

Stabilization Control Design for Polynomial Fuzzy Systems Represented in Descriptor Form

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概要

ディスクリプタ形式で表現される 多項式ファジィシステムに対する安定化制御系設計

複雑な非線形システムを多項式ファジィシステムで記述し、多項式リアプノフ関数を用いた制御系設計に関する研究が行われている。本論文では、多項式ディスクリプタシステムと sector nonlinearity に基づくファジィシステム表現を組み合わせた新しいシステム表現を提案し、ディスクリプタ形式の利点を生かしつつ、条件導出において新たな工夫やアイデアを用いることで sum-of-squares に基づく制御系設計条件の 3 つの緩和アプローチを提案する。

まず、最初の緩和アプローチとして、非線形システムを安定化するための有理多項式ファジィ制御器を提案し、多項式ファジィディスクリプタシステムで記述された閉ループ系に対して、homogenous Lyapunov 関数を用いた nonconvex 設計条件を導出する。つぎに、convex 設計条件を導出するために、ディスクリプタ形式の冗長性を利用した設計条件の導出を行い、その設計条件の緩和を試みる。最後に、ファジィシステム表現に準じた多項式スラック行列を導入することで、co-positive relaxation の適用を可能とした設計条件の緩和を試みるとともに、多項式 Lyapunov 関数に代わって多項式ファジィ Lyapunov 関数を導入することで緩和した設計条件を導出する。従来研究では、多項式ファジィ Lyapunov 関数のシステム解軌道に沿った時間微分において派生するメンバーシップ関数の時間微分の表現が sum-of-squares の枠組みでは扱えない表現となることがネックとなっていた。本論文では、メンバーシップ関数の時間微分の分解表現という新たなアイデアを提案し、この問題の解決を図っている。

最後に、複数のベンチマーク設計問題や複雑非線形システムの設計例題（自転車の安定化制御問題）を通して 3 つの緩和アプローチの有効性を明らかにする。

本論文は 6 章で構成され、概要は以下の通りである。

第 1 章では緒論を述べる。本研究の背景や目的を述べ、他の関連手法に対する本研究の位置付けを説明する。

第 2 章では、本研究の対象システムであるファジィシステム/多項式ファジィシステム、および、本論文で扱う設計条件導出や解法において重要な役割を担う sum-of-squares, co-positive relaxation について述べる。さらに、本研究で提案するメンバーシップ関数の時間微分表現の分解についてもそのアイデアを簡単に示す。

第 3 章では，多項式ファジィシステムを安定化するための有理多項式ファジィ制御器を提案し，閉ループ系のダイナミクスを多項式ファジィディスクリプタシステムで記述する．この多項式ファジィディスクリプタシステムに対して homogenous Lyapunov 関数を用い，さらに，Euler's theorem を利用することで nonconvex sum-of-squares 設計条件を導出する．ベンチマーク設計問題では，その設計条件を解くために path-following アルゴリズムを用い，先行研究との比較検討を通して提案する設計条件の有効性を明かにする．

第 4 章では，解を求めるのが困難である nonconvex 設計条件を回避するために，convex sum-of-squares 設計条件を導出する．第 3 章で提案した有理多項式ファジィ制御器に代わり多項式ファジィ制御器に限定することで，convex 設計条件を導出する．しかし，多項式行列で記述される非線形システムでは，nonconvex から convex への変換が一般には等価に行えないことが知られており，保守的となる条件導出をなるべく避けるためにいくつかの工夫がなされている．また，ディスクリプタ形式の冗長性により，システム行列のサイズは増加するが，sum-of-squares 設計条件の数が大きく減少することを示し，2つの設計例題を通して，ディスクリプタ形式導入の有用性も示している．一つはベンチマーク設計例題で，先行研究に対する優位性を検証している．もう一つは複雑非線形システムの設計例題として，自転車の安定化制御問題を取り上げており，複雑非線形システムに対しても有効な設計条件であることを示している．

