + 33.983\tilde{x}_1^3\tilde{x}_2 - 183.296\tilde{x}_1^2\tilde{x}_2^2 + 46.813\tilde{x}_1\tilde{x}_2^3 - 63.54\tilde{x}_2^4$$

$$\Pi_{2122}^{44}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2737.044x_2^4 + 263.045x_2^3\tilde{x}_1 - 180.684x_2^3\tilde{x}_2 + 2535.315x_2^2\tilde{x}_1^2 + 56.387x_2^2\tilde{x}_1\tilde{x}_2 + 2190.578x_2^2\tilde{x}_2^2 + 157.209x_2\tilde{x}_1^3 - 126.451x_2\tilde{x}_1^2\tilde{x}_2 + 139.029x_2\tilde{x}_1\tilde{x}_2^2 - 92.767x_2\tilde{x}_2^3 + 2404.877\tilde{x}_1^4 - 31.069\tilde{x}_1^3\tilde{x}_2 + 2057.51\tilde{x}_1^2\tilde{x}_2^2 - 81.7720\tilde{x}_1\tilde{x}_2^3 + 2299.597\tilde{x}_2^4$$

$$\begin{aligned}\Pi_{2211}^{11}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 36169.361x_2^4 + 744.277x_2^3\tilde{x}_1 + 2708.625x_2^3\tilde{x}_2 + 46749.873x_2^2\tilde{x}_1^2 - 2800.523x_2^2\tilde{x}_1\tilde{x}_2 + \\ & 22998.102x_2^2\tilde{x}_2^2 + 385.199x_2\tilde{x}_1^3 + 2085.406x_2\tilde{x}_1^2\tilde{x}_2 - 891.982x_2\tilde{x}_1\tilde{x}_2^2 + 1128.695x_2\tilde{x}_2^3 + \\ & 57412.888\tilde{x}_1^4 - 5592.272\tilde{x}_1^3\tilde{x}_2 + 26300.37\tilde{x}_1^2\tilde{x}_2^2 - 5167.7\tilde{x}_1\tilde{x}_2^3 + 25549.072\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi_{2211}^{21}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -1578.341x_2^4 + 2301.396x_2^3\tilde{x}_1 - 1403.427x_2^3\tilde{x}_2 - 2132.626x_2^2\tilde{x}_1^2 + 6664.693x_2^2\tilde{x}_1\tilde{x}_2 - \\ & 4438.507x_2^2\tilde{x}_2^2 + 1448.884x_2\tilde{x}_1^3 - 1010.68x_2\tilde{x}_1^2\tilde{x}_2 + 1258.485x_2\tilde{x}_1\tilde{x}_2^2 - 1130.166x_2\tilde{x}_2^3 - \\ & 3526.623\tilde{x}_1^4 + 7970.886\tilde{x}_1^3\tilde{x}_2 - 8401.518\tilde{x}_1^2\tilde{x}_2^2 + 5956.956\tilde{x}_1\tilde{x}_2^3 - 3859.652\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi_{2211}^{31}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -16539.114x_2^4 - 1273.055x_2^3\tilde{x}_1 - 2282.828x_2^3\tilde{x}_2 - 27948.609x_2^2\tilde{x}_1^2 + 493.986x_2^2\tilde{x}_1\tilde{x}_2 - \\ & 4815.416x_2^2\tilde{x}_2^2 - 355.067x_2\tilde{x}_1^3 - 1762.213x_2\tilde{x}_1^2\tilde{x}_2 + 273.169x_2\tilde{x}_1\tilde{x}_2^2 - 833.066x_2\tilde{x}_2^3 - \\ & 24971.601\tilde{x}_1^4 + 2364.566\tilde{x}_1^3\tilde{x}_2 - 7036.707\tilde{x}_1^2\tilde{x}_2^2 + 3347.507\tilde{x}_1\tilde{x}_2^3 - 2812.311\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi_{2211}^{41}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 2109.909x_2^4 + 23458.37x_2^3\tilde{x}_1 - 1604.208x_2^3\tilde{x}_2 + 3988.345x_2^2\tilde{x}_1^2 - 3887.774x_2^2\tilde{x}_1\tilde{x}_2 + \\ & 3202.342x_2^2\tilde{x}_2^2 + 20541.348x_2\tilde{x}_1^3 - 1869.323x_2\tilde{x}_1^2\tilde{x}_2 + 3866.842x_2\tilde{x}_1\tilde{x}_2^2 - 1858.573x_2\tilde{x}_2^3 + \\ & 3787.496\tilde{x}_1^4 - 6502.16\tilde{x}_1^3\tilde{x}_2 + 7402.756\tilde{x}_1^2\tilde{x}_2^2 - 4919.483\tilde{x}_1\tilde{x}_2^3 + 3197.556\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi_{2211}^{12}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -1578.341x_2^4 + 2301.396x_2^3\tilde{x}_1 - 1403.427x_2^3\tilde{x}_2 - 2132.626x_2^2\tilde{x}_1^2 + 6664.693x_2^2\tilde{x}_1\tilde{x}_2 - \\ & 4438.507x_2^2\tilde{x}_2^2 + 1448.884x_2\tilde{x}_1^3 - 1010.68x_2\tilde{x}_1^2\tilde{x}_2 + 1258.485x_2\tilde{x}_1\tilde{x}_2^2 - 1130.166x_2\tilde{x}_2^3 - \\ & 3526.623\tilde{x}_1^4 + 7970.886\tilde{x}_1^3\tilde{x}_2 - 8401.518\tilde{x}_1^2\tilde{x}_2^2 + 5956.956\tilde{x}_1\tilde{x}_2^3 - 3859.652\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi_{2211}^{22}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 29120.243x_2^4 - 1478.535x_2^3\tilde{x}_1 + 3143.758x_2^3\tilde{x}_2 + 22709.165x_2^2\tilde{x}_1^2 - 4190.026x_2^2\tilde{x}_1\tilde{x}_2 + \\ & 24923.883x_2^2\tilde{x}_2^2 - 966.743x_2\tilde{x}_1^3 + 1526.466x_2\tilde{x}_1^2\tilde{x}_2 - 1617.024x_2\tilde{x}_1\tilde{x}_2^2 + 1176.506x_2\tilde{x}_2^3 + \\ & 25159.236\tilde{x}_1^4 - 5110.099\tilde{x}_1^3\tilde{x}_2 + 26875.561\tilde{x}_1^2\tilde{x}_2^2 - 6944.945\tilde{x}_1\tilde{x}_2^3 + 26808.285\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi_{2211}^{32}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -4540.289x_2^4 - 1827.22x_2^3\tilde{x}_1 - 632.042x_2^3\tilde{x}_2 + 693.995x_2^2\tilde{x}_1^2 - 5209.894x_2^2\tilde{x}_1\tilde{x}_2 + \\ & 1664.239x_2^2\tilde{x}_2^2 - 1176.529x_2\tilde{x}_1^3 + 390.799x_2\tilde{x}_1^2\tilde{x}_2 - 840.63x_2\tilde{x}_1\tilde{x}_2^2 + 346.645x_2\tilde{x}_2^3 + \\ & 1867.566\tilde{x}_1^4 - 5044.715\tilde{x}_1^3\tilde{x}_2 + 5806.819\tilde{x}_1^2\tilde{x}_2^2 - 4255.094\tilde{x}_1\tilde{x}_2^3 + 4695.768\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi_{2211}^{42}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 1242.587x_2^4 - 1467.625x_2^3\tilde{x}_1 + 11046.84x_2^3\tilde{x}_2 - 2114.776x_2^2\tilde{x}_1^2 + 3071.853x_2^2\tilde{x}_1\tilde{x}_2 - \\ & 3408.488x_2^2\tilde{x}_2^2 - 1118.177x_2\tilde{x}_1^3 + 5123.157x_2\tilde{x}_1^2\tilde{x}_2 - 3922.438x_2\tilde{x}_1\tilde{x}_2^2 + 3393.788x_2\tilde{x}_2^3 - \\ & 2854.099\tilde{x}_1^4 + 4660.604\tilde{x}_1^3\tilde{x}_2 - 6849.496\tilde{x}_1^2\tilde{x}_2^2 + 5761.136\tilde{x}_1\tilde{x}_2^3 - 3834.937\tilde{x}_2^4\end{aligned}$$

