Robust Optimization Approach to Network Congestion and Power Efficient Network Problems

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Robust Optimization Approach to Network Congestion and Power Efficient Network Problems

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和文概要

本研究は、通信ネットワークの問題に対してロバスト最適化の手法を適用することをテーマとしている。

1 つ目の題材は、基幹ネットワークが混雑しないようにルーティングを定める問題である。基本的かつ重要な問題である。既存研究では、通信需要が正確にわかっているモデルや末端ノードにおける入出力量が制限されているモデルなど、さまざまなモデルが提案されている。この問題において通信需要が正確にはわかっていない場合を想定し、真の通信需要が楕円体と多面体の交わりに含まれているという仮定のもとでロバスト最適化の手法を適用し、その問題を 2 次錐計画問題として定式化した。いくつかの例に対して数値実験を行い、提案したモデルは汎用ソルバーを用いて合理的な時間内で求解可能であることを示し、また、従来のモデルとの比較検討を行なった。

2 つ目の題材は、エネルギーの節約のために不要なネットワークのリンクの電源を落とす問題である。このとき、真の通信需要が正確にはわからない状況は容易に起こりうるし、ネットワーク全体が通信需要のゆらぎに対して頑健であることが求められる。真の通信需要が楕円体と多面体の交わりに含まれているという仮定のもとでロバスト最適化の手法を適用し、その問題を整数 2 次錐計画問題として定式化し、実際に汎用最適化ソルバーで求解できることを示した。

Abstract

This thesis focuses on providing robust optimization models for minimization of the network congestion ratio and design of power efficient network that can handle fluctuation in traffic demands between source-destination pairs in the networks. It has become essential to design networks that are robust to different traffic conditions.

In the first part of the thesis, we propose robust optimization models to minimize congestion ratio for better performance of the network. The simplest and widely used model to minimize the congestion ratio, called the pipe model, is based on precisely specified traffic demands. However, in practice, network operators are often unable to estimate exact traffic demands as they can fluctuate due to unpredictable factors. To overcome this weakness, we apply robust optimization to the problem of minimizing the network congestion ratio. First, we review existing models as robust counterparts of certain uncertainty sets. Then we consider robust optimization assuming ellipsoidal uncertainty sets; the total amount of squared errors in traffic demands is bounded by a positive constant which represents the total admissible fluctuations over the network, and derive a tractable optimization problem in the form of second-order cone programming (SOCP). Furthermore, we take uncertainty sets to be the intersection of ellipsoid and polyhedral sets, and considering the mirror subproblems inherent in the models, obtain tractable optimization problems, again in SOCP form. Compared to the previous model that assumes an error interval on each coordinate, our models have the advantage of being able to cope with the total amount of errors by setting a parameter that determines the volume of the ellipsoid.

In the second part of the thesis, a green and robust optimization model is proposed to minimize the network power consumption. There are several researches that assume fluctuation in the traffic-demand matrix, our model is based on the idea of the green hose model where the knowledge of an exact traffic-demand matrix is not required. In the green hose model, the traffic is bounded by just total outgoing and incoming amount at each node. To allow fluctuations in traffic demands, here we also consider the same uncertainty set and subproblems as we did in the first part and formulate the green hose ellipsoid (green HE) model in the form of mixed-integer second-order cone programming (MISOCP) problem whose objective is to reduce the total energy by allowing some links to be put into the sleep mode. Furthermore, we establish a relationship between our model and the green HLT model, formulated from an extended version of the hose model called the *hose model with bound of link traffic* (HLT).

Numerical results demonstrate that our proposed robust optimization models for congestion ratio and power efficient network achieves the performances with traffic fluctuations comparable to the previous studies in terms of congestion ratio, computation time and power efficiency.

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Chapter 1

Introduction

Nowadays, it is important to ensure an appropriate routing scheme in a network so that an operator can perform better in the case of traffic fluctuations due to various reasons and users' need. In a backbone network, sending too much information on just a few of the links causes serious network congestion that results in greater end-to-end delay, packet loss, and decreasing the throughput. Network congestion can severely degrade the performance of the network [1]. Fortunately, setting an appropriate routing can enlarge the network resource utilization rate and the throughput [9]. Finding such a proper routing is a major concern for network operators whose goal is to provide better network performance.

The maximum link utilization rate among all links in a network is called the *network congestion ratio* [2]. Minimizing network congestion ratio is equivalent to maximizing additional admissible traffic [21].

Energy efficiency network is also a major concern in modern communication systems. Specially for economical and environmental reasons, energy-efficient networking has received a remarkable research interest over the last decade. The huge amount of power is consumed by the information and communication technology (ICT) sector and it can save the worldwide energy by 2% to 10% [3], [4].

There are several researches on minimizing network power consumption that have been presented in the history. Most of them are presented under the assumption that the traffic-demand matrix, the set of traffic demands, is known and there are some bounds on the total outgoing/incoming traffic from/to node.

Or, in addition to these bounds, traffic demands between each source and destination pair are bounded by upper and lower bounds [56]-[60]. There are also some studies on estimating the traffic-demand matrix, which makes it easy for network operators to avoid frequent dynamic route changes [61]-[63]. Most of the previous studies have a common objective to turn off some links for green computing.

1.1 Fluctuations of traffic demands

The outgoing and incoming amount at each node in the network is defined as traffic. The traffic demand is denoted as the traffic volume that a source node requests to send to a destination node and the link capacity is the maximal volume of traffic that a link can accommodate. The traffic demands and link capacities are expressed in bits per seconds (bps). The set of demands which describes all the traffic demands between each source-destination pair in the network is called the traffic-demand matrix. When the network size is large, it is not easy for network operators to know the actual traffic matrix. In other words, since traffic demand between source and destination nodes easily fluctuates depending on the users' needs, it is difficult task for network operators to compute the exact traffic-demand matrix. Therefore, it is challenging work for network operators to deal with the uncertain nature of traffic demands, if there are some errors in traffic demands. In the Figure 1.1, the matrix T which describes the traffic demands, d_{ij} for $i = 1, 2, 3$ and $j = 1, 2, 3$, is the traffic matrix. If there are some errors in d_{ij}, then it is not so easy for operators to deal with this matirx.

1.2 Related work

1.2.1 Problem of minimizing network congestion ratio

There are several studies on the problem of minimizing the network congestion ratio. Wang and Wang [2] state that the problem of finding flows that minimize the network congestion ratio can be cast into a linear programming (LP) problem with the assumption that the traffic demand d_{pq} for every pair of source node p

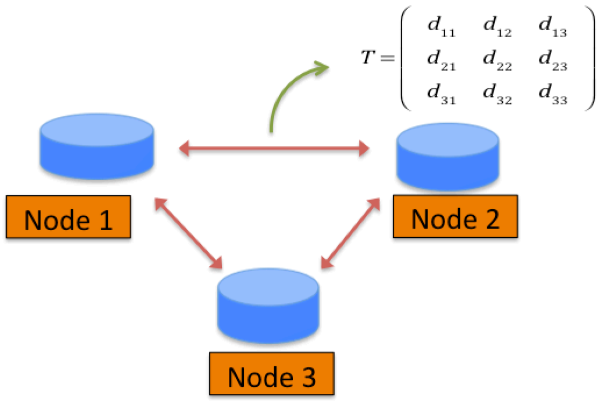

Figure 1.1: Measurement of traffic demands.

and destination node q, (p, q), is exactly known. This traffic-demand model is equivalent to the *pipe model* presented in [17], [18].

In practice, obtaining the exact traffic-demand matrix is difficult, if not impossible, for network operators. In a typical situation, the network operators can only estimate the total outgoing/incoming traffic from/to each node rather than the actual traffic-demand matrix [13], [15]. Chu et al. [13] formulated the problem of minimizing the network congestion ratio under this situation into an LP problem. Their traffic-demand model is called the *hose model* to contrast it with the pipe model [17], [18]. The hose model rids network operators of the heavy task of estimating the traffic-demand matrix exactly.

In general, the hose model has much lower routing performance than the pipe model. Based on the idea of additionally bounding the traffic demands in the hose model, Oki and Iwaki [20], [21] introduced another traffic-demand model that sets upper and lower bounds on the traffic demand between each pair of source and destination nodes in the network. They state that the problem of minimizing the network congestion ratio using their model can also be formulated as an LP problem. We call this model the *hose-rectangle model* in this paper as the bounded area for the traffic-demand matrix is contained in a rectangle.

1.2.2 Power efficient network problem

In this thesis, we also propose another robust optimization model for minimization of the network power consumption. Minimizing the network power consumption is also ongoing research and there are lots of studies that have been presented in the history.

To minimize the power consumption in networks, Bianzino et al. [56] introduced the green pipe model based on the knowledge of previously specified traffic-demand matrix. The green pipe model achieves the high performance due to exact traffic demands. However, it may not be fully applicable in the case where the traffic demands often fluctuate.

In green computing, Ouédraogo and Oki [30] applied the idea of the hose model introduced by Duffield et al. [17]. The hose model is contrary to the pipe model introduced by Wang and Wang [2] and does not require the exact traffic-demand matrix as the pipe model does. In the hose model, they bound the traffic with total outgoing and incoming amount of traffic at each node. The hose model is robust against traffic uncertainty and it is called a flexible service model.

Ouédraogo and Oki [30] introduced the green HLT model under traffic uncertainty to achieve power saving in networks. The green HLT model is proposed on the idea of a model called the hose model with bound of link traffic (HLT) [31], which is the developed version of the hose model. In HLT, the authors assume that network operators are able to impose additional bounds on the traffic passing through each link. The traffic bound for each link is determined by total amount of traffic measured on that link. Since the performance of the model depends on the maximum amount of traffic measured on each link, In the green HLT model, the authors proposed a parameter which indicates uncertainty to maximum amount of traffic measured on each link in order to maintain the robustness in the model. In the green HLT model, the additional bound on link fixes the range of traffic demand described by the hose model and makes it close to the pipe model.

1.3 Problem statement

Based on the previous work related to the congestion ratio problem and power efficient networks, using the formulations based on the knowledge of the exact traffic information (pipe model) makes it possible to achieve significant performance in both cases. However, these approaches, apart from the difficulty of traffic prediction for operators, have weaknesses in face of traffic fluctuation. A scheme applying the hose model, on the other hand, provides robustness against traffic uncertainty, but meager routing and energy performance compared to the pipe model. The limitation with the hose model, which is highly conservative, is due to the wide range of traffic specification considered. On the other hand, the hose-rectangle model is also robust and provides better routing performance by adding additional bounds to the traffic demand for each pair. The hose-rectangle model has a weakness, that is, it does not deal with total fluctuation over the network except fluctuation in each pair.

In the case of power efficient network, the green HLT model is also robust against traffic uncertainty. In the green HLT model, the authors bound the traffic passing through each link by total amount of traffic measured on that link as an additional bound to the hose model. The performance of the green HLT is closed to the green pipe model due to additional bound for each link. But the green HLT requires initial routing to compute the total traffic measured on each link and the routes need to be changed after solving the optimization problem which is not preferable for additional operating procedures needed to maintain network stability.

Our goal is to provide robust optimization models in the form of SOCP for minimization of congestion ratio and power efficient network, where we can use an ellipsoidal uncertainty set to allow total amount of fluctuation over the network without any additional bound and enhance the performances compared to the pervious studies. This is the objective in this thesis.

1.4 Contributions

This thesis introduces robust optimization models to minimize the network congestion ratio and for the design of power efficient networks allowing fluctuations in traffic demands between source-destination pair in the network. The assumption is that the network operator can estimate the traffic-demand matrix, but the estimated values may contain some errors where the total amount of error is bounded by a predefined constant. In other words, we assume that the traffic-demand matrix is contained in some uncertainty set whose 'center' is the estimated traffic-demand matrix and whose volume is bounded by some constant. This is a relevant situation if we remember that estimating the traffic-demand matrix exactly is a virtually impossible task. Our models allow the network operator to roughly estimate the traffic-demand matrix, and indicate the amount of admissible error.

In the first part of this thesis (Chapter 5), we apply robust optimization to minimize the network congestion ratio with traffic fluctuation. To do this, we consider an ellipsoidal uncertainty set, different from the hose and hose-rectangle uncertainty sets. Here we propose two models. The ellipsoid model considers only the ellipsoidal uncertainty set, and the hose-ellipsoid model considers both the hose and ellipsoidal uncertainty sets. We derive robust counterparts for these models which turn out to be SOCP problems. To derive the robust counterpart of the hose-ellipsoid model, we exploit the duality of conic linear programming in the presence of polyhedral cones [41].

In contrast to the hose-rectangle model, which considers the error of each source/destination pair, our ellipsoidal uncertainty set allows us to manage just the total amount of error among all pairs. The ellipsoidal uncertainty set is especially appropriate when the network-wide demands are governed by a multivariate gaussian probability distribution whose average is the estimated traffic-demand matrix.

It is well-known that, in general, a robust counterpart of LP with an ellipsoidal uncertainty set becomes an SOCP [24]. Our first model, the ellipsoid model, follows this logic; the original model is an LP, and we consider the ellipsoidal uncertainty set used to derive its robust counterpart as an SOCP. However, in the second model, the hose-ellipsoid model, we consider the intersection of polyhedral and ellipsoidal sets as an uncertainty set. Deriving a robust counterpart in such a case is non-obvious, but we are able to derive it as an SOCP by applying the duality theorem of SOCP. This is our major contribution.

In the second part of this thesis (Chapter 6), we propose a green and robust optimization model to the problem of minimizing the network power consumption using the same ellipsoidal uncertainty set that can also deal with traffic fluctuations. Our model introduced in that chapter is based on the hose-model-based optimization in green research applying second-order cone programming (SOCP) [16]. Our ultimate goal is to reduce the power consumption in networks by turning off some unnecessary links under traffic fluctuation. To do this, we use an ellipsoidal uncertainty set in addition to the hose model traffic bounds, where we assume that the true traffic-demand matrix is contained in the uncertainty set whose center is the estimated traffic-demand matrix and whose volume is bounded by some constant which indicates the total amount of fluctuation over the network

For robust optimization, we consider subproblems for the worst-case traffic scenario taking the uncertainty set which is the intersection of ellipsoidal and polyhedral sets. The subproblem is changed into a tractable SOCP problem. Finally using the knowledge of conic duality of the subproblem, we formulate the green hose ellipsoid (green HE) model again in mixed-integer second-order cone programming (MISOCP) form. Our proposed model allows the network operator to roughly estimate the traffic-demand matrix and represents the amount of permissible errors in the traffic demands.

1.5 Organization of the thesis

In Chapter 2, we discuss network communication with components of network, network model, and different types of network. Basic concept of conic programming and its dual are described in Chapter 3. In the later part of Chapter 3, we briefly discuss about mixed-integer second-order cone programming. The robust optimization with uncertainty sets are described in Chapter 4.

Chapter 5 presents the robust optimization model for the problem of minimizing the network congestion ratio. In this chapter, first we explain the pipe model, which demands the exact traffic-demand matrix, and then, we show that the hose and hose-rectangle models can be viewed as robust counterparts of the pipe model. After introducing the ellipsoidal uncertainty set that we use in our models, we formulate the ellipsoid model, the hose-ellipsoid model, and derive their robust counterparts.

In Chapter 6, another robust optimization model is presented in order to minimize the network power consumption for green computing considering the traffic uncertainty. The power model used to build the model is first introduced. Then the steps, which lead to the MISOCP formulation of green HE, are explained in details using the same uncertainty set as we used for formulation of the HE model in Chapter 5. These steps include the formulation for known traffic demands (green pipe model) and the formulation of the green hose model and green HLT models using the respective uncertainty sets.

The performances of our proposed models for congestion ratio and power efficient network are shown in experiments sections at the end of Chapter 5 and Chapter 6, respectively. We have conducted the experiments to check whether the proposed models can be solved in a reasonable time or not. In particular, in the case of MISOCP, there is a possibility that the problem cannot be solved in a reasonable time because it is NP-hard. The another reason for experiments is to observe some properties of the proposed models with respect to other models numerically. Some properties are proved in theory, but some are not proved.

Chapter 7 concludes the thesis by summarizing the findings and showing the direction for future work.

Chapter 2

Network communications

A communication network is an infrastructure that allows two or more nodes to communicate each other. The network accomplishes this by arranging a set of rules for communication, which are called protocols that should be observed by all participating nodes. In this chapter we review the fundamental concept of network communication. We will first discuss the constituent network components and various types of network, and then introduce the network model and backbone network.

2.1 Components of communication network

2.1.1 Node

A node is a point or vertex where more than two branches (links) meet. In a general communication network, there may be a large number of nodes and it is not necessary that each of them is connected to all others. The nodes typically handle the network protocols and provide switching capabilities. A node is usually itself a computer (general or special) which runs specific network software. The computers are also called the hosts of the network. The function of a network node is to make a connection between the output and incoming paths so that the signal can be switched to the desired path for onward transmission. In a telephone network, the telephone exchange (circuit switch) works as a node. In a data network, packet switch is known as node and is usually indicated to as a router. There are some nodes, like the message and packet switching ones, that have buffers and storage for messages. These kinds of nodes work as store-and-forward switches. There are also other functions of nodes like determining the incoming message, testing the free outlet, signaling, etc.

In a communication network, there are two kinds of nodes: edge node and intermediary node. The function of an edge node is to admit the data into the network and forward the data from one network to another network. The function of an intermediary node is to receive and forward data from one node to another node.

2.1.2 Link

The transmission medium of a communication network is known as a link which can be either a wire or a radio channel. There are different kinds of wired transmission medium such as a co-axial cable, an optical fiber, pair of copper wires or a multi-pair cable. The above transmission medium broadcasts the signals from one node to the other. The wireless transmission medium is a part of electromagnetic spectrum ranging from very low frequency to ultra high frequency including millimeter and optical waves. The wireless and wire channels can support data rates ranging from a few bits per second to many Giga/Peta bits per second since they have a very wide range of bandwidth. Due to various reasons such as dispersion and attenuation, the lengths of these transmission links are limited. All kinds of links in a communication networkIt are not necessary to be of the same type. Some of them can be wireless and some other may be wired.

2.2 Network model

A network interconnects many nodes through which a desired entity flows. The network is expressed as a directed graph $G(V, A)$, where A is the set of links and V is the set of nodes. Let the set $Q \subseteq V$ represents the set of edge nodes through which data is admitted into and going outside the network. In this thesis, an edge-node pair of $p \in Q$ and $q \in Q$, where $p \neq q$, is denoted by $(p, q) \in W$, where W indicates the set of edge-node pairs (p, q). The link from node $i \in V$ to node $j \in V \setminus \{i\}$ is represented as $(i, j) \in A$, $i \neq j$. Here c_{ij} and u_{ij} represent the capacity and the flow on $(i, j) \in A$, respectively. We consider the full duplex links in this research and both directions are powered on when there is data in one direction. Deactivating a link between node i and node j means deactivating both directions (i, j) and (j, i). The binary variable b_{ij} represents the on/off status of the links. The orientation of link remains different in the mathematical formulation although both directions (i, j) and (j, i) are powered on or off simultaneously. Power consumption of $(i, j) \in A$ is an affine function of its usage (ratio between its load and capacity). Here E_{fij} is the slope of the function and E_{0ij} is the constant term. The consumption of a link is equal to E_{0ij} when it is powered on and does not hold any traffic. The set of traffic demands is represented by $\boldsymbol{T} = \{d_{pq}\}$. And x_{ij}^{pq}, where $0 \leq x_{ij}^{pq} \leq 1$, is the portion of the traffic from $p \in Q$ to $q \in Q \setminus \{p\}$ routed through $(i, j) \in A$. Figure 2.1 describes a network topology with five nodes A, B, C, D, and E. These nodes are connected with the links L_{AB}, L_{BC}, L_{DC}, L_{DE}, etc.

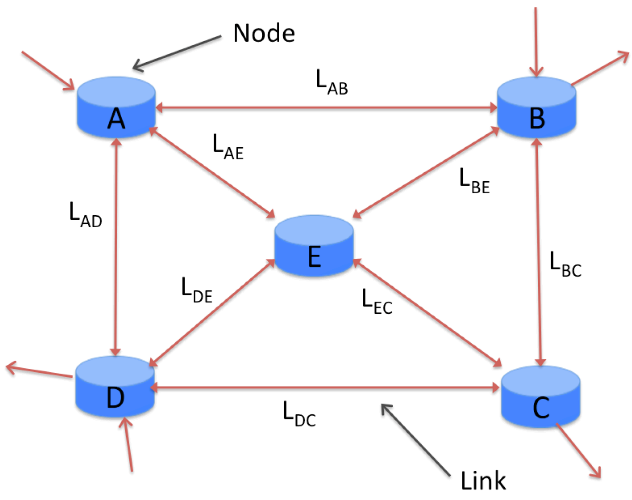

Figure 2.1: Network topology.