第 5 章では，新しい概念として，ファジィシステム表現に準じた多項式スラック行列を定義し，緩和した sum-of-squares 設計条件を導出している．第 4 章では，ディスクリプタ形式の冗長性を利用することで，システム行列のサイズは増加するが，sum-of-squares 設計条件の数が大きく減少することを示し，設計条件の可解領域の拡大に成功した．一方で，ディスクリプタ形式を用いたことで，多項式ファジィシステム制御で活用してきた co-positive relaxation の適用が困難となっている．本論文では，この多項式スラック行列を導入することで，閉ループ系のディスクリプタ記述に対しても co-positive relaxation を適用可能となり，さらなる設計条件の緩和に成功している．加えて，多項式 Lyapunov 関数に代わって，多項式ファジィ Lyapunov 関数を導入し，さらなる設計条件の緩和を実現している．とくに，従来研究では，多項式ファジィ Lyapunov 関数のシステム解軌道に沿った時間微分において派生するメンバーシップ関数の時間微分の表現が sum-of-squares の枠組みでは扱えない表現となることがネックとなっていた．本論文では，メンバーシップ関数の時間微分の分解表現という新たなアイデアを提案し，この問題の解決を図っている．2つの設計例題を通して提案した設計条件の有効性を示している．

第 6 章では，結論を述べる．本研究のまとめと問題点，および，今後の展望について述べる

ABSTRACT

This thesis presents a stabilization control design for polynomial fuzzy systems represented in descriptor form. At the very first stage of this research, a closed-loop polynomial fuzzy system with the controller using rational functions is presented in descriptor form. The stabilization control design is constructed in the operation domain and presented in sum-of-squares (SOS) conditions. However, the stabilization criteria are nonconvex with the bilinear terms that have to be solved with the path-following method.

Thus, in the second method, the concept of the parallel distributed compensation (PDC) controller is employed to design a polynomial-based fuzzy controller. The closed-loop system is presented in descriptor form. A commonly used Lyapunov function for polynomial fuzzy-model-based (FMB) system is applied in stabilization, and the concept of PDC controller made the stabilization criterion convex. Compared with the polynomial FMB control design without descriptor form, this method obtains less conservative results though the SOS conditions are reduced.

Based on the second method, the third approach is launched. In this approach, the slack matrices are adopted, aiming to obtain more relaxed stabilization criteria. Because of the fuzzy slack matrices, more relaxed results are obtained. Though the double fuzzy summation problem is the side effect, this can be solved by co-positivity relaxation. In the special case that all membership functions are functions of the states being not related to the inputs, this thesis proposes the fourth method by applying a novel fuzzy Lyapunov function to further make the conditions more relaxed. Since the novel fuzzy Lyapunov function is applied, the time derivative of membership function (MF) with sector nonlinearity technique is also applied. Since the commonly used Lyapunov function can be seen as a special case of novel fuzzy Lyapunov function, the last method can always obtain more satisfactory results than previous methods. However, the last method can only be applied in special cases while the others do not have such the limit.

The six chapters contained in this thesis are as follow:

Chapter 1 is the introduction which includes the research background, motivations, and the position of this research.

Chapter 2 are the preliminaries, in which definitions, mathematical tools, and relaxation

tools are introduced.

Chapter 3 proposes a polynomial fuzzy descriptor system approach for rational fuzzy control design. A polynomial fuzzy model with the controller using rational function is presented and transformed into the descriptor form. However, when presenting the stabilization criterion in the SOS conditions, the bilinear terms seem something that can not be removed. Thus, the path-following method is applied to solve the bilinear issue. A design example is presented to show the contrast and comparison between the proposed method and the previous study.

Chapter 4 presents a descriptor system approach for polynomial FMB control design. Instead of the rational controller, the technique of PDC is applied, and the polynomial FMB closed-loop system with such the polynomial-based fuzzy controller is adopted in the descriptor form so that the nonconvex conditions met in **Chapter 3** are avoided. A commonly used Lyapunov function for polynomial FMB control system is applied for stabilization analysis. The redundancy of the descriptor form will raise the dimension of the matrices, though the SOS conditions decline. To illustrate, two examples are presented. The first is a numerical one, making a comparison between the proposed method and the previous study. The second one shows how the proposed method in this chapter is applied to a bicycle's dynamic system.