$$\Pi^{13}_{2211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -16539.114x_2^4 - 1273.055x_2^3\tilde{x}_1 - 2282.828x_2^3\tilde{x}_2 - 27948.609x_2^2\tilde{x}_1^2 + 493.986x_2^2\tilde{x}_1\tilde{x}_2 - 4815.416x_2^2\tilde{x}_2^2 - 355.067x_2\tilde{x}_1^3 - 1762.213x_2\tilde{x}_1^2\tilde{x}_2 + 273.169x_2\tilde{x}_1\tilde{x}_2^2 - 833.066x_2\tilde{x}_2^3 - 24971.601\tilde{x}_1^4 + 2364.566\tilde{x}_1^3\tilde{x}_2 - 7036.707\tilde{x}_1^2\tilde{x}_2^2 + 3347.507\tilde{x}_1\tilde{x}_2^3 - 2812.311\tilde{x}_2^4$$

$$\Pi^{23}_{2211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -4540.289x_2^4 - 1827.22x_2^3\tilde{x}_1 - 632.042x_2^3\tilde{x}_2 + 693.995x_2^2\tilde{x}_1^2 - 5209.894x_2^2\tilde{x}_1\tilde{x}_2 + 1664.239x_2^2\tilde{x}_2^2 - 1176.529x_2\tilde{x}_1^3 + 390.799x_2\tilde{x}_1^2\tilde{x}_2 - 840.63x_2\tilde{x}_1\tilde{x}_2^2 + 346.645x_2\tilde{x}_2^3 + 1867.566\tilde{x}_1^4 - 5044.715\tilde{x}_1^3\tilde{x}_2 + 5806.819\tilde{x}_1^2\tilde{x}_2^2 - 4255.094\tilde{x}_1\tilde{x}_2^3 + 4695.768\tilde{x}_2^4$$

$$\Pi^{33}_{2211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 83082.403x_2^4 + 760.549x_2^3\tilde{x}_1 + 3384.47x_2^3\tilde{x}_2 + 73292.893x_2^2\tilde{x}_1^2 + 1162.337x_2^2\tilde{x}_1\tilde{x}_2 + 56831.356x_2^2\tilde{x}_2^2 + 615.163x_2\tilde{x}_1^3 + 1579.121x_2\tilde{x}_1^2\tilde{x}_2 + 272.61x_2\tilde{x}_1\tilde{x}_2^2 - 344.042x_2\tilde{x}_2^3 + 68462.723\tilde{x}_1^4 - 348.139\tilde{x}_1^3\tilde{x}_2 + 50108.74\tilde{x}_1^2\tilde{x}_2^2 - 1755.299\tilde{x}_1\tilde{x}_2^3 + 66237.157\tilde{x}_2^4$$

$$\Pi^{43}_{2211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -3217.034x_2^4 - 21944.909x_2^3\tilde{x}_1 - 5648.646x_2^3\tilde{x}_2 - 2880.643x_2^2\tilde{x}_1^2 + 2796.307x_2^2\tilde{x}_1\tilde{x}_2 - 2784.614x_2^2\tilde{x}_2^2 - 17470.755x_2\tilde{x}_1^3 - 138.542x_2\tilde{x}_1^2\tilde{x}_2 - 4389.831x_2\tilde{x}_1\tilde{x}_2^2 + 156.329x_2\tilde{x}_2^3 - 1937.35\tilde{x}_1^4 + 3756.764\tilde{x}_1^3\tilde{x}_2 - 5314.938\tilde{x}_1^2\tilde{x}_2^2 + 3513.259\tilde{x}_1\tilde{x}_2^3 - 4458.443$$

$$\Pi^{14}_{2211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2109.909x_2^4 + 23458.37x_2^3\tilde{x}_1 - 1604.208x_2^3\tilde{x}_2 + 3988.345x_2^2\tilde{x}_1^2 - 3887.774x_2^2\tilde{x}_1\tilde{x}_2 + 3202.342x_2^2\tilde{x}_2^2 + 20541.348x_2\tilde{x}_1^3 - 1869.323x_2\tilde{x}_1^2\tilde{x}_2 + 3866.842x_2\tilde{x}_1\tilde{x}_2^2 - 1858.573x_2\tilde{x}_2^3 + 3787.496\tilde{x}_1^4 - 6502.16\tilde{x}_1^3\tilde{x}_2 + 7402.756\tilde{x}_1^2\tilde{x}_2^2 - 4919.483\tilde{x}_1\tilde{x}_2^3 + 3197.556\tilde{x}_2^4$$

$$\Pi^{24}_{2211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 1242.587x_2^4 - 1467.625x_2^3\tilde{x}_1 + 11046.84x_2^3\tilde{x}_2 - 2114.776x_2^2\tilde{x}_1^2 + 3071.853x_2^2\tilde{x}_1\tilde{x}_2 - 3408.488x_2^2\tilde{x}_2^2 - 1118.177x_2\tilde{x}_1^3 + 5123.157x_2\tilde{x}_1^2\tilde{x}_2 - 3922.438x_2\tilde{x}_1\tilde{x}_2^2 + 3393.788x_2\tilde{x}_2^3 - 2854.099\tilde{x}_1^4 + 4660.604\tilde{x}_1^3\tilde{x}_2 - 6849.496\tilde{x}_1^2\tilde{x}_2^2 + 5761.136\tilde{x}_1\tilde{x}_2^3 - 3834.937\tilde{x}_2^4$$

$$\Pi^{34}_{2211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -3217.034x_2^4 - 21944.909x_2^3\tilde{x}_1 - 5648.646x_2^3\tilde{x}_2 - 2880.643x_2^2\tilde{x}_1^2 + 2796.307x_2^2\tilde{x}_1\tilde{x}_2 - 2784.614x_2^2\tilde{x}_2^2 - 17470.755x_2\tilde{x}_1^3 - 138.542x_2\tilde{x}_1^2\tilde{x}_2 - 4389.831x_2\tilde{x}_1\tilde{x}_2^2 + 156.329x_2\tilde{x}_2^3 - 1937.35\tilde{x}_1^4 + 3756.764\tilde{x}_1^3\tilde{x}_2 - 5314.938\tilde{x}_1^2\tilde{x}_2^2 + 3513.259\tilde{x}_1\tilde{x}_2^3 - 4458.443\tilde{x}_2^4$$