2.3 Different types of network

Based on the following four criteria the networks are divided into different types.

2.3.1 Geographic spread of nodes and hosts

Local area network (LAN): A network is said to be a Local Area Network (LAN) if the physical distance between the hosts is within a few kilometers. LANs are usually used to connect a set of hosts within the same building e.g., an office environment or a set of closely-located buildings e.g., a university campus.
Metropolitan area network (MAN): For larger distances, the network is said to be a metropolitan area network (MAN) or a Wide Area Network (WAN). MANs interconnect hosts across a city and cover distances of up to a few hundred kilometers.

2.3.2 Restricted access network

Private network: The network where the users are supposed to use the service for their private or business purpose. Networks maintained by banks, insurance companies, airlines, hospitals, and most other businesses are of this nature.
Public network: The network where the users have to complete required registration and have to pay the connection fees to get access to the network. Internet is the well known example of public networks.

2.3.3 Communication model employed by the nodes

Point-to-point model network: In this network, to get access from one node to another the message or information has to follow a specific route.
Broadcast model network: In this network, all nodes share the same communication medium. Therefore, a message or information transmitted by any node can be received by all other nodes. A part of the message (an address) shows for which node the message is designed and the nodes ignore the message if it does not match their own address.

2.3.4 Switching model employed by the nodes

Circuit switching network: In circuit switching network, a dedicated communication path is allocated between two hosts to communicate each other in the network, via a set of intermediate nodes. The information is sent through the path as a continuous stream of bits. This path is maintained for the duration of communication between two nodes, and is then released.
Packet switching network: To send information form one node to another, packet switching network divides the information into packets and also uses intermediate nodes to pass the information. Each intermediate node temporarily stores the packet and waits for the receiving node to become available to receive it. Since data is sent in packets, it is not necessary to reserve a path across the network for the duration of communication between two nodes. In order to enhance the performance of the network, different packets can be routed differently to spread the load between the nodes. However, this requires packets to allow additional admissible information.

2.4 Backbone network

The backbone network interconnects various local networks and provides paths for the exchange of information between them. It can tie together diverse networks in the same area, in different areas, or over wide areas. A large corporation that has many locations may have its own backbone network that ties all of the locations together. The network transmission medium such as ethernet, wire or wireless connections that bring these location together is often mentioned as network backbone. It is important to consider the network congestion while designing the backbones for better performance of the network.

Chapter 3

Second-order cone programming

The importance of second-order cone programming (SOCP) and mixed-integer second-order cone programming (MISOCP) problems has long been a focus of the mathematical programming society. In this chapter, we discuss basic concepts of SOCP and MISOCP with related propositions and theorems to read this thesis.

3.1 Preliminaries

In this subsection, we list some definitions needed for the following discussions.

Definition 1 A set $C \subseteq \mathbb{R}^n$ is a *cone* if and only if for any positive scalar λ and for any $\mathbf{x} \in C$ it holds that

$$\lambda \mathbf{x} \in C.$$

Definition 2 Let $C \subseteq \mathbb{R}^n$. Then C is *convex* if for any two points $\mathbf{x}_1, \mathbf{x}_2 \in C$, it holds that,

$$\theta \mathbf{x}_1 + (1-\theta)\mathbf{x}_2 \in C, \qquad \forall\ 0 \leq \theta \leq 1.$$

Let C be a convex set in $\mathbb{R}^n$. A function $f : C \to \mathbb{R}$ is said to be a *convex function* if $\forall \mathbf{x}_1, \mathbf{x}_2 \in C$ it holds that

$$f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \leq \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2), \qquad \forall\ 0 \leq \lambda \leq 1.$$

Definition 3 A set $S \subseteq \mathbb{R}^n$ is *affine* if $\mathbf{x}_1, \mathbf{x}_2 \in S$ implies that

$$t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \in S$$

for every real number t. Geometrically, a set is affine if whenever two points are in the set, the entire line through these points is in the set. An affine combination of vectors is a special kind of linear combination. For given vectors, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_m$ an *affine combination* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is a linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$ such that $c_1 + c_2 + \cdots + c_m = 1$.

Definition 4 An *open ball* of radius ϵ is the set of points of distance less ϵ from a fixed point. The open ball centered at $\mathbf{x}$ and radius ϵ is defined by

$$B(\mathbf{x}, \epsilon) = \{\mathbf{y} : ||\mathbf{y} - \mathbf{x}|| < \epsilon\}.$$

Definition 5 The set of all affine combinations of vectors in a set S is called the *affine hull* of S, and it is denoted by $\mathrm{aff}(S)$.

Definition 6 A point $\mathbf{x} \in C$ is a *relative interior point* of C if there exists $\epsilon > 0$ such that $B(\mathbf{x}, \epsilon) \cap \mathrm{aff}(C) \subseteq C$. The set of relative interior points of C is the *relative interior* of C, denoted by $\mathrm{ri}(C)$.

Definition 7 An optimization problem of the form

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) && && (3.1a) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, && i = 1, 2, \ldots, m, && (3.1b) \end{aligned}$$

is said to be a *convex optimization problem* if the functions $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ are convex.

Definition 8 Given an optimization problem P, we call a solution $\mathbf{x}$ *feasible* for P if it satisfies all the constraints of P. When f is the objective function of the optimization problem P, we define

$$\mathrm{val}(P) = \inf\{f(\mathbf{x}) : \mathbf{x} \text{ is feasible}\},$$

which will be called the *optimal value*, or *value* of the optimization problem P.

3.2 Second-order cone programming

The second-order cone (SOC) of dimension $l + 1$ is defined as

$$\mathrm{SOC}(l+1) = \left\{ \begin{pmatrix} t \\ \mathbf{u} \end{pmatrix} : \mathbf{u} \in \mathbb{R}^l, t \in \mathbb{R}, ||\mathbf{u}|| \leq t \right\}, \qquad (3.2)$$

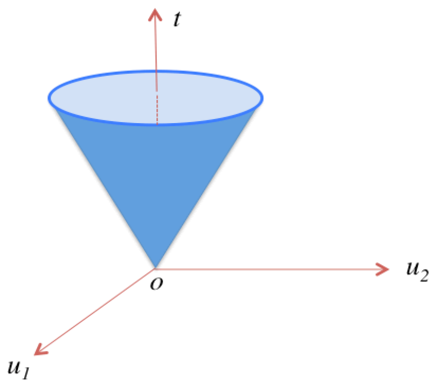

Figure 3.1: Second-order cone in 3 dimension.

which is also called the quadratic, ice-cream, or Lorentz cone. A second-order cone of dimension 3 is presented in Figure 3.1. The unit second-order cone of dimension 1 is defined as

$$\mathrm{SOC}(1) = \{t : t \in \mathbb{R}, 0 \leq t\}.$$

It is easy to verify that SOC($l+1$) is a convex cone. The proof is presented below using the following Lemma 1:

Lemma 1 . *Suppose C is a cone. Then C is convex if and only if*

$$\forall \mathbf{x}, \mathbf{y} \in C \text{ and } \forall \lambda, \mu > 0, \lambda \mathbf{x} + \mu \mathbf{y} \in C. \tag{3.3}$$

Proof: If (3.3) holds, then C is obviously convex. Suppose C is a cone and convex, and $\mathbf{x}, \mathbf{y} \in C$ and $\lambda, \mu > 0$ are given. Since C is a cone, we have

$$\frac{\lambda}{\lambda + \mu}\mathbf{x} \in C \text{ and } \frac{\mu}{\lambda + \mu}\mathbf{y} \in C.$$

Since C is convex, we have

$$\frac{\lambda}{\lambda + \mu}\mathbf{x} + \frac{\mu}{\lambda + \mu}\mathbf{y} \in C.$$

Again since C is a cone, we can write $\lambda \mathbf{x} + \mu \mathbf{y} \in C$. Therefore, (3.3) holds.

□

Proposition 1 SOC(l+1) *is convex.*

Proof: Let $\mathbf{x} = \begin{pmatrix} t_1 \\ \mathbf{u}_1 \end{pmatrix} \in \text{SOC}(l+1)$ and $\mathbf{y} = \begin{pmatrix} t_2 \\ \mathbf{u}_2 \end{pmatrix} \in \text{SOC}(l+1)$. We have to show $\forall \lambda, \mu \geq 0$, $||\lambda \mathbf{x} + \mu \mathbf{y}|| \leq \lambda t_1 + \mu t_2$.

For every, $\lambda, \mu \geq 0$ we can write

$$\begin{aligned} ||\lambda \mathbf{x} + \mu \mathbf{y}||^2 &= \lambda^2 ||\mathbf{x}||^2 + 2\lambda\mu \mathbf{x}^T \mathbf{y} + \mu^2 ||\mathbf{y}||^2 \\ &\leq \lambda^2 ||\mathbf{x}||^2 + 2\lambda\mu ||\mathbf{x}|| \; ||\mathbf{y}|| + \mu^2 ||\mathbf{y}||^2 \\ &\leq \lambda^2 t_1^2 + 2\lambda\mu t_1 t_2 + \mu^2 t_2^2 = (\lambda t_1 + \mu t_2)^2, \end{aligned}$$

which shows that $\begin{pmatrix} \lambda t_1 + \mu t_2 \\ \lambda \mathbf{x} + \mu \mathbf{y} \end{pmatrix} \in \text{SOC}(l+1)$. Therefore, SOC($l+1$) is convex.

□

Let K be a closed convex cone in $\mathbb{R}^n$. Then $K^* = \{\mathbf{s} : \mathbf{s}^T \mathbf{x} \geq 0 \; (\forall \mathbf{x} \in K)\}$ is called the *dual cone* of K. It is easy to see that K^* is in fact a cone. Furthermore, we have the following property.

Theorem 1 (Separation theorem (cone version)) [39]
Let K be a closed convex cone and $\mathbf{z} \notin K$. Then, there exists $\mathbf{y}$ such that

$$\mathbf{y}^T \mathbf{z} < 0 \leq \mathbf{y}^T \mathbf{x}, \qquad \forall \mathbf{x} \in K.$$

Proposition 2 *If K is a closed convex cone, then $K^{**} = K$.*

Proof: Let $\mathbf{x} \in K$. Then for every $\mathbf{y} \in K^*$, $\mathbf{x}^T \mathbf{y} \geq 0$. So $\mathbf{x} \in K^{**}$.

Conversely, let $\bar{\mathbf{x}} \in K^{**}$. Then for every $\mathbf{y} \in K^*$, $\bar{\mathbf{x}}^T \mathbf{y} \geq 0$. Suppose that $\bar{\mathbf{x}} \notin K$. Since K is a closed convex cone, according to the Theorem 1, there exists a vector $\bar{\mathbf{y}}$ such that

$$\bar{\mathbf{y}}^T \bar{\mathbf{x}} < 0 \leq \bar{\mathbf{y}}^T \mathbf{x}, \qquad \forall \mathbf{x} \in K.$$

The right inequality implies that $\bar{\mathbf{y}} \in K^*$. Then $\bar{\mathbf{y}}^T \bar{\mathbf{x}} < 0$ contradicts the fact that $\bar{\mathbf{x}} \in K^{**}$. Therefore, $\bar{\mathbf{x}} \in K$.

□

The dual of a second-order cone is itself, i.e., SOC$(l+1)^*$ =SOC$(l+1)$ [55]. Such a cone is called self-dual.

Second-order cone programming (SOCP) is a convex optimization problem in which a linear function is minimized over the intersection of an affine set and a direct product of second-order cones. The following is called the equality standard form second-order cone programming problem:

$$\text{minimize} \quad \mathbf{c}^T\mathbf{x} \tag{3.4a}$$

$$\text{s.t.} \quad A\mathbf{x} = \mathbf{b}, \tag{3.4b}$$

$$\mathbf{x} \in K, \tag{3.4c}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the decision variable, $K \subseteq \mathbb{R}^n$ is a direct product of second-order cones, and the other problem data are $\mathbf{c} \in \mathbb{R}^n,\ A \in \mathbb{R}^{m\times n},$ and $\mathbf{b} \in \mathbb{R}^m.$ To fully understand the properties of SOCP, please consult [55] or textbooks such as [35], [52], [53], [54].

An SOCP can be solved efficiently by the primal-dual interior-point methods [48], [49], [50], [51]. In fact, Gurobi [38] can be considered as one of the standard solvers, such as CPLEX [43] and SCIP [42]. CPLEX and SCIP also support SOCP.

3.3 Duality of conic linear programming

SOCP is a subcategory of more general class of optimization problems called conic linear programming. In this section, we briefly review conic linear programming and its duality, and derive a dual of SOCP in a special form. This dual relationship is used to derive the dual of the problem $S(\mathbf{x}_{ij})$ in Section 5.2.3 of Chapter 5.

We start with considering two closed convex cones $C \subseteq \mathbb{R}^m$ and $K \subseteq \mathbb{R}^n$. With $\bar{A} \in \mathbb{R}^{m\times n}$, $\bar{\mathbf{b}} \in \mathbb{R}^m$, and $\bar{\mathbf{c}} \in \mathbb{R}^n$, we consider an optimization problem of the form:

$$\begin{aligned} (P):\ \min\ & \bar{\mathbf{c}}^T\mathbf{x} \\ \text{s.t.}\ & \bar{A}\mathbf{x} - \bar{\mathbf{b}} \in C, \\ & \mathbf{x} \in K. \end{aligned}$$

This problem is called conic linear programming problem. Obviously, SOCP (3.4) is a special case of conic linear programming where $C = \{\mathbf{0}\}$ and K is a direct product of second-order cones. The (conic) dual of (P) [40] is

$$\begin{aligned}(D): \ \max \ & \bar{\mathbf{b}}^T\mathbf{y} \\ \text{s.t.} \ & \bar{\mathbf{c}} - \bar{A}^T\mathbf{y} \in K^*, \\ & \mathbf{y} \in C^*.\end{aligned}$$

It is easy to see that the weak duality holds between (P) and (D).

Theorem 2 (*Weak Duality*) *If* $\mathbf{x}$ *is feasible for* (P) *and* $\mathbf{y}$ *for* (D)*, then we have*

$$\bar{\mathbf{c}}^T\mathbf{x} \geq \bar{\mathbf{b}}^T\mathbf{y}.$$

Proof: We can write

$$\begin{aligned}\bar{\mathbf{c}}^T\mathbf{x} - \bar{\mathbf{b}}^T\mathbf{y} &= \bar{\mathbf{c}}^T\mathbf{x} - \mathbf{y}^T\bar{A}\mathbf{x} + \mathbf{y}^T\bar{A}\mathbf{x} - \bar{\mathbf{b}}^T\mathbf{y} \\ &= (\bar{\mathbf{c}} - \bar{A}\mathbf{y})^T\mathbf{x} + (\bar{A}\mathbf{x} - \bar{\mathbf{b}})^T\mathbf{y} \\ &\geq 0,\end{aligned}$$

since $\bar{\mathbf{c}} - \bar{A}\mathbf{y} \in K^*, \mathbf{x} \in K, \bar{A}\mathbf{x} - \bar{\mathbf{b}} \in C$, and $\mathbf{y} \in C^*$.

□

If $C = \{\mathbf{0}\}$ and K is a closed convex cone, then the primal-dual pair of conic linear programming becomes

$$(P_0): \begin{cases} \min & \bar{\mathbf{c}}^T\mathbf{x} \\ \text{s.t.} & \bar{A}\mathbf{x} = \bar{\mathbf{b}}, \\ & \mathbf{x} \in K \end{cases} \overset{dual}{\longleftrightarrow} (D_0): \begin{cases} \max & \bar{\mathbf{b}}^T\mathbf{y} \\ \text{s.t.} & \bar{\mathbf{c}} - \bar{A}^T\mathbf{y} \in K^*. \end{cases}$$

The problem (P_0), sometimes called the *equality standard form* of conic linear programming, is widely used in the literature. Note that, although the rest of this section writes the theorems in terms of (P_0) and (D_0), all are also valid on (P) and (D).

We say (P_0) satisfies *Slater's condition* if there exists a feasible solution $\bar{\mathbf{x}}$ such that

$$\bar{\mathbf{x}} \in \text{ri}K.$$

Similarly, we say that (D_0) satisfies Slater's condition if there exists a feasible solution $\bar{\mathbf{y}}$ of (D_0) such that

$$\bar{\mathbf{c}} - \bar{A}^T\bar{\mathbf{y}} \in \mathrm{ri}K^*.$$

In conic linear programming, we need Slater's condition to state the strong duality.

Theorem 3 (*Strong Duality*) [41]

1. *If* (P_0) *satisfies Slater's condition, and* (D_0) *has a feasible solution, then* $\mathrm{val}(P_0) = \mathrm{val}(D_0)$*, and* (D_0) *has an optimal solution.*

2. *If* (D_0) *satisfies Slater's condition, and* (P_0) *has a feasible solution, then* $\mathrm{val}(P_0) = \mathrm{val}(D_0)$*, and* (P_0) *has an optimal solution.*

Using the duality of (P) and (D), we can easily show that the following theorem holds.

Theorem 4 *Suppose that* K *is a direct product of SOCs, and* $\mathbf{b} \in \mathbb{R}^m, A \in \mathbb{R}^{m\times n_1}, B \in \mathbb{R}^{m\times n_2}, \mathbf{c} \in \mathbb{R}^{n_1}, \mathbf{f} \in \mathbb{R}^{n_2}$*. Then the dual of*

$$\begin{aligned} (P_1): \ & \max\ \mathbf{b}^T\mathbf{y} \\ & s.t.\ A^T\mathbf{y} \le \mathbf{c}, \\ & \quad\ \ B^T\mathbf{y} = \mathbf{f}, \\ & \quad\ \ \mathbf{y} \in K \end{aligned}$$

is

$$\begin{aligned} (D_1): \ & \min\ \mathbf{c}^T\mathbf{x} + \mathbf{f}^T\mathbf{w} \\ & s.t.\ A\mathbf{x} + B\mathbf{w} - \mathbf{b} \in K, \\ & \quad\ \ \mathbf{x} \ge \mathbf{0}. \end{aligned}$$

Proof: The problem (P_1) can be written as

$$\begin{aligned} & \max\ \mathbf{b}^T\mathbf{y} \\ & \text{s.t.}\ \ \mathbf{c} - A^T\mathbf{y} \in \mathbb{R}^{n_1}_+, && (3.5) \\ & \qquad \mathbf{f} - B^T\mathbf{y} \in \{\mathbf{0}\}, && (3.6) \end{aligned}$$

$$\mathbf{y} \in K. \tag{3.7}$$

Using the direct product $\mathbb{R}_+^{n_1} \times \{\mathbf{0}\}$, we can write the above problem as

$$\begin{aligned} (P_2) : \max\ & \mathbf{b}^T\mathbf{y} \\ \text{s.t.}\ & \begin{pmatrix} \mathbf{c} \\ \mathbf{f} \end{pmatrix} - \begin{pmatrix} A^T \\ B^T \end{pmatrix} \mathbf{y} \in \mathbb{R}_+^{n_1} \times \{\mathbf{0}\}, \\ & \mathbf{y} \in K. \end{aligned}$$

If we compare the problem (P_2) with problem (D), then we obtain the following correspondence:

$$\bar{A} = \begin{pmatrix} A^T \\ B^T \end{pmatrix}^T = (A\ B) \in \mathbb{R}^{m \times (n_1+n_2)}, \bar{\mathbf{c}} = \begin{pmatrix} \mathbf{c} \\ \mathbf{f} \end{pmatrix} \in \mathbb{R}_+^{(n_1+n_2)}, \text{and } C = K.$$

Since K is self-dual, the dual of the problem (P_2) is as follows:

$$\begin{aligned} \min\ & \begin{pmatrix} \mathbf{c} \\ \mathbf{f} \end{pmatrix}^T \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} \\ \text{s.t.}\ & (A\ \ B) \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} - \mathbf{b} \in K \\ & \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} \in \mathbb{R}_+^{n_1} \times \mathbb{R}^{n_2} \end{aligned}$$

$$\Leftrightarrow \quad \begin{aligned} \min\ & \mathbf{c}^T\mathbf{x} + \mathbf{f}^T\mathbf{w} \\ \text{s.t.}\ & A\mathbf{x} + B\mathbf{w} - \mathbf{b} \in K, \\ & \mathbf{x} \in \mathbb{R}_+^{n_1}, \mathbf{w} \in \mathbb{R}^{n_2} \end{aligned}$$

$$\Leftrightarrow \quad \begin{aligned} \min\ & \mathbf{c}^T\mathbf{x} + \mathbf{f}^T\mathbf{w} \\ \text{s.t.}\ & A\mathbf{x} + B\mathbf{w} - \mathbf{b} \in K, \\ & \mathbf{x} \geq 0, \end{aligned}$$

which is the final dual form of (P_2).

□

3.4 Mixed-integer second-order cone programming (MISOCP)

An MISOCP is an optimization problem which involves some real and integer variables to optimize a linear objective function subject to some linear and second-order cone constraints. It is well known that MISOCP is NP-hard and it is not so easy to solve MISOCP like LP or SOCP, because MISOCP involves integer variables in addition to the properties of LP or SOCP. Modern optimization software like Gurobi [38] and CPLEX [43] can handle MISOCPs but generally it takes longer computation time compared to LP or SOCP of a similar size.

Mixed-integer second-order cone problems have various applications in finance and engineering. For example, MISOCP have been used to model and solve many challenging applied problems such as transmission in cellular networks [5], power distribution [6], portfolio optimization [7], battery swapping stations on freeway networks [8], telecommunication network design [30], etc.

Chapter 4

Robust optimization

In the last few decades, the idea of robust optimization has emerged as a key trend in the field of optimization [22], [37]. In this chapter, we discuss the basic concept of robust optimization that we apply in this research.

4.1 General form of robust optimization

If some information of an optimization problem is not defined exactly, but it is said to belong to a predefined set ,and we want to optimize the problem in the worst case with respect to the set, then the resulting optimization problem is called a robust optimization problem ([23], [39], [52], [54]). The objective of robust optimization is to optimize the problem in the worst case wherein the problem data lie in a set. In such cases, the set where parameters are supposed to fall in is called uncertainty set. In general, the uncertainty set contains infinitely many points. This means that the robust optimization problem can be categorized as an instance of semi-infinite programming, which is inherently intractable. Thanks to the progress in conic linear programming, robust optimization is now being used in many real-world applications such as finance [44], mechanics [45], and control [46], [47] etc.

The general form of robust optimization problem can be stated as follows:

$$\text{minimize} \quad f_0(\mathbf{x}) \tag{4.1a}$$

$$\text{s.t.} \quad f_i(\mathbf{x}, \mathbf{d}_i) \leq 0, \qquad \forall \mathbf{d}_i \in \Omega_i, \; i = 1, 2, ..., m, \tag{4.1b}$$

where $\mathbf{x} \in \mathbb{R}^n$ is a vector of decision variables, $f_0, f_i : \mathbb{R}^n \to \mathbb{R}$ are functions, $\mathbf{d}_i \in \mathbb{R}^{k_i}$ are parameter uncertainties, and Ω_i are uncertainty sets. We can always take the objective function to have no uncertainty by introducing a new constraint if necessary. The aim of the problem (4.1) is to compute minimum solutions $\mathbf{x}^*$ among all those solutions which are feasible for all realizations of the $\mathbf{d}_i \in \Omega_i$. The optimization problem (4.1) offers some measure of feasibility protection for optimization problems containing parameters which are not known exactly. Although a robust optimization problem is in general intractable, in some cases, we can reformulate it as a conic linear programming problem which is tractable, and this is the case we present in the thesis. In such a case, the reformulated optimization problem is called robust counterpart.

In the following sections, we consider robust optimization for LP. We consider two uncertainty sets: a polyhedral uncertainty set and a ball uncertainty set. We present how we can derive robust counterparts of the robust linear programming. In the latter chapters, similar arguments will be used to obtain robust counterparts of various models.

4.2 Robust optimization for linear programming with polyhedral uncertainty sets

The linear programming (LP) problem can be represented as

$$\min_{\mathbf{x}} \ \mathbf{c}^T\mathbf{x} \tag{4.2a}$$

$$\text{s.t. } \mathbf{a}_i^T\mathbf{x} \le b_i, \qquad i = 1, 2, \ldots, m, \tag{4.2b}$$

where $\mathbf{a}_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, and $\mathbf{c} \in \mathbb{R}^n$ are given, and $\mathbf{x}$ is the decision variable.

A robust linear programming problem can be stated as

$$\min_{\mathbf{x}} \ \mathbf{c}^T\mathbf{x} \tag{4.3a}$$

$$\text{s.t. } \mathbf{a}_i^T\mathbf{x} \le b_i, \qquad \forall \mathbf{a}_i \in U_{\mathbf{a}_i}, \forall b_i \in U_{b_i}, i = 1, 2, \ldots, m, \tag{4.3b}$$

where $U_{\mathbf{a}_i} \subseteq \mathbb{R}^n$, and $U_{b_i} \subseteq \mathbb{R}$ are given uncertainty sets.

Robust linear programming with polyhedral uncertainty sets is the special case of the problem (4.3) where

$$U_{\mathbf{a}_i} = \{\mathbf{a}_i : D_i\mathbf{a}_i \leq \mathbf{d}_i\}, \tag{4.4}$$

where $D_i \in \mathbb{R}^{k_i \times n}$ and $\mathbf{d}_i \in \mathbb{R}^{k_i}$ are given, and U_{b_i} is a given interval in $\mathbb{R}$ [23]. It is clear that for the optimal value of the problem (4.3), we can get rid of the uncertainty in b_i because the worst-case scenario is obtained at the infimum of the interval. Therefore the problem (4.3) becomes

$$\min_{\mathbf{x}} \ \mathbf{c}^T\mathbf{x} \tag{4.5a}$$

$$\text{s.t. } \mathbf{a}_i^T\mathbf{x} \leq b_i, \qquad \forall \mathbf{a}_i \in U_{\mathbf{a}_i}, i = 1, 2, \ldots, m, \tag{4.5b}$$

where $U_{\mathbf{a}_i} = \{\mathbf{a}_i : D_i\mathbf{a}_i \leq \mathbf{d}_i\}$ and each b_i denotes the infimum of each interval. Equivalently, the robust linear programming problem (4.5) can be written as

$$\min_{\mathbf{x}} \ \mathbf{c}^T\mathbf{x} \tag{4.6a}$$

$$\text{s.t. } \max_{\mathbf{a}_i \in U_{\mathbf{a}_i}} \ \mathbf{a}_i^T\mathbf{x} \leq b_i, \qquad i = 1, 2, \ldots.., m. \tag{4.6b}$$

Our goal is to convert the min-max problem to a min-min problem so that we can combine the two minimization problems. If we are given $\mathbf{x}$, then the left-hand side of (4.6b) is the optimal value of

$$\max \ \mathbf{a}_i^T\mathbf{x} \tag{4.7a}$$

$$\text{s.t. } D_i\mathbf{a}_i \leq \mathbf{d}_i, \tag{4.7b}$$

where $\mathbf{a}_i$ is the decision variable. Now using the duality of equality standard form of conic linear programming as describe above, the dual of the subproblem (4.7) can be expressed as:

$$\min \ \mathbf{p}_i^T\mathbf{d}_i \tag{4.8a}$$

$$\text{s.t. } D_i^T\mathbf{p}_i = \mathbf{x}, \tag{4.8b}$$

$$\mathbf{p}_i \geq \mathbf{0}. \tag{4.8c}$$

By the strong duality theorem, the primal (4.7) and the dual (4.8) have the same optimal value. So, we can replace the left-hand side of the constraints (4.6b) by the optimal value of (4.8).

As a result, the problem (4.6) can be written as

$$\begin{aligned}
\min_{\mathbf{x},\mathbf{p}_i}\ & \mathbf{c}^T\mathbf{x} \\
\text{s.t. } & \mathbf{p}_i^T\mathbf{d}_i \le b_i, \qquad i = 1,2,\ldots,m, \\
& D_i^T\mathbf{p}_i = \mathbf{x}, \qquad i = 1,2,\ldots,m, \\
& \mathbf{p}_i \ge \mathbf{0}, \qquad i = 1,2,\ldots,m.
\end{aligned} \tag{4.9}$$

We have seen that the robust optimization of LP with a polyhedral uncertainty set is formulated into another LP . Obviously, LP can be solved efficiently, and so, (4.9) is the robust counterpart of (4.5).

4.3 Robust linear programming with ball uncertainty sets

Here, we consider the following optimization problem:

$$\min\ \mathbf{c}^T\mathbf{x} \tag{4.10a}$$

$$\text{s.t. } \mathbf{a}_i^T\mathbf{x} \le b_i, \qquad \forall \mathbf{a}_i \in U_{\mathbf{a}_i}, \quad i = 1,2,\ldots,m, \tag{4.10b}$$

where the uncertainty sets,

$$U_{\mathbf{a}_i} = \{\bar{\mathbf{a}}_i + \mathbf{u} : ||\mathbf{u}|| \le \epsilon\}. \tag{4.11}$$

Here, $\bar{\mathbf{a}}_i \in \mathbb{R}^n$ and ϵ are given, and $\mathbf{u}$ is restricted in a ball whose radius is ϵ. We call this uncertainty set a ball uncertainty set.

Again, using the ball uncertainty sets, we rewrite the problem (4.10) as

$$\begin{aligned}
\min\ & \mathbf{c}^T\mathbf{x} \\
\text{s.t. } & \max_{\mathbf{a}_i \in U_{\mathbf{a}_i}} \mathbf{a}_i^T\mathbf{x} \le b_i, \qquad i = 1,2,\ldots,m.
\end{aligned} \tag{4.12}$$

The left-hand side of the constraint (4.12) can be written as:

$$\begin{aligned}
\max\left\{\mathbf{a}_i^T\mathbf{x} : \mathbf{a}_i \in U_{\mathbf{a}_i}\right\} &= \max\left\{(\bar{\mathbf{a}}_i + \mathbf{u})^T\mathbf{x} : ||\mathbf{u}|| \le \epsilon\right\} \\
&= \bar{\mathbf{a}}_i^T\mathbf{x} + \max\left\{\mathbf{u}^T\mathbf{x} : ||\mathbf{u}|| \le \epsilon\right\} \\
&= \bar{\mathbf{a}}_i^T\mathbf{x} + \epsilon||\mathbf{x}||,
\end{aligned}$$

where the last equality is obtained by observing that the optimal solution of the optimization problem $\max\{\mathbf{u}^T\mathbf{x} : ||\mathbf{u}|| \leq \epsilon\}$ is $\epsilon\mathbf{x}/||\mathbf{x}||$.

Then, the problem (4.12) can be written as

$$\min \mathbf{c}^T\mathbf{x} \tag{4.13}$$

$$\text{s.t. } \bar{\mathbf{a}}_i^T\mathbf{x} + \epsilon||\mathbf{x}|| \leq b_i, \qquad i = 1, 2, \ldots, m, \tag{4.14}$$

which represents an SOCP problem because it minimizes the linear objective function subject to the SOC constraints (4.14). Therefore, a robust LP problem with ball uncertainty sets can be cast into an SOCP problem which can be solved efficiently by modern solvers.

Some existing studies on network communication related to this research used polyhedral uncertainty sets to allow some errors in traffic demands. In our cases, we used ellipsoidal uncertainty set to make some fluctuations in traffic demands where the total amount of squared errors bounded by a positive constant.

Chapter 5

Minimizing network congestion ratio with traffic fluctuations

The maximum link utilization rate among all links in a network is called the *network congestion ratio* [2]. If some links or nodes broadcast too much information, it causes serious network congestion that results in greater end-to-end delay and packet loss or decreases in the throughput. Network congestion can severely degrade the performance of the network [1]. Fortunately, setting an appropriate routing can enlarge the network resource utilization rate and the throughput [9]. Finding such a proper routing is a major concern for network operators whose goal is to provide better network performance. Since traffic fluctuates due to users' needs, we can allow additional traffic in the network by minimizing network congestion ratio. [21]. There are a lot of studies in the history to minimize the network congestion ratio, but most of them are presented under the assumption that the traffic-demand matrix is exactly known or there are some bounds in the traffic demands. In this chapter, we introduce robust optimization models to minimize the congestion ratio that can deal with fluctuations of traffic. Numerical results show that our proposed models can achieve the performance compared to the previous studies.

5.1 Minimizing network congestion ratio and existing robust optimization models

5.1.1 Problem formulation

The network we consider is represented as a directed graph $G(V, \mathcal{A})$, where V is the set of nodes and $\mathcal{A}$ is the set of links. A link from $i \in V$ to $j \in V \setminus \{i\}$ is denoted by $(i,j) \in \mathcal{A}$. The capacity of link $(i,j) \in \mathcal{A}$ is c_{ij}. Let $Q \subseteq V$ be the set of *edge nodes* through which traffic enters and leaves the network. We denote by W the set of edge-node pairs (p,q), i.e,

$$W = \{(p,q) \in Q \times Q : p \neq q\}.$$

We assume that traffic, which is allowed to be split into any portion, can take any route. This is, for example, executed by the Multi-Protocol Label Switching (MPLS) Traffic-Engineering (TE) technology [10], [11]. For $(p,q) \in W$ and $(i,j) \in \mathcal{A}$, the ratio of traffic from p to q routed on (i,j) with respect to the total amount sent from p to q is denoted by x_{ij}^{pq}, where $0 \leq x_{ij}^{pq} \leq 1$. When demand $T = \{d_{pq} : (p,q) \in W\}$ is given, the amount of traffic sent on link (i,j) is $\sum_{(p,q)\in W} d_{pq} x_{ij}^{pq}$. Therefore, the network congestion ratio is defined by

$$\max \left\{ \frac{\sum_{(p,q)\in W} d_{pq} x_{ij}^{pq}}{c_{ij}} \ : \ (i,j) \in \mathcal{A} \right\}.$$

Assuming that traffic demand $T = \{d_{pq} : (p,q) \in W\}$ is known, Wang and Wang [2] formulated an explicit routing problem whose objective is to minimize the network congestion ratio by solving the following LP problem:

$$\min r \tag{5.1a}$$

$$\text{s.t.} \sum_{j:(i,j)\in\mathcal{A}} x_{ij}^{pq} - \sum_{j:(j,i)\in\mathcal{A}} x_{ji}^{pq} = 1, \quad \forall (p,q) \in W, i = p, \tag{5.1b}$$

$$\sum_{j:(i,j)\in\mathcal{A}} x_{ij}^{pq} - \sum_{j:(j,i)\in\mathcal{A}} x_{ji}^{pq} = 0, \quad \forall (p,q) \in W, \forall i \in V \setminus \{p,q\}, \tag{5.1c}$$

$$\sum_{(p,q)\in W} d_{pq} x_{ij}^{pq} \leq c_{ij} r, \qquad \forall (i,j) \in \mathcal{A}, \tag{5.1d}$$

$$0 \leq x_{ij}^{pq} \leq 1, \qquad \forall (p,q) \in W, \forall (i,j) \in \mathcal{A}, \tag{5.1e}$$

$$0 \le r \le 1. \tag{5.1f}$$

Here the constraints (5.1b) and (5.1c) represent the flow conservation law. The constraint (5.1b) indicates that the total of traffic flow portions leaving node $i(=p)$ equals 1 while (5.1c) states that the total portion of traffic entering node i must be the same as that leaving from node i if node i is neither a source nor destination for the traffic flow. Constraint (5.1d) and the objective function ensure that r becomes the network congestion ratio if an optimal solution is obtained.

Note that, at the destination node ($i = q$), the condition to maintain the flow is

$$\sum_{j:(i,j)\in A} x_{ij}^{pq} - \sum_{j:(j,i)\in A} x_{ji}^{pq} = -1 \quad \forall (p,q) \in W, i = q. \tag{5.2}$$

The destination node must satisfy the constraint (5.2), but this constraint can be obtained by using the constraints (5.1b) and (5.1c). Therefore, the constraint (5.2) is guaranteed by the constraints (5.1b) and (5.1c), which is proved in Appendix A.

This model is called the pipe model in [17] and [18], where the following hose model was presented for virtual private networks. Note that the pipe model is valid only if we know the complete traffic-demand matrix exactly.

In this thesis, we use the robustness in the sense that the true traffic-demand matrix can be different from the estimated one. However, we impose that the difference is small and bounded by a constant.

5.1.2 Hose model

This and the next subsections give a fresh appraisal of two existing models associated with minimizing the congestion ratio; each can be viewed as an application of robust optimization for the pipe model, and their differences come from the uncertainty sets selected.

It is often an impossible task for network operators to measure and predict the actual traffic data, T, accurately, but sometimes they can easily specify just the total outgoing/incoming traffic from/to node p to q. The total outgoing traffic from node p is represented as

$$\sum_{q} d_{pq} \leq \alpha_p, \tag{5.3}$$

where α_p is the maximum amount of traffic that node p can send into the network. In such a case the total incoming traffic to node q is represented as

$$\sum_{p} d_{pq} \leq \beta_q, \tag{5.4}$$

where β_q is the maximum amount of traffic that node q can receive from the network. The traffic-demand model having such upper bounds is called the hose model [16], [17], [19].

With regard to the robust optimization viewpoint, their work can be regarded as follows. First, consider the uncertainty set:

$$\mathcal{H} = \left\{ \mathbf{d} \in \mathbb{R}^W \;:\; \begin{array}{r} \sum_{q \in Q} d_{pq} \leq \alpha_p, \quad \forall p \in Q, \\ \sum_{p \in Q} d_{pq} \leq \beta_q, \quad \forall q \in Q, \\ d_{pq} \geq 0, \quad \forall (p,q) \in W \end{array} \right\}, \tag{5.5}$$

which we call the *hose uncertainty set* in the following. Next, we amend the pipe model so that (5.1d) holds for every demand $\mathbf{d} \in \mathcal{H}$ to obtain

$$(H) : \min r \tag{5.6a}$$

$$\text{s.t. Eqs. (5.1b), (5.1c),} \tag{5.6b}$$

$$\max_{\mathbf{d} \in \mathcal{H}} \max_{(i,j) \in \mathcal{A}} \sum_{(p,q) \in W} d_{pq} x_{ij}^{pq} / c_{ij} \leq r, \tag{5.6c}$$

$$\text{Eqs. (5.1e), (5.1f).} \tag{5.6d}$$

This is robust version for the pipe model with respect to the hose uncertainty set. Exchanging the two max operators in (5.6c), we obtain the condition

$$\max_{\mathbf{d} \in \mathcal{H}} \sum_{(p,q) \in W} d_{pq} x_{ij}^{pq} \leq c_{ij} r, \ \forall (i,j) \in \mathcal{A}, \tag{5.7}$$

which is equivalent to (5.6c).

Note that the left hand side of (5.7) is the optimal value of the following LP problem:

$$\max \sum_{(p,q)\in W} d_{pq} x_{ij}^{pq} \tag{5.8a}$$

$$\text{s.t.} \sum_{q\in Q} d_{pq} \le \alpha_p, \qquad \forall p \in Q, \tag{5.8b}$$

$$\sum_{p\in Q} d_{pq} \le \beta_q, \qquad \forall q \in Q, \tag{5.8c}$$

$$d_{pq} \ge 0, \qquad \forall (p,q) \in W. \tag{5.8d}$$

The dual of the problem (5.8) is as follows:

$$\min \sum_{p\in Q} \alpha_p \pi_{ij}(p) + \sum_{q\in Q} \beta_q \lambda_{ij}(q) \tag{5.9a}$$

$$\text{s.t. } \pi_{ij}(p) + \lambda_{ij}(q) \ge x_{ij}^{pq}, \qquad \forall (p,q) \in W, \forall (i,j) \in \mathcal{A}, \tag{5.9b}$$

$$\pi_{ij}(p), \lambda_{ij}(q) \ge 0, \qquad \forall (p,q) \in W, \forall (i,j) \in \mathcal{A}. \tag{5.9c}$$

Since the dual of the LP problem has the same optimal value as the primal,[1] it is possible to reformulate the left hand side of (5.7) by the dual problem yielding a robust counterpart of (H) as follows:

$$(H) : \min r \tag{5.10a}$$

$$\text{s.t. Eqs. } (5.1\text{b}), (5.1\text{c}), \tag{5.10b}$$

$$\sum_{p\in Q} \alpha_p \pi_{ij}(p) + \sum_{q\in Q} \beta_q \lambda_{ij}(q) \le c_{ij} r, \qquad \forall (i,j) \in \mathcal{A}, \tag{5.10c}$$

$$x_{ij}^{pq} \le \pi_{ij}(p) + \lambda_{ij}(q), \quad \forall (p,q) \in W, \forall (i,j) \in \mathcal{A}, \tag{5.10d}$$

$$\pi_{ij}(p), \lambda_{ij}(q) \ge 0, \quad \forall (p,q) \in W, \forall (i,j) \in \mathcal{A}, \tag{5.10e}$$

$$\text{Eqs. } (5.1\text{e}), (5.1\text{f}). \tag{5.10f}$$

This is the hose model presented in [13], [16], [17], and [19]. The hose model is known as more flexible than the pipe model because we can choose parameters to bound the total amounts of inputs/outputs. On the other hand, it tends to

[1] Strictly speaking this holds if at least one of the primal or dual problem is feasible, and it is easy to show that the condition holds with a mild assumption such as Assumption 1 which will be presented in Section 5.2.3.

allow big errors in the traffic demands. The hose model is well performed in the case of highly varied traffic conditions and large network. However, the routing performance of the hose model is much lower than that of the pipe model, because this model tends to show much lower performance in the experiments [20] possibly because of its loose bounds.