$$\Pi^{44}_{2211}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 77047.103x_2^4 + 5406.603x_2^3\tilde{x}_1 - 13091.502x_2^3\tilde{x}_2 + 49238.113x_2^2\tilde{x}_1^2 - 6085.848x_2^2\tilde{x}_1\tilde{x}_2 + 33339.305x_2^2\tilde{x}_2^2 + 3682.288x_2\tilde{x}_1^3 - 8726.107x_2\tilde{x}_1^2\tilde{x}_2 + 7202.013x_2\tilde{x}_1\tilde{x}_2^2 - 5210.363x_2\tilde{x}_2^3 + 33669.058\tilde{x}_1^4 - 5277.382\tilde{x}_1^3\tilde{x}_2 + 28168.589\tilde{x}_1^2\tilde{x}_2^2 - 7024.796\tilde{x}_1\tilde{x}_2^3 + 27433.192\tilde{x}_2^4$$

$$\Pi^{11}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2596.255x_2^4 + 304.117x_2^3\tilde{x}_1 + 136.691x_2^3\tilde{x}_2 + 2791.183x_2^2\tilde{x}_1^2 + 144.985x_2^2\tilde{x}_1\tilde{x}_2 + 2022.336x_2^2\tilde{x}_2^2 + 293.762x_2\tilde{x}_1^3 + 159.598x_2\tilde{x}_1^2\tilde{x}_2 + 42.278x_2\tilde{x}_1\tilde{x}_2^2 + 16.944x_2\tilde{x}_2^3 + 2991.941\tilde{x}_1^4 + 49.545\tilde{x}_1^3\tilde{x}_2 + 2023.559\tilde{x}_1^2\tilde{x}_2^2 - 39.821\tilde{x}_1\tilde{x}_2^3 + 2261.425\tilde{x}_2^4$$

$$\Pi^{21}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 111.023x_2^4 + 131.388x_2^3\tilde{x}_1 + 75.984x_2^3\tilde{x}_2 + 167.522x_2^2\tilde{x}_1^2 + 203.331x_2^2\tilde{x}_1\tilde{x}_2 - 16.139x_2^2\tilde{x}_2^2 + 87.44x_2\tilde{x}_1^3 + 84.38x_2\tilde{x}_1^2\tilde{x}_2 + 27.385x_2\tilde{x}_1\tilde{x}_2^2 + 1.376x_2\tilde{x}_2^3 + 26.335\tilde{x}_1^4 + 146.321\tilde{x}_1^3\tilde{x}_2 - 73.056\tilde{x}_1^2\tilde{x}_2^2 + 65.77\tilde{x}_1\tilde{x}_2^3 - 32.859\tilde{x}_2^4$$

$$\Pi^{31}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -970.118x_2^4 - 945.408x_2^3\tilde{x}_1 - 358.299x_2^3\tilde{x}_2 - 2148.59x_2^2\tilde{x}_1^2 - 396.363x_2^2\tilde{x}_1\tilde{x}_2 - 226.157x_2^2\tilde{x}_2^2 - 854.627x_2\tilde{x}_1^3 - 420.744x_2\tilde{x}_1^2\tilde{x}_2 - 118.192x_2\tilde{x}_1\tilde{x}_2^2 - 40.068x_2\tilde{x}_2^3 - 1122.044\tilde{x}_1^4 - 164.877\tilde{x}_1^3\tilde{x}_2 - 229.793\tilde{x}_1^2\tilde{x}_2^2 + 30.902\tilde{x}_1\tilde{x}_2^3 - 41.35\tilde{x}_2^4$$

$$\Pi^{41}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 183.689x_2^4 + 521.672x_2^3\tilde{x}_1 + 102.904x_2^3\tilde{x}_2 + 369.081x_2^2\tilde{x}_1^2 + 18.622x_2^2\tilde{x}_1\tilde{x}_2 + 91.859x_2^2\tilde{x}_2^2 + 505.407x_2\tilde{x}_1^3 + 101.948x_2\tilde{x}_1^2\tilde{x}_2 + 114.095x_2\tilde{x}_1\tilde{x}_2^2 - 8.855x_2\tilde{x}_2^3 + 116.481\tilde{x}_1^4 - 64.411\tilde{x}_1^3\tilde{x}_2 + 111.531\tilde{x}_1^2\tilde{x}_2^2 - 52.16\tilde{x}_1\tilde{x}_2^3 + 34.443\tilde{x}_2^4$$

$$\Pi^{12}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 111.023x_2^4 + 131.388x_2^3\tilde{x}_1 + 75.984x_2^3\tilde{x}_2 + 167.522x_2^2\tilde{x}_1^2 + 203.331x_2^2\tilde{x}_1\tilde{x}_2 - 16.139x_2^2\tilde{x}_2^2 + 87.44x_2\tilde{x}_1^3 + 84.38x_2\tilde{x}_1^2\tilde{x}_2 + 27.385x_2\tilde{x}_1\tilde{x}_2^2 + 1.376x_2\tilde{x}_2^3 + 26.335\tilde{x}_1^4 + 146.321\tilde{x}_1^3\tilde{x}_2 - 73.056\tilde{x}_1^2\tilde{x}_2^2 + 65.77\tilde{x}_1\tilde{x}_2^3 - 32.859\tilde{x}_2^4$$

$$\Pi^{22}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2343.806x_2^4 + 56.451x_2^3\tilde{x}_1 + 52.824x_2^3\tilde{x}_2 + 2009.828x_2^2\tilde{x}_1^2 + 30.769x_2^2\tilde{x}_1\tilde{x}_2 + 1999.81x_2^2\tilde{x}_2^2 + 15.987x_2\tilde{x}_1^3 + 46.392x_2\tilde{x}_1^2\tilde{x}_2 + 11.34x_2\tilde{x}_1\tilde{x}_2^2 + 0.178x_2\tilde{x}_2^3 + 2267.573\tilde{x}_1^4 - 27.705\tilde{x}_1^3\tilde{x}_2 + 2007.028\tilde{x}_1^2\tilde{x}_2^2 - 70.923\tilde{x}_1\tilde{x}_2^3 + 2273.6\tilde{x}_2^4$$

$$\Pi^{32}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -349.429x_2^4 - 383.851x_2^3\tilde{x}_1 - 210.362x_2^3\tilde{x}_2 - 443.819x_2^2\tilde{x}_1^2 - 415.122x_2^2\tilde{x}_1\tilde{x}_2 - 4.16x_2^2\tilde{x}_2^2 - 232.987x_2\tilde{x}_1^3 - 233.811x_2\tilde{x}_1^2\tilde{x}_2 - 61.613x_2\tilde{x}_1\tilde{x}_2^2 - 8.8379x_2\tilde{x}_2^3 - 78.0070\tilde{x}_1^4 - 250.77\tilde{x}_1^3\tilde{x}_2 + 81.345\tilde{x}_1^2\tilde{x}_2^2 - 80.681\tilde{x}_1\tilde{x}_2^3 + 56.813\tilde{x}_2^4$$

$$\Pi^{42}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 24.812x_2^4 + 133.53x_2^3\tilde{x}_1 + 192.965x_2^3\tilde{x}_2 + 59.979x_2^2\tilde{x}_1^2 + 123.797x_2^2\tilde{x}_1\tilde{x}_2 - 54.108x_2^2\tilde{x}_2^2 + 55.302x_2\tilde{x}_1^3 + 150.989x_2\tilde{x}_1^2\tilde{x}_2 - 29.491x_2\tilde{x}_1\tilde{x}_2^2 + 60.937x_2\tilde{x}_2^3 - 17.698\tilde{x}_1^4 + 63.776\tilde{x}_1^3\tilde{x}_2 - 78.977\tilde{x}_1^2\tilde{x}_2^2 + 72.345\tilde{x}_1\tilde{x}_2^3 - 43.445\tilde{x}_2^4$$

$$\Pi^{13}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -970.118x_2^4 - 945.408x_2^3\tilde{x}_1 - 358.299x_2^3\tilde{x}_2 - 2148.59x_2^2\tilde{x}_1^2 - 396.363x_2^2\tilde{x}_1\tilde{x}_2 - 226.157x_2^2\tilde{x}_2^2 - 854.627x_2\tilde{x}_1^3 - 420.744x_2\tilde{x}_1^2\tilde{x}_2 - 118.192x_2\tilde{x}_1\tilde{x}_2^2 - 40.068x_2\tilde{x}_2^3 - 1122.044\tilde{x}_1^4 - 164.877\tilde{x}_1^3\tilde{x}_2 - 229.793\tilde{x}_1^2\tilde{x}_2^2 + 30.902\tilde{x}_1\tilde{x}_2^3 - 41.35\tilde{x}_2^4$$

$$\Pi^{23}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -349.429x_2^4 - 383.851x_2^3\tilde{x}_1 - 210.362x_2^3\tilde{x}_2 - 443.819x_2^2\tilde{x}_1^2 - 415.122x_2^2\tilde{x}_1\tilde{x}_2 - 4.16x_2^2\tilde{x}_2^2 - 232.987x_2\tilde{x}_1^3 - 233.811x_2\tilde{x}_1^2\tilde{x}_2 - 61.613x_2\tilde{x}_1\tilde{x}_2^2 - 8.8379x_2\tilde{x}_2^3 - 78.0070\tilde{x}_1^4 - 250.77\tilde{x}_1^3\tilde{x}_2 + 81.345\tilde{x}_1^2\tilde{x}_2^2 - 80.681\tilde{x}_1\tilde{x}_2^3 + 56.813\tilde{x}_2^4$$

$$\Pi^{33}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 5606.859x_2^4 + 3241.144x_2^3\tilde{x}_1 + 945.667x_2^3\tilde{x}_2 + 8088.731x_2^2\tilde{x}_1^2 + 1111.219x_2^2\tilde{x}_1\tilde{x}_2 + 2674.514x_2^2\tilde{x}_2^2 + 2641.082x_2\tilde{x}_1^3 + 1124.635x_2\tilde{x}_1^2\tilde{x}_2 + 301.661x_2\tilde{x}_1\tilde{x}_2^2 + 70.14x_2\tilde{x}_2^3 + 5270.637\tilde{x}_1^4 + 476.658\tilde{x}_1^3\tilde{x}_2 + 2530.093\tilde{x}_1^2\tilde{x}_2^2 + 31.56\tilde{x}_1\tilde{x}_2^3 + 2529.069\tilde{x}_2^4$$

$$\Pi^{43}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -551.123x_2^4 - 1226.256x_2^3\tilde{x}_1 - 308.317x_2^3\tilde{x}_2 - 1085.896x_2^2\tilde{x}_1^2 - 183.63x_2^2\tilde{x}_1\tilde{x}_2 - 194.978x_2^2\tilde{x}_2^2 - 1042.718x_2\tilde{x}_1^3 - 252.033x_2\tilde{x}_1^2\tilde{x}_2 - 192.454x_2\tilde{x}_1\tilde{x}_2^2 - 14.823x_2\tilde{x}_2^3 - 289.252\tilde{x}_1^4 + 33.983\tilde{x}_1^3\tilde{x}_2 - 183.296\tilde{x}_1^2\tilde{x}_2^2 + 46.814\tilde{x}_1\tilde{x}_2^3 - 63.54\tilde{x}_2^4$$

$$\Pi^{14}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 183.689x_2^4 + 521.672x_2^3\tilde{x}_1 + 102.904x_2^3\tilde{x}_2 + 369.081x_2^2\tilde{x}_1^2 + 18.622x_2^2\tilde{x}_1\tilde{x}_2 + 91.859x_2^2\tilde{x}_2^2 + 505.407x_2\tilde{x}_1^3 + 101.948x_2\tilde{x}_1^2\tilde{x}_2 + 114.095x_2\tilde{x}_1\tilde{x}_2^2 - 8.855x_2\tilde{x}_2^3 + 116.481\tilde{x}_1^4 - 64.411\tilde{x}_1^3\tilde{x}_2 + 111.531\tilde{x}_1^2\tilde{x}_2^2 - 52.16\tilde{x}_1\tilde{x}_2^3 + 34.443\tilde{x}_2^4$$

$$\Pi^{24}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 24.812x_2^4 + 133.53x_2^3\tilde{x}_1 + 192.965x_2^3\tilde{x}_2 + 59.979x_2^2\tilde{x}_1^2 + 123.797x_2^2\tilde{x}_1\tilde{x}_2 - 54.108x_2^2\tilde{x}_2^2 + 55.302x_2\tilde{x}_1^3 + 150.989x_2\tilde{x}_1^2\tilde{x}_2 - 29.491x_2\tilde{x}_1\tilde{x}_2^2 + 60.937x_2\tilde{x}_2^3 - 17.698\tilde{x}_1^4 + 63.776\tilde{x}_1^3\tilde{x}_2 - 78.977\tilde{x}_1^2\tilde{x}_2^2 + 72.345\tilde{x}_1\tilde{x}_2^3 - 43.445\tilde{x}_2^4$$

$$\Pi^{34}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -551.123x_2^4 - 1226.256x_2^3\tilde{x}_1 - 308.317x_2^3\tilde{x}_2 - 1085.896x_2^2\tilde{x}_1^2 - 183.63x_2^2\tilde{x}_1\tilde{x}_2 - 194.978x_2^2\tilde{x}_2^2 - 1042.718x_2\tilde{x}_1^3 - 252.033x_2\tilde{x}_1^2\tilde{x}_2 - 192.454x_2\tilde{x}_1\tilde{x}_2^2 - 14.823x_2\tilde{x}_2^3 - 289.252\tilde{x}_1^4 + 33.983\tilde{x}_1^3\tilde{x}_2 - 183.296\tilde{x}_1^2\tilde{x}_2^2 + 46.814\tilde{x}_1\tilde{x}_2^3 - 63.54\tilde{x}_2^4$$

$$\Pi^{44}_{2212}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2737.043x_2^4 + 263.045x_2^3\tilde{x}_1 - 180.684x_2^3\tilde{x}_2 + 2535.316x_2^2\tilde{x}_1^2 + 56.388x_2^2\tilde{x}_1\tilde{x}_2 + 2190.578x_2^2\tilde{x}_2^2 + 157.21x_2\tilde{x}_1^3 - 126.45x_2\tilde{x}_1^2\tilde{x}_2 + 139.029x_2\tilde{x}_1\tilde{x}_2^2 - 92.767x_2\tilde{x}_2^3 + 2404.876\tilde{x}_1^4 - 31.07\tilde{x}_1^3\tilde{x}_2 + 2057.51\tilde{x}_1^2\tilde{x}_2^2 - 81.7720\tilde{x}_1\tilde{x}_2^3 + 2299.597\tilde{x}_2^4$$