5.1.3 Hose-rectangle model

Oki and Iwaki [20] introduced another model where, in addition to the hose model, each demand d_{pq} should be bounded below and above by positive constants. Specifically, given δ_{pq} and γ_{pq} for every $(p, q) \in W$, we consider the following uncertainty set:

$$\mathcal{R} = \left\{ \mathbf{d} \in \mathbb{R}^W \ : \ \delta_{pq} \le d_{pq} \le \gamma_{pq}, \ (p, q) \in W \right\}, \tag{5.11}$$

which we call the *rectangle uncertainty set* hereafter. If we consider robust optimization with respect to the uncertainty set that is the intersection of the hose and rectangle uncertainty sets we obtain

$$(R) : \min r \tag{5.12}$$

$$\text{s.t. Eqs. (5.1b), (5.1c),} \tag{5.13}$$

$$\max_{\mathbf{d} \in \mathcal{H} \cap \mathcal{R}} \sum_{(p,q) \in W} d_{pq} x_{ij}^{pq} \le c_{ij} r, \quad \forall (i, j) \in \mathcal{A}, \tag{5.14}$$

$$\text{Eqs. (5.1e), (5.1f).} \tag{5.15}$$

Again, this problem can be converted into an LP, and the resulting problem is

$$(I_1) : \min r \tag{5.16a}$$

$$\text{s.t. Eqs. (5.1b), (5.1c),} \tag{5.16b}$$

$$\sum_{p \in Q} \alpha_p \pi_{ij}(p) + \sum_{q \in Q} \beta_q \lambda_{ij}(q) + \sum_{p \in Q} \sum_{q \in Q} [\gamma_{pq} \eta_{ij}(p, q) - \delta_{pq} \theta_{ij}(p, q)] \le c_{ij} r, \quad \forall (i, j) \in \mathcal{A}, \tag{5.16c}$$

$$x_{ij}^{pq} \le \pi_{ij}(p) + \lambda_{ij}(q) + \eta_{ij}(p, q) - \theta_{ij}(p, q), \quad \forall (p, q) \in W, \forall (i, j) \in \mathcal{A}, \tag{5.16d}$$

$$\pi_{ij}(p), \lambda_{ij}(q), \eta_{ij}(p, q), \theta_{ij}(p, q) \ge 0, \quad \forall (p, q) \in W, \forall (i, j) \in \mathcal{A}, \tag{5.16e}$$

$$\text{Eqs. } (5.1e), (5.1f), \tag{5.16f}$$

where $\pi_{ij}(p)$, $\lambda_{ij}(q)$, $\eta_{ij}(p,q)$ and $\theta_{ij}(p,q)$ are additional variables introduced by the dual of the subproblems. See [20] for details.

We point out that the rectangle uncertainty set can be viewed as follows. Set, for each $(p,q) \in W$,

$$\bar{d}_{pq} = (\delta_{pq} + \gamma_{pq})/2, \text{and } \epsilon_{pq} = (\gamma_{pq} - \delta_{pq})/2, \tag{5.17}$$

so that $\bar{d}_{pq}$ is the midpoint of the interval $[\delta_{pq}, \gamma_{pq}]$. This allows us to write the condition in $\mathcal{R}$ as

$$\left|d_{pq} - \bar{d}_{pq}\right| \leq \epsilon_{pq}, \ \forall (p,q) \in W. \tag{5.18}$$

We can say that their model presumes an *estimated value* $\bar{d}_{pq}$ for each $(p,q) \in W$ and allows some fluctuation in real demand d_{pq} from $\bar{d}_{pq}$ up to ϵ_{pq}. Specifically, we can write:

$$\mathcal{R} = \left\{\mathbf{d} \in \mathbb{R}^W \ : \ |d_{pq} - \bar{d}_{pq}| \leq \epsilon_{pq}\right\}. \tag{5.19}$$

Since they use a rectangle uncertainty set together with the hose uncertainty set, we call their model the *hose-rectangle* model in this thesis. The hose-rectangle model can set the range of errors of traffic demands for each pair in the network specified by the hose model. The network operators can predict the fluctuation of each traffic demand and can determine the total fluctuations in some part of the network using the hose-rectangle model [21]. The hose-rectangle model narrows the range of traffic conditions specified by the hose model by adding additional bounds to traffic demands for each source-destination pair in the network and offers better routing performance than the hose model.

5.2 Robust optimization models using ellipsoidal uncertainty set

5.2.1 Ellipsoidal uncertainty set

In contrast to the hose uncertainty set where fluctuation in total input/output of source nodes is captured and in rectangle uncertainty set the fluctuation is locally considered for each demand pair, we try to capture the fluctuation over the network in total. To this end, we bound the total amount of squared errors in $\bar{d}_{pq}$ for all $(p, q) \in W$ by a positive constant ϵ, and the true demand is contained in the ellipsoid:

$$\Theta_\epsilon = \left\{ \mathbf{d} : \sqrt{\sum_{(p,q)\in W} \rho_{pq}(d_{pq} - \bar{d}_{pq})^2} \leq \epsilon \right\}, \tag{5.20}$$

where $\rho_{pq} > 0$ for every $(p, q) \in W$ is a weight that indicates the significance of pair (p, q) in terms of the fluctuation. If ρ_{pq} is large, then we will not allow a large fluctuation on the link, while if it is small, we allow generous fluctuation. Of particular interest, we put $\rho_{pq} = 1$ for every $(p, q) \in W$ if we do not know any information on the fluctuations of the pairs in advance. In this case, ϵ is the single network-wide parameter to be adjusted.

It is easy to derive the following inclusion relationships between the two sets.

Proposition 3 *1. If $\epsilon_{pq} = \epsilon\rho_{pq}^{-1/2}$ for every $(p, q) \in W$, then $\Theta_\epsilon \subseteq \mathcal{R}$.*

2. If $\epsilon_{pq} = \epsilon|W|^{-1/2}\rho_{pq}^{-1/2}$ for every $(p, q) \in W$, then $\Theta_\epsilon \supseteq \mathcal{R}$.

Proof: We have to show that $\mathbf{d} \in \Theta_\epsilon$ where $\sqrt{\sum_{(p,q)\in W} \rho_{pq}(d_{pq} - \bar{d}_{pq})^2} \leq \epsilon$ implies $\mathbf{d} \in \mathcal{R}$ if $\epsilon_{pq} = \epsilon\rho_{pq}^{-1/2}$ for every $(p, q) \in W$.

Suppose $\mathbf{d} \in \Theta_\epsilon$ such that $\sqrt{\sum_{(p,q)\in W} \rho_{pq}(d_{pq} - \bar{d}_{pq})^2} \leq \epsilon$. For every $(p, q) \in W$ we can write $\sqrt{\rho_{pq}(d_{pq} - \bar{d}_{pq})^2} \leq \epsilon$. Since $\epsilon_{pq} = \epsilon\rho_{pq}^{-1/2}$, this can be written as

$$\begin{aligned} &\sqrt{\rho_{pq}(d_{pq} - \bar{d}_{pq})^2} \leq \epsilon_{pq}\rho_{pq}^{1/2} \\ \Leftrightarrow &(d_{pq} - \bar{d}_{pq})^2 \leq \epsilon_{pq}^2 \\ \Leftrightarrow &|d_{pq} - \bar{d}_{pq}| \leq \epsilon_{pq} \quad \forall (p, q) \in W, \end{aligned}$$

which implies $\mathbf{d} \in \mathcal{R}$. Therefore, $\Theta_\epsilon \subseteq \mathcal{R}$ if $\epsilon_{pq} = \epsilon\rho_{pq}^{-1/2}$ for every $(p, q) \in W$.

Similarly, suppose $\mathbf{d} \in \mathcal{R}$ such that $|d_{pq} - \bar{d}_{pq}| \leq \epsilon_{pq}$ for every $(p, q) \in W$. We have to show $\mathbf{d} \in \Theta_\epsilon$. Since $\epsilon_{pq} = \epsilon|W|^{-1/2}\rho_{pq}^{-1/2}$ for every $(p, q) \in W$, we have

$$|d_{pq} - \bar{d}_{pq}| \leq \epsilon|W|^{-1/2}\rho_{pq}^{-1/2} \Leftrightarrow \rho_{pq}^{1/2}|d_{pq} - \bar{d}_{pq}| \leq \frac{\epsilon}{|W|^{1/2}},$$

$$\Leftrightarrow \rho_{pq}(d_{pq} - \bar{d}_{pq})^2 \leq \frac{\epsilon^2}{|W|}, \quad \forall (p,q) \in W$$
$$\Rightarrow \sum_{(p,q)\in W} \rho_{pq}(d_{pq} - \bar{d}_{pq})^2 \leq \sum_{(p,q)\in W} \frac{\epsilon^2}{|W|} = \epsilon^2,$$

which shows that $\mathbf{d} \in \Theta_\epsilon$ if $\epsilon_{pq} = \epsilon|W|^{-1/2}\rho_{pq}^{-1/2}$ for every $(p,q) \in W$. Therefore, $\Theta_\epsilon \supseteq \mathcal{R}$.

□

The ellipsoidal uncertainty set is more appropriate than the rectangle uncertainty set in the cases where total amount of fluctuation is given for all pairs in the network, traffic demands fluctuate independently, and we want to capture the error in terms of variance. The use of the ellipsoidal uncertainty set is not appropriate in the case where some source-destination pairs have large errors simultaneously, because we consider that the difference between true and estimated demand is small and it is contained in a set. On the other hand, if fluctuation is given for each source-destination pair in the network, the errors are correlated and there are cases where most of the demands have large deviations simultaneously, the use of the rectangle uncertainty set is more appropriate than the ellipsoidal uncertainty set.

5.2.2 Ellipsoid model

The ellipsoid model is derived by applying the robust optimization to the pipe model with respect to the ellipsoidal uncertainty set Θ_ϵ. Assuming that constraint (5.1d) should be satisfied for every $\mathbf{d} \in \Theta_\epsilon$, we have the following condition:

$$\max_{\mathbf{d}\in\Theta_\epsilon} \max_{(i,j)\in\mathcal{A}} \sum_{(p,q)\in W} d_{pq}x_{ij}^{pq}/c_{ij} \leq r \tag{5.21}$$

for the robust model. As we can exchange the two max operators, this is equivalent with

$$\max_{\mathbf{d}\in\Theta_\epsilon} \sum_{(p,q)\in W} d_{pq}x_{ij}^{pq} \leq c_{ij}r \tag{5.22}$$

for each $(i,j) \in \mathcal{A}$. Next we evaluate the left hand side of (5.22), which yields a second-order cone constraint.

Given $\theta > 0$, define

$$\Omega_\theta = \left\{ \mathbf{v} \in \mathbb{R}^W : \|\mathbf{v}\| \leq \theta \right\}. \tag{5.23}$$

The following lemma plays a crucial role in evaluating the left hand side of (5.22).

Lemma 2 *For arbitrary* $\mathbf{a} \in \mathbb{R}^{|W|}$ *and* $\theta > 0$*, we have*

$$\max_{\mathbf{v} \in \Omega_\theta} \mathbf{a}^T \mathbf{v} = \theta \|\mathbf{a}\|. \tag{5.24}$$

The lemma is easy to show if we notice that the maximum is achieved when $\|\mathbf{a}\|\mathbf{v} = \theta \mathbf{a}$. Details of the proof of lemma 2 is presented in Appendix B.

To apply Lemma 2 when evaluating the left hand side of (5.22), we introduce a variable:

$$v_{pq} = \sqrt{\rho_{pq}}(d_{pq} - \bar{d}_{pq}) \tag{5.25}$$

for each $(p,q) \in W$. We can easily see that

$$\mathbf{d} \in \Theta_\epsilon \Leftrightarrow \mathbf{v} \in \Omega_\epsilon.$$

Therefore, using Lemma 2, we have for every $(i,j) \in \mathcal{A}$

$$\begin{aligned} &\max_{\mathbf{d} \in \Theta_\epsilon} \left(\sum_{(p,q) \in W} d_{pq} x_{ij}^{pq} \right) \\ &= \max_{\mathbf{v} \in \Omega_\epsilon} \left(\sum_{(p,q) \in W} v_{pq} \frac{x_{ij}^{pq}}{\sqrt{\rho_{pq}}} \right) + \sum_{(p,q) \in W} \bar{d}_{pq} x_{ij}^{pq} \\ &= \epsilon \sqrt{\sum_{(p,q) \in W} \frac{\left(x_{ij}^{pq}\right)^2}{\rho_{pq}}} + \sum_{(p,q) \in W} \bar{d}_{pq} x_{ij}^{pq}. \end{aligned} \tag{5.26}$$

Replacing the left hand side of (5.22) by (5.26) yields the equivalent inequality for every $(i,j) \in \mathcal{A}$:

$$\sqrt{\sum_{(p,q) \in W} \frac{\left(x_{ij}^{pq}\right)^2}{\rho_{pq}}} \leq \frac{1}{\epsilon} \left(c_{ij} r - \sum_{(p,q) \in W} \bar{d}_{pq} x_{ij}^{pq} \right). \tag{5.27}$$

The constraint in (5.27) contains a square root and can be cast into the following form using the second-order cone:

$$w_{pq}^{ij} = x_{ij}^{pq} / \sqrt{\rho_{pq}}, \tag{5.28}$$

$$w_0^{ij} = \left(c_{ij} r - \sum_{(p,q) \in W} \bar{d}_{pq} x_{ij}^{pq} \right) / \epsilon, \tag{5.29}$$

$$\begin{pmatrix} w_0^{ij} \\ \mathbf{w}^{ij} \end{pmatrix} \in \mathrm{SOC}(1 + |W|), \tag{5.30}$$

where $\mathbf{w}^{ij} = (w_{pq}^{ij})_{(p,q) \in W}$. The first two constraints are linear, and the last one is a second-order cone constraint. As a result, we obtain an SOCP as a robust optimization model of the pipe model as follows:

$$\min\ r \tag{5.31a}$$

$$\text{s.t.}\ \ \text{Eqs. } (5.1\text{b}), (5.1\text{c}), \tag{5.31b}$$

$$\text{Eqs. } (5.28), (5.29), (5.30), \qquad \forall (i,j) \in \mathcal{A} \tag{5.31c}$$

$$\text{Eqs. } (5.1\text{e}), (5.1\text{f}). \tag{5.31d}$$

We label this model the *ellipsoid model* in this thesis because it uses only the ellipsoidal uncertainty set.

The ellipsoid model, which does not consider the hose uncertainty set, can deal with total errors of traffic demands for all source-destination pairs over the network. On the other hand, the ellipsoid model has a weak point in handling the case where some source-destination pairs have large errors simultaneously, but such a case is considered to be rare.

5.2.3 Hose-ellipsoid model

Next, we implement robust optimization by applying both the hose and ellipsoidal uncertainty sets to the pipe model to minimize the network congestion ratio. The resulting model is called the *hose-ellipsoid* model.

First, we assume that the true value $\mathbf{d}$ is contained in the intersection of Θ_ϵ for some $\epsilon > 0$ and the hose uncertainty set. Given that (5.1d) should be satisfied for every $\mathbf{d} \in \Theta_\epsilon \cap \mathcal{H}$, we obtain the following robust version of (5.1d) as

$$\max_{\mathbf{d}\in\Theta_\epsilon\cap\mathcal{H}} \max_{(i,j)\in\mathcal{A}} \sum_{(p,q)\in W} d_{pq} x_{ij}^{pq} / c_{ij} \leq r. \tag{5.32}$$

Again, exchanging the two max operators yields the robust constraint as

$$\max_{\mathbf{d}\in\Theta_\epsilon\cap\mathcal{H}} \sum_{(p,q)\in W} d_{pq} x_{ij}^{pq} \leq c_{ij} r \tag{5.33}$$

for every $(i,j) \in \mathcal{A}$.

Therefore, for $(i,j) \in \mathcal{A}$, we consider the following problem:

$$\begin{aligned}
S(\mathbf{x}_{ij}) : \max \quad & \sum_{(p,q)\in W} d_{pq} x_{ij}^{pq} && && (5.34a)\\
\text{s.t.} \quad & \sum_{q\in Q} d_{pq} \leq \alpha_p, && \forall p \in Q, && (5.34b)\\
& \sum_{p\in Q} d_{pq} \leq \beta_q, && \forall q \in Q, && (5.34c)\\
& \sqrt{\sum_{(p,q)\in W} \rho_{pq}(d_{pq} - \bar{d}_{pq})^2} \leq \epsilon, && && (5.34d)\\
& d_{pq} \geq 0, && \forall (p,q) \in W. && (5.34e)
\end{aligned}$$

In problem $S(\mathbf{x}_{ij})$, routing $\mathbf{x}_{ij} = (x_{ij}^{pq})_{(p,q)\in W}$, is assumed given and $d_{pq}, \forall (p,q) \in W$ is the decision variable. The problem $S(\mathbf{x}_{ij})$ finds a traffic demand $T = \{d_{pq} : (p,q) \in W\}$ that maximizes the link load on $(i,j) \in \mathcal{A}$ for the given routing $\{x_{ij}^{pq}\}$.

In the following, for an optimization problem (P), we denote its optimal value by val(P). Then, our robust optimization problem for minimizing the congestion ratio with consideration of the error of (5.20) is

$$\begin{aligned}
\min \quad & r && && (5.35a)\\
\text{s.t.} \quad & \text{Eqs. (5.1b), (5.1c)}, && && (5.35b)\\
& \text{val}(S(\mathbf{x}_{ij})) \leq c_{ij} r, && \forall (i,j) \in \mathcal{A}, && (5.35c)\\
& \text{Eqs. (5.1e), (5.1f)}. && && (5.35d)
\end{aligned}$$

At a glance, this problem seems difficult to solve because of the constraint (5.35c). Our solution is to reformulate this constraint as second-order cone constraints with linear constraints, and obtain a robust counterpart of (5.35).

First, we derive the dual of $S(\mathbf{x}_{ij})$, which is

$$\begin{aligned}
\min \ & \sum_{p\in Q} \alpha_p \pi_{ij}(p) + \sum_{q\in Q} \beta_q \lambda_{ij}(q) - \sum_{(p,q)\in W} \sqrt{\rho_{pq}} \bar{d}_{pq} \mu_{ij}^{pq} + \epsilon\theta_{ij} && (5.36a)\\
\text{s.t.} \ & \pi_{ij}(p) + \lambda_{ij}(q) - \sqrt{\rho_{pq}}\mu_{ij}^{pq} - x_{ij}^{pq} \geq 0, \quad \forall (p,q) \in W, && (5.36b)\\
& \begin{pmatrix} \theta_{ij} \\ \boldsymbol{\mu}_{ij} \end{pmatrix} \in \mathrm{SOC}(1+|W|), && (5.36c)\\
& \pi_{ij}(p) \geq 0, \qquad \forall p \in Q, && (5.36d)\\
& \lambda_{ij}(q) \geq 0, \qquad \forall q \in Q. && (5.36e)
\end{aligned}$$

The derivation of the dual of $S(\mathbf{x}_{ij})$ is described in detail in Appendix C.