$$\Pi^{11}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 35633.27x_2^4 + 1331.475x_2^3\tilde{x}_1 + 2939.487x_2^3\tilde{x}_2 + 48363.139x_2^2\tilde{x}_1^2 - 1560.988x_2^2\tilde{x}_1\tilde{x}_2 + 23187.646x_2^2\tilde{x}_2^2 + 1914.335x_2\tilde{x}_1^3 + 2326.594x_2\tilde{x}_1^2\tilde{x}_2 - 713.701x_2\tilde{x}_1\tilde{x}_2^2 + 1036.577x_2\tilde{x}_2^3 + 80325.42999999999\tilde{x}_1^4 - 5157.781\tilde{x}_1^3\tilde{x}_2 + 26238.64\tilde{x}_1^2\tilde{x}_2^2 - 4419.183\tilde{x}_1\tilde{x}_2^3 + 25209.773\tilde{x}_2^4$$

$$\Pi^{21}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -570.291x_2^4 + 2439.438x_2^3\tilde{x}_1 - 1036.957x_2^3\tilde{x}_2 - 1107.739x_2^2\tilde{x}_1^2 + 6995.958x_2^2\tilde{x}_1\tilde{x}_2 - 3821.972x_2^2\tilde{x}_2^2 + 1748.515x_2\tilde{x}_1^3 - 850.277x_2\tilde{x}_1^2\tilde{x}_2 + 1027.058x_2\tilde{x}_1\tilde{x}_2^2 - 944.52x_2\tilde{x}_2^3 - 3436.72\tilde{x}_1^4 + 8969.415\tilde{x}_1^3\tilde{x}_2 - 7827.317\tilde{x}_1^2\tilde{x}_2^2 + 5540.234\tilde{x}_1\tilde{x}_2^3 - 3615.242\tilde{x}_2^4$$

$$\Pi^{31}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -23252.97x_2^4 - 4797.138x_2^3\tilde{x}_1 - 4108.097x_2^3\tilde{x}_2 - 39697.915x_2^2\tilde{x}_1^2 + 45.324x_2^2\tilde{x}_1\tilde{x}_2 - 4612.717x_2^2\tilde{x}_2^2 - 3822.407x_2\tilde{x}_1^3 - 3460.386x_2\tilde{x}_1^2\tilde{x}_2 + 548.968x_2\tilde{x}_1\tilde{x}_2^2 - 1286.437x_2\tilde{x}_2^3 - 34541.658\tilde{x}_1^4 + 1844.462\tilde{x}_1^3\tilde{x}_2 - 7522.163\tilde{x}_1^2\tilde{x}_2^2 + 3064.372\tilde{x}_1\tilde{x}_2^3 - 2831.628\tilde{x}_2^4$$

$$\Pi^{41}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2178.773x_2^4 + 22113.184x_2^3\tilde{x}_1 - 154.604x_2^3\tilde{x}_2 + 3723.647x_2^2\tilde{x}_1^2 - 4056.24x_2^2\tilde{x}_1\tilde{x}_2 + 2997.065x_2^2\tilde{x}_2^2 + 23471.808x_2\tilde{x}_1^3 - 892.149x_2\tilde{x}_1^2\tilde{x}_2 + 4533.081x_2\tilde{x}_1\tilde{x}_2^2 - 1708.885x_2\tilde{x}_2^3 + 4989.998\tilde{x}_1^4 - 7258.235\tilde{x}_1^3\tilde{x}_2 + 6908.477\tilde{x}_1^2\tilde{x}_2^2 - 4668.478\tilde{x}_1\tilde{x}_2^3 + 3112.61\tilde{x}_2^4$$

$$\Pi^{12}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -570.291x_2^4 + 2439.438x_2^3\tilde{x}_1 - 1036.957x_2^3\tilde{x}_2 - 1107.739x_2^2\tilde{x}_1^2 + 6995.958x_2^2\tilde{x}_1\tilde{x}_2 - 3821.972x_2^2\tilde{x}_2^2 + 1748.515x_2\tilde{x}_1^3 - 850.277x_2\tilde{x}_1^2\tilde{x}_2 + 1027.058x_2\tilde{x}_1\tilde{x}_2^2 - 944.52x_2\tilde{x}_2^3 - 3436.72\tilde{x}_1^4 + 8969.415\tilde{x}_1^3\tilde{x}_2 - 7827.317\tilde{x}_1^2\tilde{x}_2^2 + 5540.234\tilde{x}_1\tilde{x}_2^3 - 3615.242\tilde{x}_2^4$$

$$\Pi^{22}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 31095.314x_2^4 - 1069.804x_2^3\tilde{x}_1 + 3956.937x_2^3\tilde{x}_2 + 22809.936x_2^2\tilde{x}_1^2 - 3508.77x_2^2\tilde{x}_1\tilde{x}_2 + 26829.197x_2^2\tilde{x}_2^2 - 829.846x_2\tilde{x}_1^3 + 1318.388x_2\tilde{x}_1^2\tilde{x}_2 - 1373.62x_2\tilde{x}_1\tilde{x}_2^2 + 1081.748x_2\tilde{x}_2^3 + 25105.836\tilde{x}_1^4 - 4321.263\tilde{x}_1^3\tilde{x}_2 + 26277.9\tilde{x}_1^2\tilde{x}_2^2 - 6533.962\tilde{x}_1\tilde{x}_2^3 + 27244.577\tilde{x}_2^4$$

$$\Pi^{32}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -4426.597x_2^4 - 3700.633x_2^3\tilde{x}_1 + 160.899x_2^3\tilde{x}_2 - 153.701x_2^2\tilde{x}_1^2 - 5897.107x_2^2\tilde{x}_1\tilde{x}_2 + 2627.995x_2^2\tilde{x}_2^2 - 2276.409x_2\tilde{x}_1^3 + 543.231x_2\tilde{x}_1^2\tilde{x}_2 - 1450.453x_2\tilde{x}_1\tilde{x}_2^2 + 712.386x_2\tilde{x}_2^3 + 1547.202\tilde{x}_1^4 - 6477.996\tilde{x}_1^3\tilde{x}_2 + 5969.423\tilde{x}_1^2\tilde{x}_2^2 - 4080.91\tilde{x}_1\tilde{x}_2^3 + 6051.827\tilde{x}_2^4$$

$$\Pi^{42}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 1101.158x_2^4 - 36.715x_2^3\tilde{x}_1 + 13769.672x_2^3\tilde{x}_2 - 2161.441x_2^2\tilde{x}_1^2 + 2858.362x_2^2\tilde{x}_1\tilde{x}_2 - 4524.189x_2^2\tilde{x}_2^2 - 501.974x_2\tilde{x}_1^3 + 5592.168x_2\tilde{x}_1^2\tilde{x}_2 - 3505.308x_2\tilde{x}_1\tilde{x}_2^2 + 4902.966x_2\tilde{x}_2^3 - 2891.562\tilde{x}_1^4 + 4072.666\tilde{x}_1^3\tilde{x}_2 - 6501.32\tilde{x}_1^2\tilde{x}_2^2 + 5527.458\tilde{x}_1\tilde{x}_2^3 - 4307.453\tilde{x}_2^4$$

$$\Pi^{13}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -23252.97x_2^4 - 4797.138x_2^3\tilde{x}_1 - 4108.097x_2^3\tilde{x}_2 - 39697.915x_2^2\tilde{x}_1^2 + 45.324x_2^2\tilde{x}_1\tilde{x}_2 - 4612.717x_2^2\tilde{x}_2^2 - 3822.407x_2\tilde{x}_1^3 - 3460.386x_2\tilde{x}_1^2\tilde{x}_2 + 548.968x_2\tilde{x}_1\tilde{x}_2^2 - 1286.437x_2\tilde{x}_2^3 - 34541.658\tilde{x}_1^4 + 1844.462\tilde{x}_1^3\tilde{x}_2 - 7522.163\tilde{x}_1^2\tilde{x}_2^2 + 3064.372\tilde{x}_1\tilde{x}_2^3 - 2831.628\tilde{x}_2^4$$

$$\Pi^{23}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -4426.597x_2^4 - 3700.633x_2^3\tilde{x}_1 + 160.899x_2^3\tilde{x}_2 - 153.701x_2^2\tilde{x}_1^2 - 5897.107x_2^2\tilde{x}_1\tilde{x}_2 + 2627.995x_2^2\tilde{x}_2^2 - 2276.409x_2\tilde{x}_1^3 + 543.231x_2\tilde{x}_1^2\tilde{x}_2 - 1450.453x_2\tilde{x}_1\tilde{x}_2^2 + 712.386x_2\tilde{x}_2^3 + 1547.202\tilde{x}_1^4 - 6477.996\tilde{x}_1^3\tilde{x}_2 + 5969.423\tilde{x}_1^2\tilde{x}_2^2 - 4080.91\tilde{x}_1\tilde{x}_2^3 + 6051.827\tilde{x}_2^4$$

$$\Pi^{33}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 147838.012x_2^4 + 26231.26x_2^3\tilde{x}_1 + 8214.768x_2^3\tilde{x}_2 + 129836.83x_2^2\tilde{x}_1^2 + 628.414x_2^2\tilde{x}_1\tilde{x}_2 + 59264.923x_2^2\tilde{x}_2^2 + 9248.106x_2\tilde{x}_1^3 + 8604.431x_2\tilde{x}_1^2\tilde{x}_2 - 1324.173x_2\tilde{x}_1\tilde{x}_2^2 - 882.019x_2\tilde{x}_2^3 + 126450.841\tilde{x}_1^4 - 2526.682\tilde{x}_1^3\tilde{x}_2 + 53064\tilde{x}_1^2\tilde{x}_2^2 + 243.457\tilde{x}_1\tilde{x}_2^3 + 70349.888\tilde{x}_2^4$$

$$\Pi^{43}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -4686.747x_2^4 - 26107.946x_2^3\tilde{x}_1 - 5463.576x_2^3\tilde{x}_2 - 6427.802x_2^2\tilde{x}_1^2 + 1149.92x_2^2\tilde{x}_1\tilde{x}_2 - 2307.233x_2^2\tilde{x}_2^2 - 24847.23x_2\tilde{x}_1^3 - 955.371x_2\tilde{x}_1^2\tilde{x}_2 - 3425.48x_2\tilde{x}_1\tilde{x}_2^2 + 359.134x_2\tilde{x}_2^3 - 3676.09\tilde{x}_1^4 + 4089.616\tilde{x}_1^3\tilde{x}_2 - 5109.025\tilde{x}_1^2\tilde{x}_2^2 + 2933.132\tilde{x}_1\tilde{x}_2^3 - 5680.085\tilde{x}_2^4$$

$$\Pi^{14}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 2178.773x_2^4 + 22113.184x_2^3\tilde{x}_1 - 154.604x_2^3\tilde{x}_2 + 3723.647x_2^2\tilde{x}_1^2 - 4056.24x_2^2\tilde{x}_1\tilde{x}_2 + 2997.065x_2^2\tilde{x}_2^2 + 23471.808x_2\tilde{x}_1^3 - 892.149x_2\tilde{x}_1^2\tilde{x}_2 + 4533.081x_2\tilde{x}_1\tilde{x}_2^2 - 1708.885x_2\tilde{x}_2^3 + 4989.998\tilde{x}_1^4 - 7258.235\tilde{x}_1^3\tilde{x}_2 + 6908.477\tilde{x}_1^2\tilde{x}_2^2 - 4668.478\tilde{x}_1\tilde{x}_2^3 + 3112.61\tilde{x}_2^4$$

$$\Pi^{24}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 1101.158x_2^4 - 36.715x_2^3\tilde{x}_1 + 13769.672x_2^3\tilde{x}_2 - 2161.441x_2^2\tilde{x}_1^2 + 2858.362x_2^2\tilde{x}_1\tilde{x}_2 - 4524.189x_2^2\tilde{x}_2^2 - 501.974x_2\tilde{x}_1^3 + 5592.168x_2\tilde{x}_1^2\tilde{x}_2 - 3505.308x_2\tilde{x}_1\tilde{x}_2^2 + 4902.966x_2\tilde{x}_2^3 - 2891.562\tilde{x}_1^4 + 4072.666\tilde{x}_1^3\tilde{x}_2 - 6501.32\tilde{x}_1^2\tilde{x}_2^2 + 5527.458\tilde{x}_1\tilde{x}_2^3 - 4307.453\tilde{x}_2^4$$

$$\Pi^{34}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = -4686.747x_2^4 - 26107.946x_2^3\tilde{x}_1 - 5463.576x_2^3\tilde{x}_2 - 6427.802x_2^2\tilde{x}_1^2 + 1149.92x_2^2\tilde{x}_1\tilde{x}_2 - 2307.233x_2^2\tilde{x}_2^2 - 24847.23x_2\tilde{x}_1^3 - 955.371x_2\tilde{x}_1^2\tilde{x}_2 - 3425.48x_2\tilde{x}_1\tilde{x}_2^2 + 359.134x_2\tilde{x}_2^3 - 3676.09\tilde{x}_1^4 + 4089.616\tilde{x}_1^3\tilde{x}_2 - 5109.025\tilde{x}_1^2\tilde{x}_2^2 + 2933.132\tilde{x}_1\tilde{x}_2^3 - 5680.085\tilde{x}_2^4$$

$$\Pi^{44}_{2221}(\boldsymbol{x},\tilde{\boldsymbol{x}}) = 76909.779x_2^4 + 3750.547x_2^3\tilde{x}_1 - 16616.395x_2^3\tilde{x}_2 + 47530.638x_2^2\tilde{x}_1^2 - 3763.642x_2^2\tilde{x}_1\tilde{x}_2 + 38231.588x_2^2\tilde{x}_2^2 + 3270.553x_2\tilde{x}_1^3 - 9300.713x_2\tilde{x}_1^2\tilde{x}_2 + 6592.611x_2\tilde{x}_1\tilde{x}_2^2 - 7398.407x_2\tilde{x}_2^3 + 34027.825\tilde{x}_1^4 - 4257.673\tilde{x}_1^3\tilde{x}_2 + 28279.225\tilde{x}_1^2\tilde{x}_2^2 - 6616.586\tilde{x}_1\tilde{x}_2^3 + 28745.858\tilde{x}_2^4$$