With this knowledge of the duals in mind, we introduce the hose-ellipsoid model below:

$$\begin{aligned}
\min \ & r && (5.37a)\\
\text{s.t.} \ & \text{Eqs. } (5.1b), (5.1c), && (5.37b)\\
& \sum_{p\in Q} \alpha_p \pi_{ij}(p) + \sum_{q\in Q} \beta_q \lambda_{ij}(q) - \sum_{(p,q)\in W} \sqrt{\rho_{pq}} \bar{d}_{pq} \mu_{ij}^{pq} + \epsilon\theta_{ij} \leq c_{ij} r, \\
& \qquad\qquad \forall (i,j) \in \mathcal{A}, && (5.37c)\\
& \pi_{ij}(p) + \lambda_{ij}(q) - \sqrt{\rho_{pq}}\mu_{ij}^{pq} - x_{ij}^{pq} \geq 0, \quad \forall (i,j) \in \mathcal{A}, \forall (p,q) \in W, && (5.37d)\\
& \begin{pmatrix} \theta_{ij} \\ \boldsymbol{\mu}_{ij} \end{pmatrix} \in \mathrm{SOC}(1+|W|), \ \forall (i,j) \in \mathcal{A}, && (5.37e)\\
& \pi_{ij}(p) \geq 0, \qquad \forall (i,j) \in \mathcal{A}, \forall p \in Q, && (5.37f)\\
& \lambda_{ij}(q) \geq 0, \qquad \forall (i,j) \in \mathcal{A}, \forall q \in Q, && (5.37g)\\
& \text{Eqs. } (5.1e), (5.1f). && (5.37h)
\end{aligned}$$

In the hose-ellipsoid model, variables μ_{ij}^{pq} and θ_{ij} are newly introduced with parameters $\bar{d}_{pq}$ and ϵ; $\pi_{ij}(p)$ and $\lambda_{ij}(q)$ are used in the hose model. Note that the hose-ellipsoid model represents a regular SOCP problem; it contains $|\mathcal{A}|$ second-order cone constraints, and all other constraints are linear. Therefore, (5.37) is tractable, and this is a robust counterpart of (5.35).

Note that both our models, the ellipsoid and the hose-ellipsoid need the *estimated value* denoted by $\bar{d}_{pq}$ for every $(p, q) \in W$ explicitly, unlike the hose-rectangle or the hose model. However, the hose-rectangle model is considered to use $\bar{d}_{pq}$ implicitly through the infimum and supremum bounds, as we have explained above.

In order to narrow the range of traffic conditions specified by the hose model, the hose-ellipsoid model considers the ellipsoidal uncertainty set in addition to the hose uncertainty set. The ellipsoidal uncertainty set allows us to consider the total errors of traffic demands over the network, whereas in the hose-rectangle model, the rectangle uncertainty set allows the same errors for each source-destination pair in the network which degrade the routing performance compared to the hose-ellipsoid model.

Now we analyze the property of (5.37). To do this, we formally make the following assumption.

Assumption 1 *The estimated value* $\bar{d}_{pq}$, $\forall(p, q) \in W$ *is a feasible solution of* $S(\mathbf{x}_{ij})$.

Since the feasible region of $S(\mathbf{x}_{ij})$ does not depend on $\mathbf{x}_{ij}$, the assumption looks natural. In fact, the assumption is equivalent to

$$\sum_{q \in Q} \bar{d}_{pq} \leq \alpha_p, \quad \forall p \in Q, \tag{5.38a}$$

$$\sum_{p \in Q} \bar{d}_{pq} \leq \beta_q, \quad \forall q \in Q, \tag{5.38b}$$

$$\bar{d}_{pq} \geq 0, \qquad \forall (p, q) \in W. \tag{5.38c}$$

Lemma 3 *If Assumption 1 holds, then both* $S(\mathbf{x}_{ij})$ *and its dual* (5.36) *have optimal solutions, and their optimal values coincide.*

We provide a proof of Lemma 3 in the Appendix D together with the necessary theorems from the theory of conic linear programming duality.

Theorem 5 *If Assumption 1 holds, then* $\mathrm{val}(5.35) = \mathrm{val}(5.37)$.

Proof: What we need to prove is that, given $\mathbf{x}_{ij}, (i, j) \in \mathcal{A}$, $(S(\mathbf{x}_{ij}))$ equals the left-hand side of (5.37c) if $\pi_{ij}(p)$, $\lambda_{ij}(q)$, θ_{ij}, and μ_{ij}^{pq} satisfy (5.37d) to (5.37g). This is true due to Lemma 3.

□

5.2.4 Characteristic of the considered models

To compare the performances of our proposed models, we consider different models in this thesis related to our work. The correctness, importance of these models, and used uncertainty sets are summarized in Table 5.1.

5.3 Numerical experiments

5.3.1 Experiments settings

In this section, we numerically compare five models: the pipe, hose, hose-rectangle, ellipsoid, and hose-ellipsoid models.

The networks used in the experiments, shown in Figure 5.1, are typical backbone networks used in [30] (Networks 1-3) and Japan Photonic Network 12 (JPN 12) [36]. In all experiments, we generate 100 instances for each network structure. We randomly generated traffic demands $\bar{d}_{pq}$ with a uniform distribution in the range of $(0, 100)$; the link capacities lie in the range of $(2000, 3000)$ for all networks. Bounds used in the hose constraints are set to $\alpha_p = \sum_{q \in Q} \bar{d}_{pq}$ and $\beta_q = \sum_{p \in Q} \bar{d}_{pq}$. In the models that use the ellipsoidal uncertainty set, ρ_{pq} is always set to 1 for all $(p, q) \in W$.

In the hose-rectangle, ellipsoid, and hose-ellipsoid models, we use the same parameter, $\epsilon > 0$, to define the size of the uncertainty sets. For the ellipsoid and hose-ellipsoid models, we use ϵ to define Θ_ϵ. These models are called Ellipsoid and HE, respectively, in this section. For the hose-rectangle model, we use ϵ in two ways. The first model, called HR-c, uses ϵ to define $\delta_{pq} = \bar{d}_{pq} - \epsilon$ and $\gamma_{pq} = \bar{d}_{pq} + \epsilon$, and the other, called HR-i, uses $\delta_{pq} = \bar{d}_{pq} - \epsilon/\sqrt{|W|}$ and $\gamma_{pq} = \bar{d}_{pq} + \epsilon/\sqrt{|W|}$. Here, 'c' stands for 'circumscribe', since the rectangle used by HR-c circumscribes the ellipsoid used by Ellipsoid and HE. Similarly, 'i' stands for 'inscribe'.

Table 5.1: Characteristic of models.

Model	**Uncertainty set**	**Characteristic**
Pipe	None	The traffic-demand matrix is known exactly. Provides the best routing performance thanks to the exact traffic matrix.
Hose	$\mathcal{H}$	More flexible model than others, valid in high variable traffic conditions and large networks, and lower routing performance than Pipe, Ellipsoid, Hose-Ellipsoid, and Hose-Rectangle.
Ellipsoid	Θ_ϵ	Deals with total errors of demands over the network. The optimization problem is of the form of SOCP.
Hose-Rectangle	$\mathcal{R} \cap \mathcal{H}$	Bounds the range of errors for each pair by upper and lower bounds. Trends of traffic demands and fluctuations can be predicted by network operators. Narrows the range of traffic conditions specified by the hose model. Provides better routing performance than the hose model.
Hose-Ellipsoid	$\Theta_\epsilon \cap \mathcal{H}$	Deals with total errors of demands over the network. Narrows the range of traffic conditions specified by the hose model. The optimization problem is of the form of SOCP. Improves the routing performance of Hose-Rectangle using the ellipsoidal uncertainty set.

Due to Proposition 3, HR-c has a larger uncertainty set than HE, whereas HR-i has a smaller uncertainty set than HE. Since the uncertainty set of HR-c circumscribes the uncertainty set of HE, the routing performance of HR-c is lower than or equal to that of HE. On the other hand, since the uncertainty set used in HR-i inscribes the uncertainty set of HE by dividing the errors by $\sqrt{|W|}$ as described above, the routing performance of HR-i is higher than or equal to that of HE. Therefore, theoretically, the routing performance of HE is between those of HR-c and HR-i.

Note that the hose model (Hose) and the pipe model (Pipe) are independent of ϵ. Since Pipe has a no uncertainty set, the smallest among the six models, the optimal value of Pipe is the minimum of the six models. Since Hose has the largest uncertainty set among all the models except for Ellipsoid, the optimal value of Hose is the largest among the five models. In comparing the results, we always take the six optimal values by normalizing the optimal value of Hose to 1.

The experiments program is written in Python language and the optimization problems are solved by Gurobi, version 7.0.1 [38]. We use a Windows based computer with Intel(R) Core(TM) i7-4790 CPU @ 3.60 GHz and 16 GB memory.

Optimal solutions are obtained in all considered 100 problems for each model and for each network. We confirmed optimality of each solution by checking its primal feasibility, dual feasibility, and zero duality gap. Theoretically, these three conditions ensure that the obtained solution is optimal for the corresponding optimization problem. For more details about the optimality condition of conic programming, please consult the textbooks such as [35], [51], and [54].

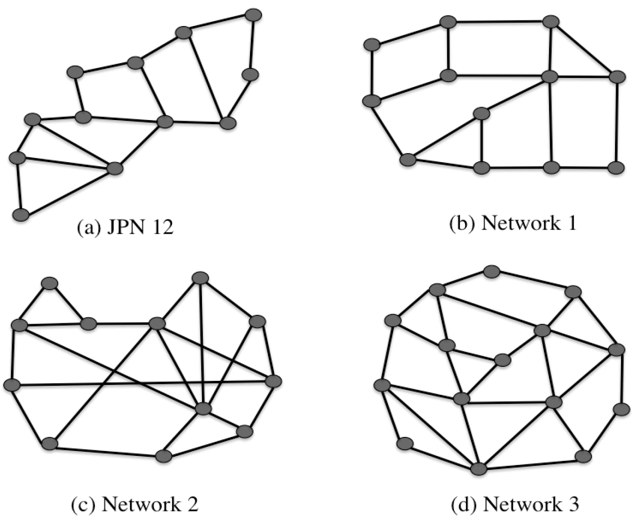

Figure 5.1: Sample networks.

5.3.2 Experiment results for fixed ϵ

The comparisons of normalized average congestion ratios for $\epsilon = 30$ are found in Figure 5.2.

Table 5.2: Network parameters considered in the experiments.

Network	No. of nodes considered	No. of links considered	$\sqrt{\lvert W\rvert}$
JPN 12	12	17	11.49
Network 1	12	18	11.49
Network 2	12	22	11.49
Network 3	15	27	14.49

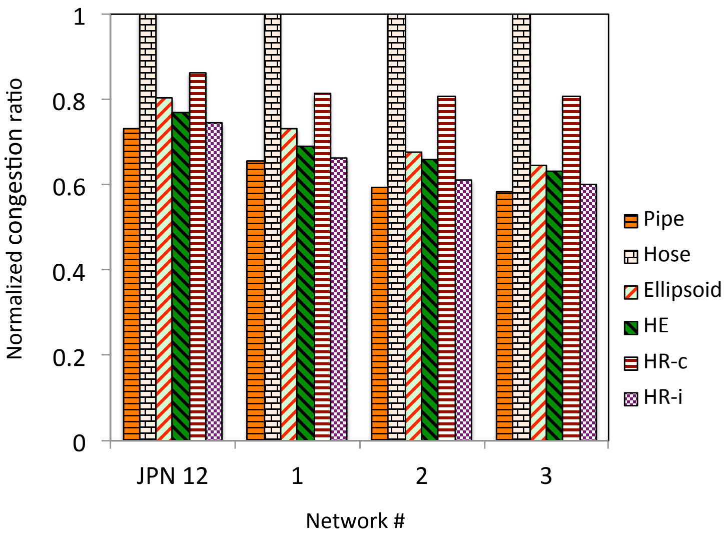

Figure 5.2: Average normalized congestion ratio for 100 randomly generated problems for $\epsilon = 30$.

First, as was theoretically expected, the normalized average congestion ratios obtained by HE, HR-c, and HR-i always lie between those obtained by Hose and Pipe in all networks considered. Furthermore, the optimal value of Ellipsoid is always less than that of HR-c and Hose. The optimal value of HE is always between those of HR-c and HR-i; this was predicted by Proposition 3.

We notice that the difference between the hose and pipe models increases with network size. Ellipsoid and HE have the same tendency as the pipe model, whereas HR-c does not change much as network size changes. In this sense, the

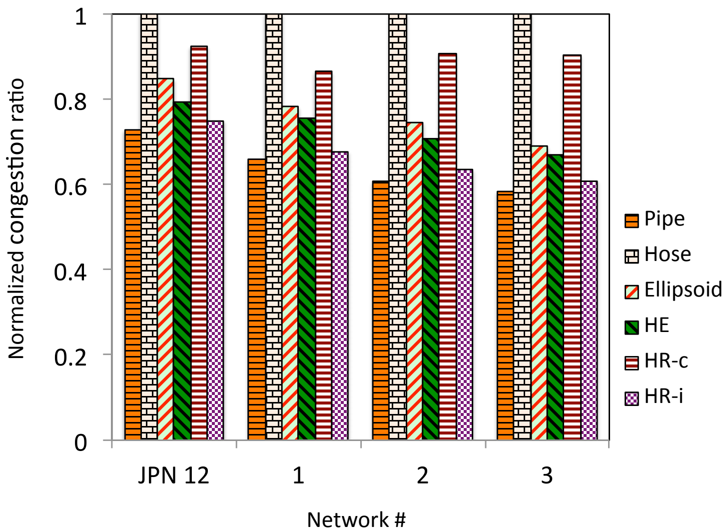

Figure 5.3: Average normalized congestion ratio for 100 randomly generated problems for $\epsilon = 50$.

behavior of HR-c is similar to that of the hose model.

HR-i actually follows the pipe model closely. We explain this in the following. Since the volume of a $|W|$-dimensional cube whose side length is $2\epsilon/\sqrt{|W|}$ is $2^{|W|}\epsilon^{|W|}|W|^{-|W|/2}$, and that of a $|W|$-dimensional ball whose radius is ϵ is

$$\frac{\pi^{|W|/2}\epsilon^{|W|}}{\Gamma(|W|/2+1)}, \tag{5.39}$$

where Γ is the gamma function, their ratio is

$$\frac{2^{|W|}\Gamma(|W|/2+1)}{(|W|\pi)^{|W|/2}}. \tag{5.40}$$

When $|W| = 132$ as true for JPN 12, Network 1, and Network 2, a rough evaluation finds that this value is less than 2^{-66}. This analysis suggests that the uncertainty set of HR-i is so small that it loses robustness compared to HE or Ellipsoid. The same behaviors describe above for $\epsilon = 30$ appear for $\epsilon = 50$ and $\epsilon = 10$ which are depicted in Figure 5.3 and Figure 5.4, respectively.

Next, we compare the models in terms of computation time. The minimum, average, maximum and variance of the computation times of 100 random prob-

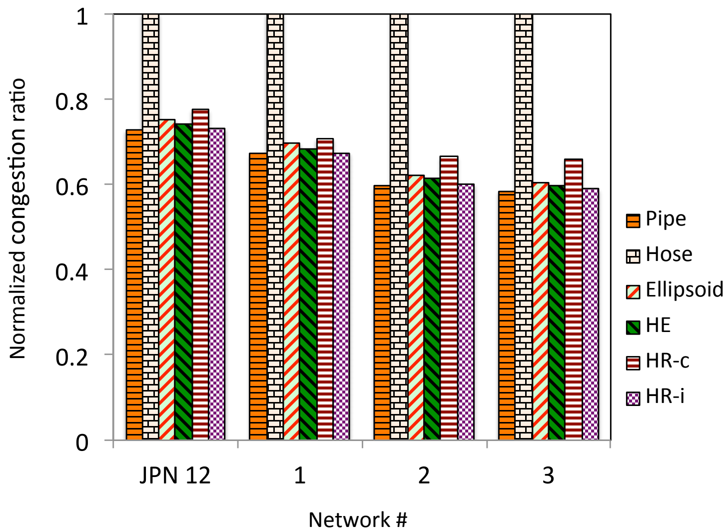

Figure 5.4: Average normalized congestion ratio for 100 randomly generated problems for $\epsilon = 10$.

lems for the six models are listed in Table 5.3 and comparisons of average computation times between the considered models are presented in Figure 5.5.

Pipe is far faster than the other five models in every network, and Hose is the second fastest. This seems natural if we recall that both Pipe and Hose are LP problems, and Pipe has the least number of decision variables followed by Hose in second.

Computation times of HE, HR-c, HR-i are all comparable; Among the other four models, Ellipsoid always has the longest computation time. The ratio of Ellipsoid with respect to the other four models depends on the network.

We note that, although the average of HE is smaller than those of Ellipsoid, HR-c and HR-i, the variance of HE for Network 3 together with the maximum computation time is the largest. We do not know the exact reason for this. What we do know is that the SOCP solver is relatively young, and probably not as stable as the LP solver, which has seen continuous development over 40 years.

Note that, from the complexity point of view, the HE model is the most complex, because it has $|\mathcal{A}|$ second-order cone constraints which are equivalent to quadratic constraints. In the considered models, the HE model also has the

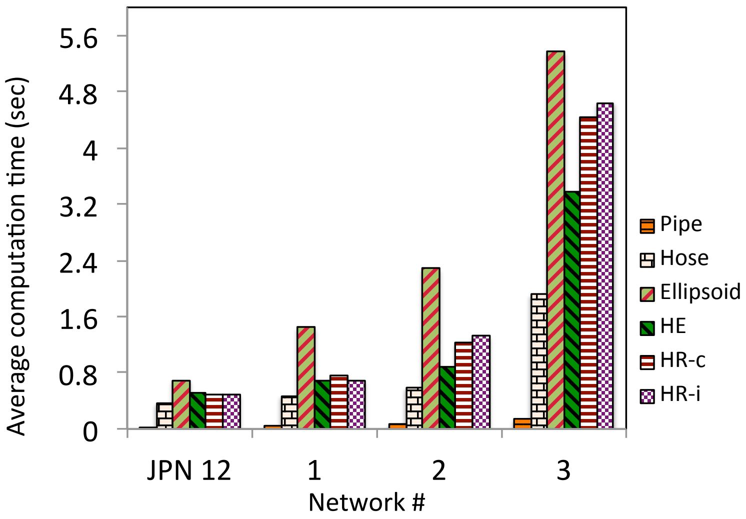

Figure 5.5: Comparisons of average computation time of 100 randomly generated problems for $\epsilon = 30$.

largest number of decision variables and constraints; the HR-c and HR-i model are the second most complex in terms of number of variables and constraints. However, experimental results for computation time illustrates that the HE model has shorter computation time on average than the HR-c, HR-i and Ellipsoid models in the four sample networks. We do not have the correct answer of this question. Generally, LP is easier than SOCP, but that does not mean SOCP always takes longer computation time than L. The computation time depends on various factors.

5.3.3 ϵ dependency of the models

Figure 5.6 shows comparisons of the dependency of the network congestion ratios for 100 randomly generated problems as given by the Pipe, Hose, HE, HR-c and HR-i models for Network 3. Note that network congestion ratios for the Pipe and Hose models do not depend on ϵ.

We observe that the normalized congestion ratio of HR-c rapidly increases as ϵ increases; with $\epsilon = 50$, the ratio is close to 90 % of Hose's. We presume that

Table 5.3: Comparisons of computation time of 100 randomly generated problems for $\epsilon = 30$.

Network	Model	Minimum (sec)	Average (sec)	Maximum (sec)	Variance (sec^2)
JPN 12	**Pipe**	0.0000	0.0208	0.0400	0.0000
	Hose	0.2400	0.3713	0.4900	0.0028
	Ellipsoid	0.4100	0.6975	2.3600	0.0807
	HE	0.4100	0.5013	0.8200	0.0037
	HR-c	0.4300	0.4861	0.7100	0.0016
	HR-i	0.4100	0.4968	0.5800	0.0013
Network 1	**Pipe**	0.0200	0.0322	0.0500	0.0001
	Hose	0.4000	0.4752	0.7000	0.0022
	Ellipsoid	0.5900	1.4506	4.7800	0.7697
	HE	0.4900	0.6934	0.9600	0.0109
	HR-c	0.5100	0.7528	1.0400	0.0124
	HR-i	0.4500	0.6905	1.2700	0.0311
Network 2	**Pipe**	0.0200	0.0703	0.1200	0.0003
	Hose	0.5100	0.5911	0.7900	0.0029
	Ellipsoid	1.0300	2.2801	5.5300	1.6287
	HE	0.5900	0.8923	1.6400	0.0242
	HR-c	0.8500	1.2236	1.8500	0.0341
	HR-i	0.6700	1.3197	1.9900	0.0927
Network 3	**Pipe**	0.1000	0.1363	0.1700	0.0001
	Hose	1.6200	1.9146	2.4600	0.0318
	Ellipsoid	2.5600	5.3673	17.3600	4.9361
	HE	2.3100	3.3776	20.9000	3.6909
	HR-c	2.9500	4.4434	8.3900	0.7664
	HR-i	2.6400	4.6432	7.7500	1.0238

the uncertainty set of HR-c is too large and cannot bound the congestion ratio efficiently if ϵ is large.

The normalized congestion ratio of HR-i is, even for $\epsilon = 50$, close to that of Pipe. This is natural since the uncertainty set of HR-i is very small as we saw in the previous subsection.

On the other hand, Ellipsoid and HE always lie between HR-c and HR-i, and exhibit moderate increases, suggesting a different tendency from HR-c and HR-i.

There are also some research in the history that use robust optimization in congestion ratio problem. The objective of this research is not to show the excellence of our proposed models to the previous studies. What we want to insist is that, our proposed models can be used as an alternative or a candidate in the respective field.

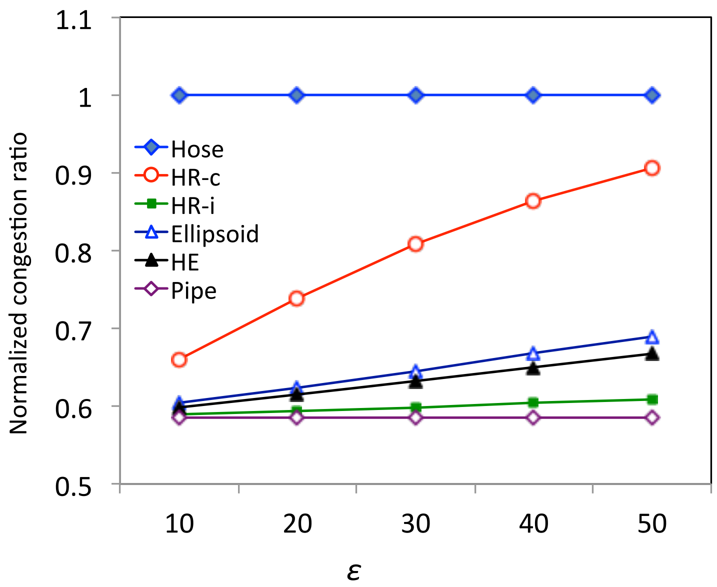

Figure 5.6: ϵ dependency of congestion ratios for 100 random problems. Congestion ratios are normalized by the hose model (Network 3).

5.4 Summary

In this chapter, we applied robust optimization to the problem of minimizing the network congestion ratio. The situation we considered was such that fluctuations or errors in traffic demands exist, but the total amount of them is limited. The uncertainty sets contained in ellipsoids are able to cope with this situation, and then we need to apply robust optimization technique to obtain robust counterparts that can be numerically computable by the standard solvers.

Specifically, we derived two robust optimization problems from the pipe model. The first one considers the ellipsoidal uncertainty set, and the second the intersection of the hose and ellipsoidal uncertainty sets. We obtained robust counterparts of these robust optimization problems, with proofs of their equivalence. The resulting robust counterparts of our models are both SOCP problems.

Both models exempt the operators from knowing the exact traffic demands by allowing them to specify merely the total error in traffic demands.

We compared our proposed models with three existing models: the pipe, hose, and hose-rectangle models. Our numerical experiments clearly showed that the SOCP problems obtained by our proposed models are tractable by modern solvers. Although the computation times are much larger than those of the pipe model, they are comparable to those of the other existing models. Even for large networks, the proposed robust optimization models for congestion ratio can be solved within reasonable time compared to some existing models.

Numerical comparisons of the dependence of ϵ showed that the proposed models share a tendency in terms of optimal values which differs from that of the hose-rectangle model. The congestion ratios achieved by our proposed models increase very slightly for increasing values of ϵ while for the hose-rectangle model, it increase rapidly and close to the hose model. We believe that the proposed models provide a new approach to the analysis of networks by existing methods.

Chapter 6

Minimizing network power consumption based on robust optimization

In this chapter, a green and robust optimization model is proposed to minimize the network power consumption allowing fluctuation in traffic demands. There are several researches on minimizing network power consumption that have been presented in the history. Most of them are presented under the assumption that the traffic-demand matrix, the set of traffic demands, is known, there are some bounds on the total outgoing/incoming traffic from/to node, or, in addition to these bounds, traffic demands between each source and destination pair are bounded by upper and lower bounds [56], [57], [58], [59], [60]. There are also some studies on estimating the traffic-demand matrix, which is easy for network operators to avoid frequent dynamic route changes [61], [62], [63]. In this chapter, we apply the idea of robust optimization and proposed a model in the form of mixed-integer second-order cone programming (MISOCP) whose objective is to reduce the total energy by allowing some links to be put into the sleep mode.

6.1 Minimizing network power consumption

6.1.1 Power model

Considering an appropriate power model is an important task to minimize the network power consumption efficiently. In our work we consider the power model called *energy aware model* proposed by [56]. In this model, we assume that the link power consumption is an affine function of its utilization. Specifically, the power consumption (PC) of a link is expressed as

$$PC = E_M x + E_0, \tag{6.1}$$

if the link is on, and 0 otherwise. Here, E_0 is the energy consumption by keeping the link on, and x indicates the total portion of the traffic passing through the link with respect to the capacity of the link. As a result, we have $0 \leq x \leq 1$. We assume that E_0 and E_M depend on link.

If $E_M = 0$, the model is called *energy agnostic*. In this case, the power consumption does not depend on the utilization. On the other hand, if $E_0 = 0$, then the model is called *fully proportional*. Although *adaptive link rate* technologies and *dynamic voltage scaling* [65] were introduced as proportional computing techniques [64], it is still difficult to develop such devices by the current technology.

In this chapter, we use the same network model as we consider in Chapter 5. The useful notation used in this chapter is summarized in Table 6.1.

6.1.2 Formulation of green pipe model

To minimize the network power consumption, Bianzino et al. [56] introduced a mixed-integer formulation known as the *green pipe* model assuming that the traffic-demand matrix $T = \{d_{pq} : (p, q) \in W\}$ is exactly known. The green pipe model is formulated as follows:

$$\min \frac{1}{2} \sum_{(i,j)\in A} \left(\frac{u_{ij} + u_{ji}}{c_{ij}} E_{fij} + b_{ij} E_{0ij} \right) \tag{6.2a}$$

$$\text{s.t.} \sum_{j:(i,j)\in A} x_{ij}^{pq} - \sum_{j:(j,i)\in A} x_{ji}^{pq} = 1, \quad \forall (p, q) \in W, i = p, \tag{6.2b}$$

Table 6.1: Summary of notations.

Parameters	**Description**
$G(V, A)$	Directed graph G with $\|V\|$ nodes and $\|A\|$ links
Q	Set of edge nodes, $Q \subseteq V$
W	Set of edge node pairs of $p \in Q$ and $q \in Q$, $p \neq q$
c_{ij}	Capacity of link $(i, j) \in A$
$\bar{d}_{pq}$	Estimated traffic demand from node p to q
$T = \{d_{pq}\}$	Traffic-demand matrix (set of traffic demands)
E_{fij}, E_{0ij}	Power model parameters
ϵ	Total error in the traffic demands
M	Large positive number, at least twice the maximum capacity in the network
α_p	Total outgoing traffic from node $p \in Q$
β_q	Total incoming traffic at node $q \in Q$
a_{ij}^{pq}	Portion of traffic from node $p \in Q$ to node $q \in Q$ through $(i, j) \in A$ (initial routing)
y_{ij}	Maximum amount of traffic measured on $(i, j) \in A$
Variables	**Description**
d_{pq}	Traffic demand from node p to q
u_{ij}	Traffic flow on $(i, j) \in A$
b_{ij}	Binary variable used to designate the state of (i, j)
x_{ij}^{pq}	Portion of traffic from $p \in Q$ to $q \in Q \setminus \{p\}$ routed through $(i, j) \in A$
$\pi_{ij}(p), \lambda_{ij}(q),$ $\theta_{ij}, \xi_{ij}(s, t), \mu_{ij}^{pq}$	Variables introduced by dual transformation

$$\sum_{j:(i,j)\in A} x_{ij}^{pq} - \sum_{j:(j,i)\in A} x_{ji}^{pq} = 0, \quad \forall (p, q) \in W, \forall i \in V \setminus \{p, q\}, \tag{6.2c}$$

$$\sum_{(p,q)\in W} d_{pq} x_{ij}^{pq} = u_{ij}, \qquad \forall (i, j) \in A, \tag{6.2d}$$

$$u_{ij} \leq c_{ij}, \qquad \forall (i, j) \in A, \tag{6.2e}$$

$$M b_{ij} \geq u_{ij} + u_{ji}, \qquad \forall (i, j) \in A, \tag{6.2f}$$

$$x_{ij}^{pq} \geq 0, \qquad \forall (p, q) \in W, \forall (i, j) \in A, \tag{6.2g}$$

$$b_{ij} \in \{0, 1\}, \qquad \forall (i, j) \in A. \tag{6.2h}$$

Here the constraints (6.2b) and (6.2c) are flow conservation constraints. The constraint (6.2b) represents that the total of portions of traffic flow outgoing from node $i (= p)$ is equal to 1 and (6.2c) states that the total portion of traffic incoming to node i must be the same as that of outgoing from node i if node i is neither a source nor destination for the traffic flow. Constraint (6.2d) defines the total traffic flow over link $(i, j) \in A$ and constraint (6.2e) is the capacity constraint. Constraint (6.2f) is used to force the link to be on when there is a traffic in one direction of the link. Here, M is chosen as a big positive number twice greater than the highest capacity in the network. The objective function (6.2a) minimizes the network power consumption by deactivating some unnecessary links, i.e., the flow through each link is minimized so that the link without any flow can be put into the sleep mode. Here, we assume that the values of c_{ij} and c_{ji} are same. In the objective, summation is divided by two to avoid counting the power consumption twice.

6.1.3 Green hose model formulation

It is often a difficult task for network operators to measure and predict the actual traffic data T, but operators could easily specify the traffic as just the total outgoing/incoming traffic from/to node p to q. The total outgoing traffic from node p is represented as

$$\sum_q d_{pq} \leq \alpha_p, \qquad \forall p \in Q, \tag{6.3}$$

where α_p is the maximum amount of traffic that node p can send into the network. The total incoming traffic to node q is represented as

$$\sum_p d_{pq} \leq \beta_q, \qquad \forall p \in Q, \tag{6.4}$$

where β_q is the maximum amount of traffic that node q can receive from the network. The traffic-demand model having such upper bounds is called the hose model in [13], [14], [16], [17].

With regard to the robust optimization point of view, their work can be regarded as follows. First, consider the uncertainty set

$$\mathcal{H} = \left\{ \mathbf{d} \in \mathbb{R}^W : \begin{array}{c} \sum_q d_{pq} \leq \alpha_p, \quad \forall p \in Q, \\ \sum_p d_{pq} \leq \beta_q, \quad \forall q \in Q, \\ d_{pq} \geq 0, \quad \forall (p,q) \in W \end{array} \right\}, \tag{6.5}$$

which we call the *hose uncertainty set* in the following.

In the green hose model, we first suppose that the routing x_{ij}^{pq} for $(i,j) \in A$ and $(p,q) \in W$ is fixed, and consider the worst case flow on each link when the true demand $\mathbf{d}$ is contained in $\mathcal{H}$:

$$u_{ij} = \max_{\mathbf{d} \in \mathcal{H}} \sum_{(p,q) \in W} d_{pq} x_{ij}^{pq}, \quad \forall (i,j) \in A.$$

Then we force u_{ij} to satisfy (6.2e) and (6.2f). Considering x_{ij}^{pq} as a variable again, we obtain the following optimization problem:

$$(H) : \min \frac{1}{2} \sum_{(i,j) \in A} \left(\frac{u_{ij} + u_{ji}}{c_{ij}} E_{fij} + b_{ij} E_{0ij} \right) \tag{6.6a}$$

$$\text{s.t. Eqs. (6.2b), (6.2c),} \tag{6.6b}$$

$$\max_{\mathbf{d} \in \mathcal{H}} \sum_{(p,q) \in W} d_{pq} x_{ij}^{pq} = u_{ij}, \quad \forall (i,j) \in A, \tag{6.6c}$$

$$\text{Eqs. (6.2e), (6.2f), (6.2g), (6.2h).} \tag{6.6d}$$

The robust counterpart of (H) can be obtained as follows by using the same technique as we did for the formulation of the hose model in Chapter 5:

$$(H) : \min \frac{1}{2} \sum_{(i,j) \in A} \left(\frac{u_{ij} + u_{ji}}{c_{ij}} E_{fij} + b_{ij} E_{0ij} \right) \tag{6.7a}$$

$$\text{s.t. Eqs. (6.2b), (6.2c),} \tag{6.7b}$$

$$\sum_{p \in Q} \alpha_p \pi_{ij}(p) + \sum_{q \in Q} \beta_q \lambda_{ij}(q) = u_{ij}, \quad \forall (i,j) \in A, \tag{6.7c}$$

$$x_{ij}^{pq} \leq \pi_{ij}(p) + \lambda_{ij}(q), \quad \forall (p,q) \in W, \forall (i,j) \in A, \tag{6.7d}$$

$$\pi_{ij}(p), \lambda_{ij}(q) \geq 0, \quad \forall (p,q) \in W, \forall (i,j) \in A, \tag{6.7e}$$

$$\text{Eqs. (6.2e), (6.2f), (6.2g), (6.2h).} \tag{6.7f}$$

6.1.4 Green HLT model formulation

In order to minimize the network power consumption, Ouédraogo and Oki [30] proposed the green HLT model based on an extended version of the hose model [31]. The traffic bounds in the green HLT model are determined as below:

$$\text{Eqs. (6.3), (6.4),} \tag{6.8a}$$

$$\sum_{(p,q)\in W} a_{ij}^{pq} d_{pq} \leq y_{ij}, \quad \forall (i,j) \in A, \tag{6.8b}$$

where y_{ij} is the maximum amount of traffic measured on link (i,j). The parameter a_{ij}^{pq} is the initial portion of traffic from node p to q using link (i,j). The value of a_{ij}^{pq} is known in advance and it is derived from an initial routing determined by the network operator, e.g., shortest path routing. The value of variable x_{ij}^{pq} that represents the proper routing is different from the value of a_{ij}^{pq} after solving the optimization problem.

The power saving by the green HLT model highly varies on the maximum amount of traffic measured on link (i,j) which is expressed by y_{ij}. Lower values of y_{ij} due to measurement errors during low-traffic durations can lead to infeasibility of the optimization problem. On the other hand, higher values of y_{ij} due to measurement errors cause to degrade the performance of the green HLT model. For the robust optimization point of view, when the traffic measurements deviate from the initial green HLT model bounds, y_{ij} varies from its original value to $y_{ij}(1+\delta)$, where δ represents the uncertainty to y_{ij} [30]. Using this parameter δ to y_{ij}, they allowed some fluctuation in the true demand d_{pq} up to δ, which can be represented as an increase in the initial value of y_{ij} due to the error set by the operator. Based on this knowledge, the green HLT model uses the following uncertainty set,

$$\mathcal{G} = \mathcal{H} \cap \mathcal{L} = \left\{ \mathbf{d} \in \mathbb{R}^W : \begin{array}{c} \sum_q d_{pq} \leq \alpha_p, \quad \forall p \in Q, \\ \sum_p d_{pq} \leq \beta_q, \quad \forall q \in Q, \\ \sum_{(p,q)\in W} a_{ij}^{pq} d_{pq} \leq y_{ij}(1+\delta), \forall (i,j) \in A, \\ d_{pq} \geq 0, \quad \forall (p,q) \in W \end{array} \right\},$$

where

$$\mathcal{L} = \left\{ \mathbf{d} \in \mathbb{R}^W : \sum_{(p,q)\in W} a_{ij}^{pq} d_{pq} \leq y_{ij}(1+\delta), \forall (i,j) \in A \right\}.$$

We call the set $\mathcal{G}$ the *green-hose uncertainty set* in this thesis.

In [30], the performance of green HLT is evaluated for different values of δ, which can be represented as errors or margin in the estimated value y_{ij} for every $(i,j) \in A$ to allow fluctuation in the real demand d_{pq} for every $(p,q) \in W$. In the green HLT model, the authors assume that the true demand $\mathbf{d}$ is contained in $\mathcal{G}$. Now the same argument with the green hose model leads to the following optimization problem:

$$\min \frac{1}{2} \sum_{(i,j)\in A} \left(\frac{u_{ij}+u_{ji}}{c_{ij}} E_{fij} + b_{ij} E_{0ij} \right) \tag{6.9a}$$

$$\text{s.t. Eqs. (6.2b), (6.2c)} \tag{6.9b}$$

$$\max_{\mathbf{d}\in\mathcal{G}} \sum_{(p,q)\in W} d_{pq} x_{ij}^{pq} = u_{ij}, \quad \forall (i,j) \in A, \tag{6.9c}$$

$$\text{Eqs. (6.2e), (6.2f), (6.2g), (6.2h).} \tag{6.9d}$$

In the above problem, the left hand side of constraint (6.9c) is the optimal value of the following problem:

$$V(\mathbf{x}_{ij}, \mathbf{a}_{ij}, y_{ij}) : \max \sum_{(p,q)\in W} d_{pq} x_{ij}^{pq} \tag{6.10a}$$

$$\text{s.t.} \sum_{q\in Q} d_{pq} \le \alpha_p, \quad \forall p \in Q, \tag{6.10b}$$

$$\sum_{p\in Q} d_{pq} \le \beta_q, \quad \forall q \in Q, \tag{6.10c}$$

$$\sum_{(p,q)\in W} a_{ij}^{pq} d_{pq} \le y_{ij}(1+\delta), \tag{6.10d}$$

$$d_{pq} \ge 0, \qquad \forall (p,q) \in W. \tag{6.10e}$$

The problem $V(\mathbf{x}_{ij}, \mathbf{a}_{ij}, y_{ij})$ is the worst-case traffic scenario of (6.9c) where the routing x_{ij}^{pq} is assumed given and d_{pq} is considered to be a decision variable. Applying the duality theory to the above problem, the resulting green HLT model is as follows:

$$\min \frac{1}{2} \sum_{(i,j)\in A} \left(\frac{u_{ij}+u_{ji}}{c_{ij}} E_{fij} + b_{ij} E_{0ij} \right) \tag{6.11a}$$

$$\text{s.t. Eqs. (6.2b), (6.2c),} \tag{6.11b}$$

$$\sum_{p \in Q} \alpha_p \pi_{ij}(p) + \sum_{q \in Q} \beta_q \lambda_{ij}(q) + \sum_{(s,t) \in A} \xi_{ij}^{st} y_{st}(1+\delta) = u_{ij}, \quad \forall (i,j) \in A, \tag{6.11c}$$

$$x_{ij}^{pq} \leq \pi_{ij}(p) + \lambda_{ij}(q) + \sum_{(s,t) \in A} a_{st}^{pq} \xi_{ij}^{st}, \quad \forall (p,q) \in W, \forall (i,j) \in A, \tag{6.11d}$$

$$\pi_{ij}(p), \lambda_{ij}(q) \geq 0, \ \forall (p,q) \in W, \forall (i,j) \in A, \tag{6.11e}$$

$$\xi_{ij}^{st} \geq 0, \qquad \forall (i,j), (s,t) \in A, \tag{6.11f}$$

$$\text{Eqs. } (6.2e), (6.2f), (6.2g), (6.2h). \tag{6.11g}$$

6.2 Green hose-ellipsoid model formulation

To minimize the network power consumption allowing fluctuation in the demands between nodes, we apply the robust optimization to the green hose model and formulate the green hose-ellipsoid model (green HE) in the form of mixed integer second-order cone programming problem (MISOCP). In this case, we consider the same uncertainty set $\Theta_\epsilon \cap \mathcal{H}$ as we used to formulate the hose-ellipsoid model in Section 5.2.3 of Chapter 5.

First, we assume that the true demand $\mathbf{d}$ is contained in $\Theta_\epsilon \cap \mathcal{H}$, for some $\epsilon > 0$. By a similar argument as that for the green hose and HLT models, we set u_{ij} as

$$\max_{\mathbf{d} \in \Theta_\epsilon \cap \mathcal{H}} \sum_{(p,q) \in W} d_{pq} x_{ij}^{pq} = u_{ij}, \quad \forall (i,j) \in A. \tag{6.12}$$

For the worst-case traffic scenario of our proposed model, for every $(i,j) \in A$, we consider the same subproblem $S(\mathbf{x}_{ij})$ (used in Section 5.2.3 of Chapter 5) in this chapter to formulate the robust optimization model.