$$\begin{aligned}\Pi^{11}_{2222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 3157.094x_2^4 + 394.056x_2^3\tilde{x}_1 + 250.704x_2^3\tilde{x}_2 + 4406.231x_2^2\tilde{x}_1^2 + 131.132x_2^2\tilde{x}_1\tilde{x}_2 + \\ & 2273.069x_2^2\tilde{x}_2^2 + 337.162x_2\tilde{x}_1^3 + 268.491x_2\tilde{x}_1^2\tilde{x}_2 + 79.949x_2\tilde{x}_1\tilde{x}_2^2 + 43.1x_2\tilde{x}_2^3 + \\ & 5004.154\tilde{x}_1^4 - 54.412\tilde{x}_1^3\tilde{x}_2 + 2475.893\tilde{x}_1^2\tilde{x}_2^2 - 142.081\tilde{x}_1\tilde{x}_2^3 + 2340.909\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{21}_{2222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 221.097x_2^4 + 187.816x_2^3\tilde{x}_1 + 133.256x_2^3\tilde{x}_2 + 291.628x_2^2\tilde{x}_1^2 + 531.415x_2^2\tilde{x}_1\tilde{x}_2 - \\ & 130.931x_2^2\tilde{x}_2^2 + 174.397x_2\tilde{x}_1^3 + 133.124x_2\tilde{x}_1^2\tilde{x}_2 + 26.425x_2\tilde{x}_1\tilde{x}_2^2 + 6.64x_2\tilde{x}_2^3 - \\ & 16.077\tilde{x}_1^4 + 590.263\tilde{x}_1^3\tilde{x}_2 - 383.079\tilde{x}_1^2\tilde{x}_2^2 + 232.979\tilde{x}_1\tilde{x}_2^3 - 124.737\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{31}_{2222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -2410.873x_2^4 - 1270.676x_2^3\tilde{x}_1 - 645.200x_2^3\tilde{x}_2 - 5402.281x_2^2\tilde{x}_1^2 - 427.078x_2^2\tilde{x}_1\tilde{x}_2 - \\ & 846.585x_2^2\tilde{x}_2^2 - 1155.685x_2\tilde{x}_1^3 - 675.857x_2\tilde{x}_1^2\tilde{x}_2 - 215.075x_2\tilde{x}_1\tilde{x}_2^2 - 116.719x_2\tilde{x}_2^3 - \\ & 4242.393\tilde{x}_1^4 - 92.0580\tilde{x}_1^3\tilde{x}_2 - 1142.57\tilde{x}_1^2\tilde{x}_2^2 + 142.021\tilde{x}_1\tilde{x}_2^3 - 189.326\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{41}_{2222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 342.053x_2^4 + 1367.641x_2^3\tilde{x}_1 + 165.585x_2^3\tilde{x}_2 + 552.346x_2^2\tilde{x}_1^2 - 247.241x_2^2\tilde{x}_1\tilde{x}_2 + \\ & 320.412x_2^2\tilde{x}_2^2 + 1611.172x_2\tilde{x}_1^3 + 131.276x_2\tilde{x}_1^2\tilde{x}_2 + 376.21x_2\tilde{x}_1\tilde{x}_2^2 - 44.698x_2\tilde{x}_2^3 + \\ & 225.237\tilde{x}_1^4 - 433.15\tilde{x}_1^3\tilde{x}_2 + 469.751\tilde{x}_1^2\tilde{x}_2^2 - 212.029\tilde{x}_1\tilde{x}_2^3 + 137.811\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{12}_{2222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 221.097x_2^4 + 187.816x_2^3\tilde{x}_1 + 133.256x_2^3\tilde{x}_2 + 291.628x_2^2\tilde{x}_1^2 + 531.415x_2^2\tilde{x}_1\tilde{x}_2 - \\ & 130.931x_2^2\tilde{x}_2^2 + 174.397x_2\tilde{x}_1^3 + 133.124x_2\tilde{x}_1^2\tilde{x}_2 + 26.425x_2\tilde{x}_1\tilde{x}_2^2 + 6.64x_2\tilde{x}_2^3 - \\ & 16.077\tilde{x}_1^4 + 590.263\tilde{x}_1^3\tilde{x}_2 - 383.079\tilde{x}_1^2\tilde{x}_2^2 + 232.979\tilde{x}_1\tilde{x}_2^3 - 124.737\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{22}_{2222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 2486.788x_2^4 + 90.852x_2^3\tilde{x}_1 + 51.144x_2^3\tilde{x}_2 + 2112.998x_2^2\tilde{x}_1^2 + 29.048x_2^2\tilde{x}_1\tilde{x}_2 + \\ & 2150.095x_2^2\tilde{x}_2^2 + 8.262x_2\tilde{x}_1^3 + 79.217x_2\tilde{x}_1^2\tilde{x}_2 + 20.742x_2\tilde{x}_1\tilde{x}_2^2 - 40.097x_2\tilde{x}_2^3 + \\ & 2327.68\tilde{x}_1^4 - 122.411\tilde{x}_1^3\tilde{x}_2 + 2250.353\tilde{x}_1^2\tilde{x}_2^2 - 286.284\tilde{x}_1\tilde{x}_2^3 + 2394.877\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{32}_{2222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -713.873x_2^4 - 612.281x_2^3\tilde{x}_1 - 349.951x_2^3\tilde{x}_2 - 669.331x_2^2\tilde{x}_1^2 - 1028.815x_2^2\tilde{x}_1\tilde{x}_2 + \\ & 189.147x_2^2\tilde{x}_2^2 - 391.158x_2\tilde{x}_1^3 - 357.081x_2\tilde{x}_1^2\tilde{x}_2 - 98.537x_2\tilde{x}_1\tilde{x}_2^2 - 14.542x_2\tilde{x}_2^3 - \\ & 10.549\tilde{x}_1^4 - 931.311\tilde{x}_1^3\tilde{x}_2 + 526.75\tilde{x}_1^2\tilde{x}_2^2 - 334.197\tilde{x}_1\tilde{x}_2^3 + 241.186\tilde{x}_2^4\end{aligned}$$

$$\begin{aligned}\Pi^{42}_{2222}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -96.312x_2^4 + 252.665x_2^3\tilde{x}_1 + 548.409x_2^3\tilde{x}_2 - 18.141x_2^2\tilde{x}_1^2 + 287.325x_2^2\tilde{x}_1\tilde{x}_2 - \\ & 293.633x_2^2\tilde{x}_2^2 + 46.48x_2\tilde{x}_1^3 + 481.001x_2\tilde{x}_1^2\tilde{x}_2 - 196.995x_2\tilde{x}_1\tilde{x}_2^2 + 214.378x_2\tilde{x}_2^3 - \\ & 95.565\tilde{x}_1^4 + 184.243\tilde{x}_1^3\tilde{x}_2 - 333.79\tilde{x}_1^2\tilde{x}_2^2 + 289.706\tilde{x}_1\tilde{x}_2^3 - 179.954\tilde{x}_2^4\end{aligned}$$

$$
\begin{aligned}
\Pi_{2222}^{13}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -2410.873x_2^4 - 1270.676x_2^3\tilde{x}_1 - 645.200x_2^3\tilde{x}_2 - 5402.281x_2^2\tilde{x}_1^2 - 427.078x_2^2\tilde{x}_1\tilde{x}_2 - \\
& 846.585x_2^2\tilde{x}_2^2 - 1155.685x_2\tilde{x}_1^3 - 675.857x_2\tilde{x}_1^2\tilde{x}_2 - 215.075x_2\tilde{x}_1\tilde{x}_2^2 - 116.719x_2\tilde{x}_2^3 - \\
& 4242.393\tilde{x}_1^4 - 92.0580\tilde{x}_1^3\tilde{x}_2 - 1142.57\tilde{x}_1^2\tilde{x}_2^2 + 142.021\tilde{x}_1\tilde{x}_2^3 - 189.326\tilde{x}_2^4 \\
\Pi_{2222}^{23}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -713.873x_2^4 - 612.281x_2^3\tilde{x}_1 - 349.951x_2^3\tilde{x}_2 - 669.331x_2^2\tilde{x}_1^2 - 1028.815x_2^2\tilde{x}_1\tilde{x}_2 + \\
& 189.147x_2^2\tilde{x}_2^2 - 391.158x_2\tilde{x}_1^3 - 357.081x_2\tilde{x}_1^2\tilde{x}_2 - 98.537x_2\tilde{x}_1\tilde{x}_2^2 - 14.542x_2\tilde{x}_2^3 - \\
& 10.549\tilde{x}_1^4 - 931.311\tilde{x}_1^3\tilde{x}_2 + 526.75\tilde{x}_1^2\tilde{x}_2^2 - 334.197\tilde{x}_1\tilde{x}_2^3 + 241.186\tilde{x}_2^4 \\
\Pi_{2222}^{33}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 10153.682x_2^4 + 4541.29x_2^3\tilde{x}_1 + 1601.163x_2^3\tilde{x}_2 + 15235.987x_2^2\tilde{x}_1^2 + 1311.776x_2^2\tilde{x}_1\tilde{x}_2 + \\
& 4750.523x_2^2\tilde{x}_2^2 + 3566.89x_2\tilde{x}_1^3 + 1668.165x_2\tilde{x}_1^2\tilde{x}_2 + 649.597x_2\tilde{x}_1\tilde{x}_2^2 + 236.72x_2\tilde{x}_2^3 + \\
& 11071.854\tilde{x}_1^4 + 491.941\tilde{x}_1^3\tilde{x}_2 + 4704.336\tilde{x}_1^2\tilde{x}_2^2 - 22.41\tilde{x}_1\tilde{x}_2^3 + 3179.1\tilde{x}_2^4 \\
\Pi_{2222}^{43}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -956.419x_2^4 - 2924.655x_2^3\tilde{x}_1 - 574.111x_2^3\tilde{x}_2 - 1552.656x_2^2\tilde{x}_1^2 + 133.376x_2^2\tilde{x}_1\tilde{x}_2 - \\
& 659.792x_2^2\tilde{x}_2^2 - 2798.712x_2\tilde{x}_1^3 - 342.641x_2\tilde{x}_1^2\tilde{x}_2 - 664.824x_2\tilde{x}_1\tilde{x}_2^2 - 26.426x_2\tilde{x}_2^3 - \\
& 443.044\tilde{x}_1^4 + 558.185\tilde{x}_1^3\tilde{x}_2 - 722.581\tilde{x}_1^2\tilde{x}_2^2 + 253.173\tilde{x}_1\tilde{x}_2^3 - 272.816\tilde{x}_2^4 \\
\Pi_{2222}^{14}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 342.053x_2^4 + 1367.641x_2^3\tilde{x}_1 + 165.585x_2^3\tilde{x}_2 + 552.346x_2^2\tilde{x}_1^2 - 247.241x_2^2\tilde{x}_1\tilde{x}_2 + \\
& 320.412x_2^2\tilde{x}_2^2 + 1611.172x_2\tilde{x}_1^3 + 131.276x_2\tilde{x}_1^2\tilde{x}_2 + 376.21x_2\tilde{x}_1\tilde{x}_2^2 - 44.698x_2\tilde{x}_2^3 + \\
& 225.237\tilde{x}_1^4 - 433.15\tilde{x}_1^3\tilde{x}_2 + 469.751\tilde{x}_1^2\tilde{x}_2^2 - 212.029\tilde{x}_1\tilde{x}_2^3 + 137.811\tilde{x}_2^4 \\
\Pi_{2222}^{24}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -96.312x_2^4 + 252.665x_2^3\tilde{x}_1 + 548.409x_2^3\tilde{x}_2 - 18.141x_2^2\tilde{x}_1^2 + 287.325x_2^2\tilde{x}_1\tilde{x}_2 - \\
& 293.633x_2^2\tilde{x}_2^2 + 46.48x_2\tilde{x}_1^3 + 481.001x_2\tilde{x}_1^2\tilde{x}_2 - 196.995x_2\tilde{x}_1\tilde{x}_2^2 + 214.378x_2\tilde{x}_2^3 - \\
& 95.565\tilde{x}_1^4 + 184.243\tilde{x}_1^3\tilde{x}_2 - 333.79\tilde{x}_1^2\tilde{x}_2^2 + 289.706\tilde{x}_1\tilde{x}_2^3 - 179.954\tilde{x}_2^4 \\
\Pi_{2222}^{34}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& -956.419x_2^4 - 2924.655x_2^3\tilde{x}_1 - 574.111x_2^3\tilde{x}_2 - 1552.656x_2^2\tilde{x}_1^2 + 133.376x_2^2\tilde{x}_1\tilde{x}_2 - \\
& 659.792x_2^2\tilde{x}_2^2 - 2798.712x_2\tilde{x}_1^3 - 342.641x_2\tilde{x}_1^2\tilde{x}_2 - 664.824x_2\tilde{x}_1\tilde{x}_2^2 - 26.426x_2\tilde{x}_2^3 - \\
& 443.044\tilde{x}_1^4 + 558.185\tilde{x}_1^3\tilde{x}_2 - 722.581\tilde{x}_1^2\tilde{x}_2^2 + 253.173\tilde{x}_1\tilde{x}_2^3 - 272.816\tilde{x}_2^4 \\
\Pi_{2222}^{44}(\boldsymbol{x},\tilde{\boldsymbol{x}}) =& 3779.996x_2^4 + 496.192x_2^3\tilde{x}_1 - 874.818x_2^3\tilde{x}_2 + 3749.128x_2^2\tilde{x}_1^2 - 41.377x_2^2\tilde{x}_1\tilde{x}_2 + \\
& 2919.519x_2^2\tilde{x}_2^2 + 263.769x_2\tilde{x}_1^3 - 716.509x_2\tilde{x}_1^2\tilde{x}_2 + 534.837x_2\tilde{x}_1\tilde{x}_2^2 - 362.968x_2\tilde{x}_2^3 + \\
& 2650.865\tilde{x}_1^4 - 168.942\tilde{x}_1^3\tilde{x}_2 + 2431.4\tilde{x}_1^2\tilde{x}_2^2 - 325.902\tilde{x}_1\tilde{x}_2^3 + 2498.844\tilde{x}_2^4
\end{aligned}
$$

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List of Publications

Journal Paper:

Alissa Ully Ashar, Motoyasu Tanaka, and Kazuo Tanaka, “Stabilization and Robust Stabilization of Polynomial Fuzzy Systems: A Piecewise Polynomial Lyapunov Function Approach,h *International Journal of Fuzzy Systems (Springer)*, **DOI**: https://doi.org/10.1007/s40815-017-0435-6, Accepted. (Impact Factor: 2.198)
(Related to the contents of Chapter 3-4)

International Conference Paper:

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Other Publication:

Alissa Ully Ashar, Muhammad Akbar Jamaluddin, M. Fadhli Zakiy, and Arief Syaichu-Rohman, "Design, Kinematic Modeling, and Implementation of Autonomous Fish Robot for Underwater Sensing," in *2013 Joint International Conference on Rural Information & Communication Technology and Electric-Vehicle Technology (rICT & ICeV-T)*, Bandung-Bali, Indonesia, 2013, Nov. 26-28, **DOI**: https://doi.org/10.1109/rICT-ICeVT.2013.6741557. Available at IEEE Xplore.