The proposed robust optimization model for minimizing network power consumption with consideration of the error (5.20) and uncertainty set $\mathcal{H}$ is

$$\min \frac{1}{2} \sum_{(i,j) \in A} \left(\frac{u_{ij} + u_{ji}}{c_{ij}} E_{fij} + b_{ij} E_{0ij} \right) \tag{6.13a}$$

$$\text{s.t. Eqs. } (6.2b), (6.2c), \tag{6.13b}$$

$$\text{val}(S(\mathbf{x}_{ij})) = u_{ij}, \quad \forall (i,j) \in A, \tag{6.13c}$$

$$\text{Eqs. } (6.2e), (6.2f), (6.2g), (6.2h). \tag{6.13d}$$

Since the left hand side of constraint (6.13c) is the optimal value of a second-order cone programming problem (SOCP), it can be evaluated by its dual optimal value. We derived the dual of $S(\mathbf{x}_{ij})$ in Chapter 5.

We present our proposed green HE model using the knowledge of dual of the problem $S(\mathbf{x}_{ij})$ as

$$\min \frac{1}{2} \sum_{(i,j)\in A} \left(\frac{u_{ij} + u_{ji}}{c_{ij}} E_{fij} + b_{ij} E_{0ij} \right) \tag{6.14a}$$

$$\text{s.t. Eqs. } (6.2b), (6.2c), \tag{6.14b}$$

$$\sum_{p\in Q} \alpha_p \pi_{ij}(p) + \sum_{q\in Q} \beta_q \lambda_{ij}(q) - \sum_{(p,q)\in W} \sqrt{\rho_{pq}} \bar{d}_{pq} \mu_{ij}^{pq} + \epsilon \theta_{ij} = u_{ij}, \quad \forall (i,j) \in A, \tag{6.14c}$$

$$\pi_{ij}(p) + \lambda_{ij}(q) - \sqrt{\rho_{pq}} \mu_{ij}^{pq} - x_{ij}^{pq} \geq 0, \qquad \forall (i,j) \in A, \forall (p,q) \in W, \tag{6.14d}$$

$$\begin{pmatrix} \theta_{ij} \\ \boldsymbol{\mu}_{ij} \end{pmatrix} \in \text{SOC}(1 + |W|), \ \forall (i,j) \in A, \tag{6.14e}$$

$$\pi_{ij}(p) \geq 0, \qquad \forall (i,j) \in A, \forall p \in Q, \tag{6.14f}$$

$$\lambda_{ij}(q) \geq 0, \qquad \forall (i,j) \in A, \forall q \in Q, \tag{6.14g}$$

$$\text{Eqs. } (6.2e), (6.2f), (6.2g), (6.2h). \tag{6.14h}$$

In the green hose-ellipsoid model, variables μ_{ij}^{pq} and θ_{ij} are newly introduced with parameters $\bar{d}_{pq}$ and ϵ, while $\pi_{ij}(p)$ and $\lambda_{ij}(q)$ are used in the hose model, too. Note that the green hose-ellipsoid model represents a regular SOCP problem; it contains $|A|$ second-order cone constraints, and all the other constraints are linear. This is an MISOCP, which can be solved by modern solvers, e.g., Gurobi.

We explicitly assume the following to analyze the property of (6.14).

Assumption 2

$$\begin{aligned} \sum_{q\in Q} \bar{d}_{pq} &\leq \alpha_p, \quad \forall p \in Q, \\ \sum_{p\in Q} \bar{d}_{pq} &\leq \beta_q, \quad \forall q \in Q, \\ \bar{d}_{pq} &\geq 0, \quad \forall (p,q) \in W. \end{aligned}$$

Note that, if Assumption 2 holds, then both Lemma 3 and Theorem 5 of Chapter 5 also hold.

6.3 Relation between ϵ and δ

In the green hose-ellipsoid model, ϵ is a parameter in the ellipsoidal uncertainty set whose center is the estimated value of $\bar{d}_{pq}$ for every $(p, q) \in W$. The parameter ϵ represents the total fluctuation over the network. To explain the role of δ in the green HLT model, let us consider the two dimensional situation given in Figure 6.1. In this figure, the shaded area is the original feasible region defined by the constraints (6.8b). The green HLT model expands the feasible region with the ratio $1 + \delta$. This figure suggests that if we take δ sufficiently large, then the ellipsoid is contained in the expanded feasible region of (6.8b). In the following, we mathematically describe this relationship.

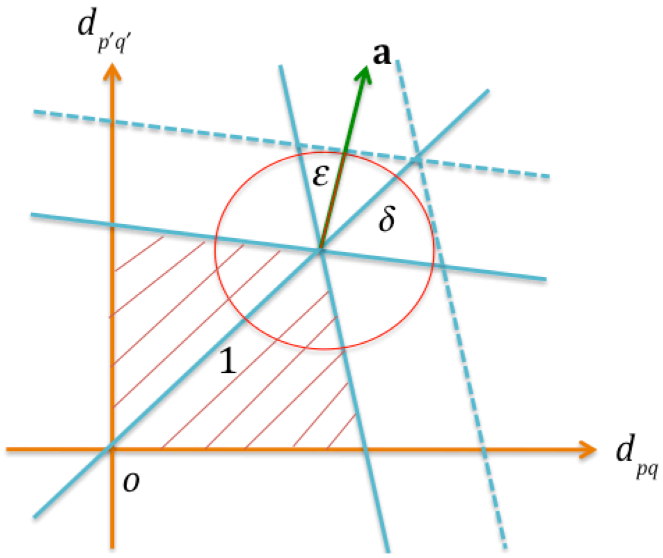

Figure 6.1: Relation between ϵ and δ.

To analyze the properties of $\mathcal{G}$ and Θ_ϵ, we make the following assumption.

Assumption 3

$$\epsilon\sqrt{\sum_{(p,q)\in W}\frac{\left(a_{ij}^{pq}\right)^2}{\rho_{pq}}} \leq \delta y_{ij}, \quad \forall(i,j)\in A. \tag{6.15}$$

Theorem 6 *If Assumption 3 holds, the feasible region of the optimization problem $S(\mathbf{x}_{ij})$ is contained in the feasible region of the problem $V(\mathbf{x}_{ij}, \mathbf{a}_{ij}, y_{ij})$.*

Theorem 6 is proved in Appendix E.

Next, given δ and y_{ij}, we consider computing ϵ and ρ_{pq} to satisfy Assumption 3. To do this, let $r = \epsilon/\delta$. Then by putting $\xi^{pq} = 1/\sqrt{\rho_{pq}}$ for every $(p,q)\in W$, we obtain that

$$\sqrt{\sum_{(p,q)\in W}\left(a_{ij}^{pq}\xi^{pq}\right)^2} \leq y_{ij}/r. \tag{6.16}$$

We consider the following optimization problem:

$$\min \ t \tag{6.17a}$$
$$\text{s.t.} \ \eta_{ij}^{pq} = a_{ij}^{pq}\xi^{pq}, \quad \forall(p,q)\in W, \forall(i,j)\in A, \tag{6.17b}$$
$$\sqrt{\sum_{(p,q)\in W}\left(\eta_{ij}^{pq}\right)^2} \leq y_{ij}t, \qquad \forall(i,j)\in A, \tag{6.17c}$$
$$t \geq 0, \tag{6.17d}$$
$$\xi^{pq} \geq 0.1, \qquad \forall(p,q)\in W. \tag{6.17e}$$

In order to avoid infinite values of ρ_{pq}, we set $\xi^{pq} \geq 0.1$ which leads values of ρ_{pq} for $(p,q)\in W$ to at most 100. The linear optimization problem (6.17) allows us to maximize r for its optimal value, where $t = 1/r$. Note that every feasible solution of (6.17) satisfies Assumption 3. The optimal solution of (6.17) achieves the largest value in ϵ.

6.4 Numerical experiments

6.4.1 Experiment settings

We compare the network power consumption obtained by the green HE model against the green pipe model, the green hose model, and the green HLT model.

The networks used in the experiments, shown in Figure 6.2, are typical backbone networks used in [66] (Networks 1 and 2) and [30] (Networks 3 and 4). The values of ρ_{pq} for all $(p,q) \in W$ are taken by solving the optimization problem (6.17) for each considered network. The traffic demands $\bar{d}_{pq}$ are randomly generated with a uniform distribution in the range of (1,150) and the link capacities are in the range of (800, 2800) for all networks. We assume that link power consumption is an affine function of its usage as we describe in Section 6.1.1. To describe the power consumption of 1 Gbps links, the values of E_{fij} and E_{0ij} are deduced from the affine function used in [56].

We calculate the power saving for each model using the following formula:

$$\text{power saving} = \frac{\text{TCP} - \text{RCP}}{\text{TCP}},$$

where TCP stands for total consumed power by the links when all considered links are on and RCP is that by the links required by the optimal solution.

The power saving obtained by the green pipe model is based on the exact traffic demands. To allow fluctuation, we need the estimated traffic demands for the green HE model. In the green HLT model, we do not need traffic demands directly, but need the estimated traffic volume y_{ij} passing through each $(i,j) \in A$. To compare the power consumption between the considered models, HLT bounds are set as follows: $\alpha_p = \sum_{q \in Q} \bar{d}_{pq}$, $\forall p \in Q$, $\beta_q = \sum_{p \in Q} \bar{d}_{pq}$, $\forall q \in Q$, and $y_{ij} = \sum_{(p,q) \in W} a_{ij}^{pq} \bar{d}_{pq}$, $\forall (i,j) \in A$. The values of a_{ij}^{pq} are obtained by initial routing which is a shortest path routing where we assume that each link in the network has a unit length.

For comparisons between the models in terms of power saving in the network, we choose ϵ and δ so that Assumption 3 holds. Specifically, we fix $\delta = 0.5$ for the HLT model in Networks 1-4. Then, using the relation as we established in the theory, we compute ϵ from

$$\epsilon = \min_{(i,j) \in A} \left\{ \frac{y_{ij}}{\sqrt{\sum_{(p,q) \in W} \left(a_{ij}^{pq}\right)^2 / \rho_{pq}}} \right\} \delta \tag{6.18}$$

for every considered network. Then, the corresponding values of ϵ stand to 24.75, 35, 95.26, and 25 in Networks 1-4, respectively.

The experiment program is written in Python language and the optimization problems are solved by using Gurobi, version 7.0.1 [38]. We use a Windows based computer with Intel(R) Core(TM) i7-4790 CPU @ 3.60 GHz and 16 GB memory.

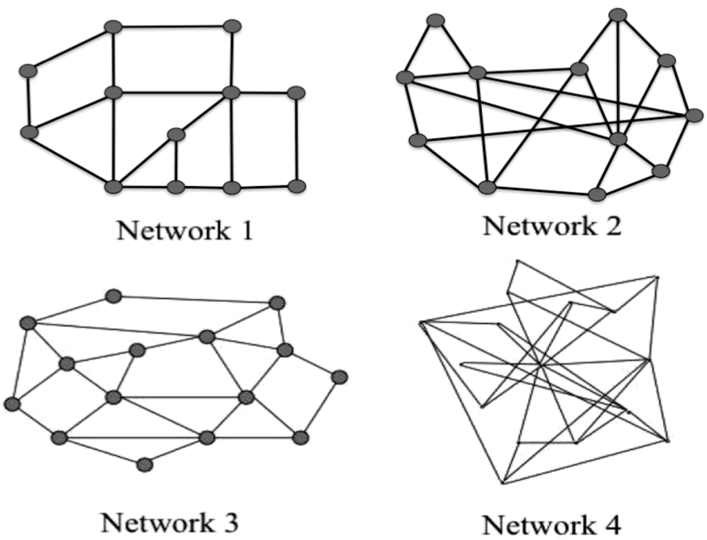

Figure 6.2: Sample networks.

Table 6.2: Network fixtures considered in experiments.

Network	No. of nodes considered	No. of links considered
Network 1	12	18
Network 2	12	22
Network 3	15	27
Network 4	16	32

6.4.2 Model comparisons

The network power consumption achieved by the green HE (G. HE) model is compared with the green pipe (G. Pipe), green HLT (G. HLT), and green hose

(G. Hose) models. The computation results for the values of ϵ and δ are reported in Figure 6.3. Note that, in G. HE, the fluctuation of traffic demands between two nodes, i.e., the error that we consider depends on ϵ, and in G. HLT, the error on maximum amount of traffic measured on each link depends on δ. However, G. Pipe and G. Hose are independent of ϵ and δ.

Figure 6.3 depicts that the power saving achieved by G. HE is ranking from 29.1% to 41.29%, whereas these values for G. HLT varies from only 18.82% to 36.3% in the considered networks for $\delta = 0.5$. The network power reduction between G. Pipe and G. HLT contrasts from 5.06% to 10.49% whereas the difference of power reduction between G. Pipe and G. HE differs from only 0.23% to 0.01% in the examined networks.

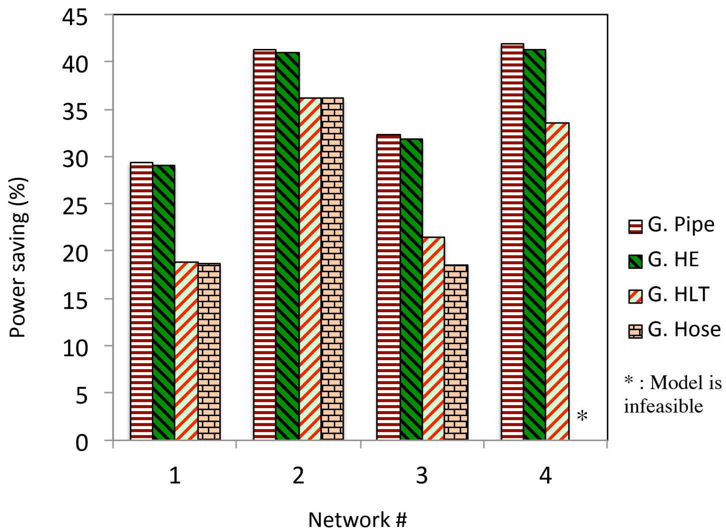

Figure 6.3: Comparisons of power saving for $\delta = 0.5, \epsilon = 24.75$ (Network 1), 35 (Network 2), 95 (Network 3), 25 (Network 4).

The numbers of deactivated links achieved by considered models are shown in Table 6.3 for $\delta = 0.5$ and the corresponding values of ϵ, which shows that links deactivated for G. Pipe and G. HE are the same for all considered networks. However, for G. Pipe and G. HLT, this value differs by 2, 1, 3, and 3 units for Networks 1, 2, 3, and 4, respectively.

Table 6.3: Number of deactivated links for $\delta = 0.5, \epsilon = 24.75$ (Network 1), 35 (Network 2), 95 (Network 3), 25 (Network 4).

Network	**Number of deactivated links**			
Type	**G. Pipe**	**G. HE**	**G. HLT**	**G. Hose**
Network 1	6	6	4	4
Network 2	10	10	9	9
Network 3	10	10	7	6
Network 4	15	15	12	*

* The model is infeasible.

6.4.3 ϵ dependency of G. HE model

Figures 6.4, 6.5, and 6.6 describe the comparisons of dependency of the power saving by the G. Hose, G. HLT, and G. HE models for Networks 1, 2, and 3. The comparisons are made by considering the different values of δ from 0 to 1 and the corresponding values of ϵ using the relation (6.18). In this case, the values of ρ_{pq} for every $(p, q) \in W$ are also chosen from the optimal solution of the problem (6.17). Note that power saving for the G. Hose model is independent of ϵ and δ, and it constructs a lower bound in each network.

Figures 6.4, 6.5, and 6.6 show that network power saving for G. HE and G. HLT are decreased for increasing values of ϵ and δ. It is also clear that for $\epsilon =$ 25, 21, and 152.4, the power saving by G. HLT for the corresponding values of δ ($\delta = 0.5, 0.3, 0.8$) equals the results of the G. Hose model in Networks 1, 2, and 3, respectively. The results also confirm that the G. HE model achieves the first position in terms of power saving for every value of ϵ in each network and decreases very slightly for increasing values of ϵ compared to the G. HLT model.

6.5 Summary

In this chapter, we have developed a mixed-integer SOCP formulation (green HE model) using the robust optimization to minimize network power consumption allowing traffic fluctuation. Compared to the previously studied model that uses exact information on traffic demands, the green HE model exempts the operators

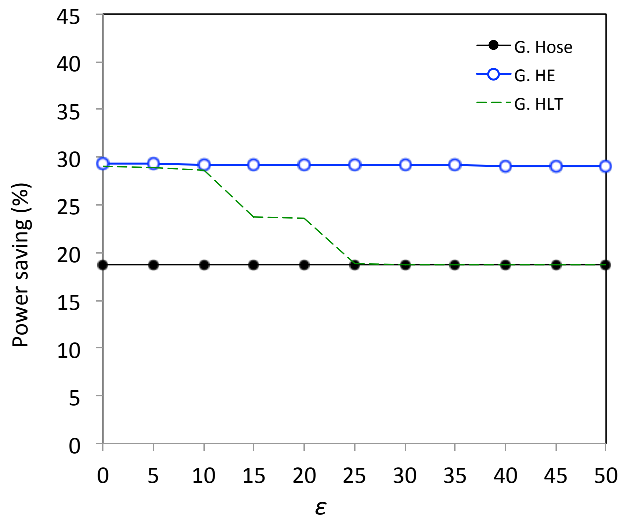

Figure 6.4: ϵ dependency of G. HE model (Network 1).

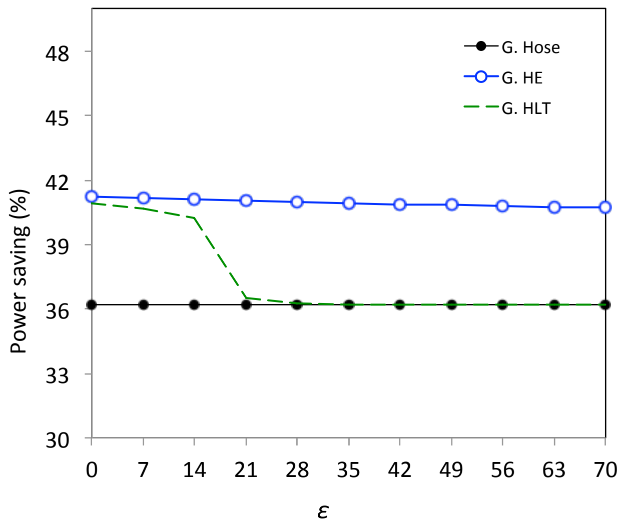

Figure 6.5: ϵ dependency of G. HE model (Network 2).

from knowing the exact traffic demands by allowing total outgoing and incoming amount of traffic at each node and the total amount of fluctuation in the estimated value over the network. We formulated this problem into MISOCP that can be

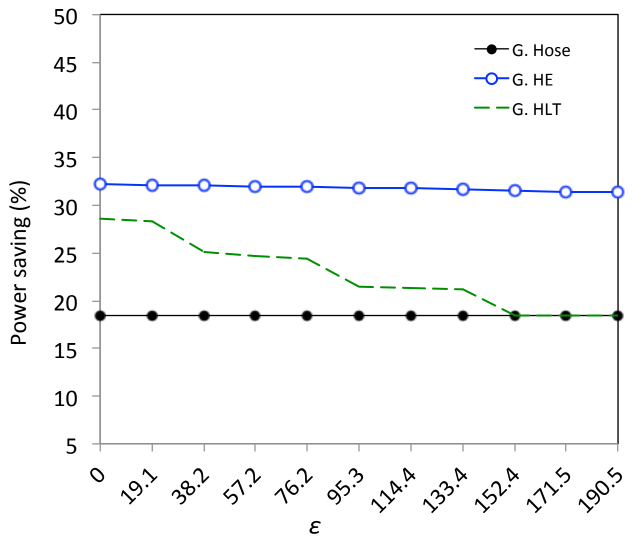

Figure 6.6: ϵ dependency of G. HE model (Network 3).

solved by the modern optimization solvers with proof of related theorems.

The power saving achieved by our proposed model are compared with those of the existing models, the green pipe, green hose and green HLT models. Note that, due to exact traffic demands, the green pipe model achieves the highest performance in terms of power saving. The numerical results showed that our proposed model provides significantly better power efficiency than that by the green HLT model and it is close to the green pipe model in every considered network. The limitation of the proposed model is that it is not easy to solve the problems in the case of large network due to MISOCP which is NP-hard.

Since practically traffic demands fluctuate, our proposed robust optimization model is an effective approach to the problem of minimizing the network power consumption allowing fluctuation in the traffic demands as much as we wish.

Chapter 7

Conclusions and future work

7.1 Conclusions

In this thesis, we applied robust optimization to the problem of minimizing the network congestion ratio and to the problem of minimizing power consumption in network. We considered the situation that there are some fluctuations or errors in traffic demands, but the total amount of errors is limited. The uncertainty sets contained in ellipsoids are able to cope with this situation, and then we need to apply robust optimization technique to obtain robust counterparts that can be numerically computable by the standard solvers.

In the first part of this thesis, we introduced two robust optimization models from the pipe model to minimize the network congestion ratio. The first one considers the ellipsoidal uncertainty set and the second the intersection of the hose and ellipsoidal uncertainty sets. We obtained robust counterparts of these robust optimization problems in the form of SOCP problems, with proofs of their equivalence. Both of our proposed models exempt the operators from knowing the exact traffic demands and can deal with total amount of fluctuations over the network.

The congestion ratios obtained by our proposed models are compared with three existing models: the pipe, hose, and hose-rectangle models. Numerical experiments confirmed that the SOCP problems obtained by our proposed models are tractable by modern solvers; although the computation times are much larger than those of the pipe model due to SOC constraints but they are comparable to those of the other existing models. Recall that due to exact traffic matrix, the pipe model achieves the first position in terms of minimizing the congestion ratio. The results also confirm that our proposed models can minimize the congestion ratios with fluctuations of traffic demands comparable to the hose and hose-rectangle models.

In the last part of the thesis, we proposed a mixed-integer SOCP formulation, the *green HE* model using the same robust optimization technique for the design of power efficient networks allowing fluctuations in traffic demands. Compared to the previous research that uses exact information on traffic demands, the green HE model releases the operators from knowing the exact traffic demands by allowing total outgoing and incoming amount of traffic at each node and the total amount of fluctuations in the estimated value over the network. Here, we proposed an MISOCP formulation that can be tracked by the modern optimization solvers.

The achieved power saving by our proposed model is compared with those of the existing models, the green pipe, green hose and green HLT models. It is noted that due to exact traffic demands, the green pipe model achieves the highest performance in terms of power saving. Since traffic demands fluctuate due to various reasons and users' needs, our proposed robust optimization model is an effective approach to the problem of minimizing the network power consumption allowing fluctuations in the traffic demands. The numerical results showed that our proposed model provides at most 11% better power efficiency than that by the green HLT model and it is close to the green pipe model in every considered network.

7.2 Future work

This research paper is on theoretical aspect. The packet level performance including delays and packet losses corresponding to the discussion in Section 6.4 of Chapter 5 should be addressed by using network simulators tools to observe the packet level behaviors. Duplex links are considered in this thesis, we also have a plan to conduct some researches considering the one directional link only.

Appendix A

Derivation of traffic flow condition for destination node

We fix $(p,q) \in W$. Then, the given traffic conditions for source and intermediary nodes are

$$\sum_{j:(i,j)\in\mathcal{A}} x_{ij}^{pq} - \sum_{j:(j,i)\in\mathcal{A}} x_{ji}^{pq} = 1, \quad \text{if } i = p, \tag{A.1}$$

$$\sum_{j:(i,j)\in\mathcal{A}} x_{ij}^{pq} - \sum_{j:(j,i)\in\mathcal{A}} x_{ji}^{pq} = 0, \quad \forall i \in V \setminus \{p,q\}. \tag{A.2}$$

The equation (A.1) can be written as

$$\sum_{j:(p,j)\in\mathcal{A}} x_{pj}^{pq} - \sum_{j:(j,p)\in\mathcal{A}} x_{jp}^{pq} = 1. \tag{A.3}$$

The equation (A.2) represents a set of $N-2$ equations for $i \in V\setminus\{p,q\}$. We obtain the following equation (A.4) by taking the sum over the left sides of equation (A.3) and $N-2$ equations expressed in equation (A.2) and a sum over the right sides of them:

$$\sum_{j:(p,j)\in\mathcal{A}} x_{pj}^{pq} + \sum_{i\in V\setminus\{p,q\}} \sum_{j:(i,j)\in\mathcal{A}} x_{ij}^{pq} - \sum_{j:(j,p)\in\mathcal{A}} x_{jp}^{pq} - \sum_{i\in V\setminus\{p,q\}} \sum_{j:(j,i)\in\mathcal{A}} x_{ji}^{pq} = 1. \tag{A.4}$$

Now, we consider the following subsets of links
$A_0 = \{(p,j) : (p,j) \in \mathcal{A}\}, A_1 = \{(q,j) : (q,j) \in \mathcal{A}\}, \text{and } A_2 = \mathcal{A} \setminus (A_0 \cup A_1)$, where $\mathcal{A} = A_0 \cup A_1 \cup A_2$. Then, the following relationship holds:

$$\sum_{j:(i,j)\in A_0} x_{ij}^{pq} + \sum_{j:(i,j)\in A_2} x_{ij}^{pq} = \sum_{j:(i,j)\in\mathcal{A}} x_{ij}^{pq} - \sum_{j:(i,j)\in A_1} x_{ij}^{pq}. \tag{A.5}$$

By substituting

$$\sum_{j:(i,j)\in A_0} x_{ij}^{pq} = \sum_{j:(p,j)\in\mathcal{A}} x_{pj}^{pq}, \sum_{j:(i,j)\in A_1} x_{ij}^{pq} = \sum_{j:(q,j)\in\mathcal{A}} x_{qj}^{pq},$$

$$\sum_{j:(i,j)\in A_2} x_{ij}^{pq} = \sum_{i\in V\setminus\{p,q\}} \sum_{j:(i,j)\in\mathcal{A}} x_{ij}^{pq}, \text{and} \sum_{j:(i,j)\in\mathcal{A}} x_{ij}^{pq} = \sum_{i\in V} \sum_{j:(i,j)\in\mathcal{A}} x_{ij}^{pq}$$

in our case, we can write the following relationships:

$$\sum_{j:(p,j)\in\mathcal{A}} x_{pj}^{pq} + \sum_{i\in V\setminus\{p,q\}} \sum_{j:(i,j)\in\mathcal{A}} x_{ij}^{pq} = \sum_{i\in V} \sum_{j:(i,j)\in\mathcal{A}} x_{ij}^{pq} - \sum_{j:(q,j)\in\mathcal{A}} x_{qj}^{pq}, \tag{A.6}$$

$$\sum_{j:(j,p)\in\mathcal{A}} x_{jp}^{pq} + \sum_{i\in V\setminus\{p,q\}} \sum_{j:(j,i)\in\mathcal{A}} x_{ji}^{pq} = \sum_{i\in V} \sum_{j:(j,i)\in\mathcal{A}} x_{ji}^{pq} - \sum_{j:(j,q)\in\mathcal{A}} x_{jq}^{pq}. \tag{A.7}$$

Using the relations (A.6) and (A.7), the equation (A.4) can be transformed to

$$\sum_{i\in V} \sum_{j:(i,j)\in\mathcal{A}} x_{ij}^{pq} - \sum_{j:(q,j)\in\mathcal{A}} x_{qj}^{pq} - \sum_{i\in V} \sum_{j:(j,i)\in\mathcal{A}} x_{ji}^{pq} - \sum_{j:(j,q)\in\mathcal{A}} x_{jq}^{pq} = 1. \tag{A.8}$$

Finally, using the condition

$$\sum_{i\in V} \sum_{j:(i,j)\in\mathcal{A}} x_{ij}^{pq} - \sum_{i\in V} \sum_{j:(j,i)\in\mathcal{A}} x_{ji}^{pq} = 0,$$

the equation (A.8) can be expressed as

$$\sum_{j:(q,j)\in\mathcal{A}} x_{qj}^{pq} - \sum_{j:(j,q)\in\mathcal{A}} x_{jq}^{pq} = -1. \tag{A.9}$$

□

Appendix B

Proof of Lemma 2

The given optimization problem is

$$\max_{\mathbf{v}\in\Omega_\theta} \mathbf{a}^T\mathbf{v}$$

$$\text{s.t. } ||\mathbf{v}|| \leq \theta.$$

The Lagrangian function of the problem is $F(\mathbf{v}, \lambda) = \mathbf{a}^T\mathbf{v} + \lambda(\theta - ||\mathbf{v}||)$. At the optimal point, the Karush-Kuhn-Tucker (K.K.T.) conditions are

$$1.\ \nabla_{\mathbf{v}} F(\mathbf{v}, \lambda) \equiv \mathbf{a} - \lambda \nabla_{\mathbf{v}}(||\mathbf{v}||) = 0, \tag{B.1}$$

$$2.\ \lambda(\theta - ||\mathbf{v}||) = 0, \tag{B.2}$$

$$3.\ (\theta - ||\mathbf{v}||) \geq 0, \tag{B.3}$$

$$4.\ \lambda \geq 0. \tag{B.4}$$

The equation (B.1) can be written as $\mathbf{a} - \lambda \frac{\mathbf{v}}{||\mathbf{v}||} = 0$, which is equivalent to

$$\mathbf{a}||\mathbf{v}|| = \lambda \mathbf{v}. \tag{B.5}$$

Here, $\lambda \neq 0$ because by condition (B.1), if $\lambda = 0$, then $\mathbf{a} = 0$. Therefore, by condition (B.2), $||\mathbf{v}|| = \theta$ and the equation (B.5) is equivalent to

$$\mathbf{a}\theta = \lambda \mathbf{v}$$

$$\Leftrightarrow \mathbf{v} = \frac{\theta}{\lambda}\mathbf{a}. \tag{B.6}$$

Here we have, $\mathbf{a}\theta = \lambda\mathbf{v} \Rightarrow ||\mathbf{a}||\theta = \lambda||\mathbf{v}|| \Leftrightarrow ||\mathbf{a}||\theta = \lambda\theta$, which implies $\lambda = ||\mathbf{a}||$. Therefore the equation (B.6) can be written as

$$\mathbf{v} = \frac{\mathbf{a}}{||\mathbf{a}||}\theta \Rightarrow \mathbf{a}^T\mathbf{v} = \frac{\mathbf{a}^T\mathbf{a}}{||\mathbf{a}||}\theta,$$

which is equivalent to

$$\mathbf{a}^T\mathbf{v} = \frac{||\mathbf{a}||^2}{||\mathbf{a}||}\theta.$$

Since K.K.T. conditions are written at optimal point, therefore

$$\max_{\mathbf{v} \in \Omega_\theta} \mathbf{a}^T\mathbf{v} = \theta||\mathbf{a}||.$$

$\square$

Appendix C

Dual transformation of the problem $S(\mathbf{x}_{ij})$

Using the vector notation $\boldsymbol{\alpha} = (\alpha_p)_{p \in Q}$, $\boldsymbol{\beta} = (\beta_q)_{q \in Q}$, and $\mathbf{x} = (x_{ij}^{pq})_{(p,q) \in W}$, we can express $S(\mathbf{x}_{ij})$ as follows:

$$
\begin{aligned}
S(\mathbf{x}_{ij}) : \ \max \ & \mathbf{x}_{ij}^T \mathbf{d} \\
\text{s.t. } & E_1^T \mathbf{d} \leq \boldsymbol{\alpha}, \\
& E_2^T \mathbf{d} \leq \boldsymbol{\beta}, \\
& v_0 = \epsilon, \\
& -\sqrt{\rho_{pq}} d_{pq} + v_{pq} = -\sqrt{\rho_{pq}} \bar{d}_{pq}, \\
& \mathbf{d} \geq \mathbf{0}, \\
& \begin{pmatrix} v_0 \\ \mathbf{v} \end{pmatrix} \in \mathrm{SOC}(1 + |W|),
\end{aligned}
$$

where $E_1 \in \mathbb{R}^{W \times Q}$ and $E_2 \in \mathbb{R}^{W \times Q}$ are

$$
(E_1)_{(p,q),j} = \begin{cases} 1 & \text{if } q = j \\ 0 & \text{otherwise,} \end{cases}
$$

$$
(E_2)_{(p,q),j} = \begin{cases} 1 & \text{if } p = j \\ 0 & \text{otherwise,} \end{cases}
$$

respectively. Let

$$
\mathbf{f} = \begin{pmatrix} \epsilon \\ \tilde{\mathbf{f}} \end{pmatrix} \in \mathbb{R}^{1+W},
$$

where $(\tilde{\mathbf{f}})_{pq} = -\sqrt{\rho_{pq}} \bar{d}_{pq}$, and $B_1 \in \mathbb{R}^{W \times W}$ is the diagonal matrix defined by

$$
(B_1)_{(p,q),(i,j)} = \begin{cases} -\sqrt{\rho_{pq}} & \text{if } (p,q) = (i,j) \\ 0 & \text{otherwise.} \end{cases}
$$

Then, the equality constraints of $S(\mathbf{x}_{ij})$ can be written as

$$\begin{pmatrix} \mathbf{0}^T & 1 & \mathbf{0}^T \\ B_1 & \mathbf{0} & I \end{pmatrix} \begin{pmatrix} \mathbf{d} \\ v_0 \\ \mathbf{v} \end{pmatrix} = \mathbf{f}.$$

Finally, $S(\mathbf{x}_{ij})$ can be written in the form:

$$\begin{aligned} S(\mathbf{x}_{ij}) : \ \max \ & \begin{pmatrix} \mathbf{x}_{ij} \\ 0 \\ \mathbf{0} \end{pmatrix}^T \begin{pmatrix} \mathbf{d} \\ v_0 \\ \mathbf{v} \end{pmatrix} \\ \text{s. t.} \ & \begin{pmatrix} E_1^T & \mathbf{0} & O \\ E_2^T & \mathbf{0} & O \end{pmatrix} \begin{pmatrix} \mathbf{d} \\ v_0 \\ \mathbf{v} \end{pmatrix} \leq \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} \\ & \begin{pmatrix} \mathbf{0}^T & 1 & \mathbf{0}^T \\ B_1 & \mathbf{0} & I \end{pmatrix} \begin{pmatrix} \mathbf{d} \\ v_0 \\ \mathbf{v} \end{pmatrix} = \mathbf{f} \\ & \begin{pmatrix} \mathbf{d} \\ v_0 \\ \mathbf{v} \end{pmatrix} \in \mathbb{R}_+^W \times \mathrm{SOC}(1 + |W|). \end{aligned}$$

Therefore, applying the duality between (P_1) and (D_1) yields the dual of $S(\mathbf{x}_{ij})$ as

$$\begin{aligned} \min \ & \boldsymbol{\alpha}^T \boldsymbol{\pi} + \boldsymbol{\beta}^T \boldsymbol{\lambda} + \epsilon\theta + \tilde{\mathbf{f}}^T \boldsymbol{\mu} \\ \text{s. t.} \ & E_1 \boldsymbol{\pi} + E_2 \boldsymbol{\lambda} + B_1 \boldsymbol{\mu} - \mathbf{x}_{ij} \geq \mathbf{0} \\ & \begin{pmatrix} \theta \\ \boldsymbol{\mu} \end{pmatrix} \in \mathrm{SOC}(1 + |W|) \\ & \boldsymbol{\pi} \geq \mathbf{0}, \ \boldsymbol{\lambda} \geq \mathbf{0}. \end{aligned}$$

□

Appendix D

Proof of Lemma 3

One can easily verify that $d_{pq} = \bar{d}_{pq}$ for every $(p,q) \in W$ forms a feasible solution of $S(\mathbf{x}_{ij})$, and that

$$\begin{aligned} \mu_{ij}^{pq} &= -x_{ij}^{pq}/\sqrt{\rho_{pq}}, \\ \pi_{ij}(p) &= \lambda_{ij}(q) = 0, \\ \theta_{ij} &= \sqrt{\sum_{(p,q)\in W} (\mu_{ij}^{pq})^2 + 1} \end{aligned}$$

for every $(p,q) \in W$ form a feasible solution of (5.36).

Unfortunately, Assumption 1 does not ensure that these feasible solutions are relative interior points of the corresponding cones because the linear inequality constraints could be satisfied with equality. Thus, we cannot apply Theorem 3 directly.

Recently, Lourenço, Muramatsu, and Tsuchiya [41] extended this theorem to the 'partially polyhedral' case where $K = K_1 \times K_2$ and K_2 is polyhedral. Note that in this case, $K^* = K_1^* \times K_2^*$, and K_2^* is polyhedral.

We recall that (D_0) satisfies the Partial Polyhedral Slater's (PPS) condition if there exists a slack $(\mathbf{s}_1, \mathbf{s}_2) = \mathbf{c} - A^T\mathbf{y}$ such that $\mathbf{s}_1 \in \mathrm{ri}K_1$ and $\mathbf{s}_2 \in K_2$. Similarly, we say that (P_0) satisfies the PPS condition if there exists a feasible solution $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$ such that $\mathbf{x}_1 \in \mathrm{ri}K_1$.

Theorem 7 *(Proposition 2 of [41])*

1. *If* (P_0) *satisfies the PPS condition and* (D_0) *is feasible, then* $\mathrm{val}(P_0) = \mathrm{val}(D_0)$ *and* (D_0) *has an optimal solution.*

2. *If* (D_0) *satisfies the PPS condition and* (P_0) *is feasible, then* val(P_0) =val(D_0) *and* (P_0) *has an optimal solution.*

Since each subvector of the feasible solutions corresponding to a second-order cone is contained in the relative interior of the second-order cone, the two feasible solutions satisfy the PPS condition. Therefore, we can apply Theorem 7 and conclude that both $S(\mathbf{x}_{ij})$ and its dual (5.36) have optimal solutions, and that their optimal values coincide.

□

Appendix E

Proof of Theorem 6

We have to show that $\mathbf{d} \in \Theta_\epsilon \cap \mathcal{H}$ implies $\mathbf{d} \in \mathcal{G}$ if $\epsilon \sqrt{\sum_{(p,q)\in W} \left(a_{ij}^{pq}\right)^2 / \rho_{pq}} \leq \delta y_{ij}$ for every $(i,j) \in A$. Suppose $\mathbf{d} \in \Theta_\epsilon \cap \mathcal{H}$. We can express $d_{pq} = \bar{d}_{pq} + \epsilon \frac{v_{pq}}{\sqrt{\rho_{pq}}}$, where $||\mathbf{v}|| \leq 1$. Then for each $(i,j) \in A$, we can write

$$\begin{aligned}
\sum_{(p,q)\in W} a_{ij}^{pq} d_{pq} &= \sum_{(p,q)\in W} a_{ij}^{pq} \left(\bar{d}_{pq} + \epsilon \frac{v_{pq}}{\sqrt{\rho_{pq}}} \right) \\
&\leq y_{ij} + \epsilon \sum_{(p,q)\in W} \frac{a_{ij}^{pq}}{\sqrt{\rho_{pq}}} v_{pq} \\
&\leq y_{ij} + \epsilon \sqrt{\sum_{(p,q)\in W} \frac{\left(a_{ij}^{pq}\right)^2}{\rho_{pq}}} ||\mathbf{v}|| \leq y_{ij} + \epsilon \sqrt{\sum_{(p,q)\in W} \frac{\left(a_{ij}^{pq}\right)^2}{\rho_{pq}}}.
\end{aligned}$$

Therefore, if $\epsilon \sqrt{\sum_{(p,q)\in W} \left(a_{ij}^{pq}\right)^2 / \rho_{pq}} \leq \delta y_{ij}$ for each $(i,j) \in A$, then $\sum_{(p,q)\in W} a_{ij}^{pq} d_{pq} \leq y_{ij}(1+\delta)$, which shows that $\mathbf{d} \in \mathcal{L}$.

□

Publications

List of Publications related to the dissertation

Journal Papers

1. Bimal Chandra Das, Satoshi Takahashi, Eiji Oki, and Masakazu Muramatsu, “Network congestion minimization models based on robust optimization,” IEICE Transaction on Communications, (accepted for publication) vol. E101-B, no. 3, pp.- , March 2018.

National Conference Paper

1. Bimal Chandra Das, Ihsen A. Ouedraogo, Eiji Oki, Masakazu Muramatsu, “A simple formulation of minimization on network congestion ratio,” State-of-the-art and future development of the optimization techniques, Kyoto University, Kyoto, Japan, August 2016.

International Conference Talk

1. Bimal Chandra Das, Ihsen A. Ouedraogo, Eiji Oki, Masakazu Muramatsu, “A Mixed-Integer SOCP Model for Robust and Power Efficient Networks,” The Fifth International Conference on Continuous Optimization, National Graduate Institute for Policy Studies (GRIPS), Tokyo, Japan, August 2016.

2. Bimal Chandra Das, Eiji Oki, Masakazu Muramatsu, “Application of SOCP in Power Efficient Networks,” SIAM Conference on Optimization (OP17), Vancouver, British Columbia, May 24, 2017.

National Conference Talk

1. Bimal Chandra Das, Satoshi Takahashi, Eiji Oki, Masakazu Muramatsu, "Network congestion minimization model based on robust optimization," Optimization: Modeling and Algorithms, The Institute of Statistical Mathematics, Tachikawa-shi, Tokyo, Japan, Mar 23. 2017.

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