Chapter 5 shows a descriptor form approach for the polynomial FMB control systems design. Through the redundancy of the descriptor form, the fuzzy slack matrices are brought into stabilization analysis. The double fuzzy summation issue arises inevitably. Nevertheless, the double fuzzy summation issue can be regarded as the co-positivity problem, which can be solved by applying the co-positivity relaxation. In addition, for the cases all membership functions are functions of the states being not related to the inputs, this thesis presents another stabilization analysis approach with the application of the novel fuzzy Lyapunov function. Based on the ground that the commonly used Lyapunov function can be seen as the special case of the novel fuzzy Lyapunov function, the stabilization criteria is more relaxed than the third one. Also taken into consideration in the stabilization analysis are the time derivatives of MF because the novel fuzzy Lyapunov function is applied. Meanwhile, the sector nonlinearity technique is applied to deal with the rest part of the MF time derivatives after polynomial common factors are extracted. Likewise, the numerical examples are presented to demonstrate the advantages of the proposed third and fourth methods over the previous studies.

Finally, **Chapter 6** gives the conclusion of the previous chapters and the prospective improvements in future research.

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LIST OF NOTATIONS

List of Acronyms

T-S fuzzy	Takagi-Sugeno fuzzy
MF	Membership Function
PDC	Parallel Distributed Compensation
FMB	Fuzzy Model Based
LMI	Linear Matrix Inequalities
BMI	Bilinear Matrix Inequalities
SDP	Semi-Definite Problem
SOS	Sum-of-Squares
LUB	Lower upper-bound

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INTRODUCTION

The main techniques considered in this thesis are "fuzzy control" and "descriptor form." It is important to have some basic knowledge of descriptor form before introducing fuzzy control system. There are many dynamic phenomena that exist in our world. To analyze them, people try to define some parameters like positions, velocity, acceleration, etc., to "describe" the dynamic phenomena, which are usually presented as nonlinear systems. Throughout history, many control models were developed and tried to represent those nonlinear systems into general form, for example, the state-space representation. Descriptor system, the product of problem formulation of the system presented in sets of equations in general form, was developed in 1977 [1]. The mentioned parameters, positions, velocity, acceleration, etc. are called "descriptor variables" in descriptor modeling. Descriptor formulations include many standard forms of controlling models as the special cases. Therefore, this method contains more general classes.

Due to the more natural description of dynamical systems than the state-space representation, the descriptor system had gotten much attention. However, an issue was found in the descriptor modeling called "impulsive mode" [2]. The impulse presented by descriptor can cause many serious problems in the control system. Hence, the designing of descriptor model should avoid the impulsive model. Some studies had generalized the conditions that can prevent impulse, for example, the rank condition given by [3], and stabilization conditions given by [4]. The descriptor system which is prevented from impulsive mode is called "impulse-free."

There are two ways to present the descriptor system. The first one is to present the nonlinear system into descriptor system directly ([1], [35], [5], et. al.) The second one is to represent the existing standard form models into descriptor form ([6], [7], [23], et. al.) The thesis focuses on the second one. After introducing the background of descriptor system, the introduction of fuzzy control will be given.

Giving a general form for presenting nonlinear dynamical systems is the biggest achievement from Takagi-Sugeno (T-S) fuzzy model [8]. In 1985, Tomohiro Takagi and Michio Sugeno tried to develop a tool that builds a fuzzy model of a system. The so-called "fuzzy control" are the studies that discuss how to implicate fuzzy logic for expressing the control rules. At that time, the linguistic variables, compositional rule, and unimodal fuzzy models were considered for multivariable control. However, the system required a large amount of input space. The problem was solved by using multidimensional matrices for fuzzy reasoning, which reduced the number of implications and simplified the reasoning. Due to the fact that fuzzy implications and reasoning-based modeling is one of the most important things in fuzzy systems' studies, they tried to deal with the dynamic system in general by considering multidimensional reasoning method. The membership function of the fuzzy set was denoted in their research, which means all fuzzy sets are being related to linear membership functions. The fuzzy implication was presented in the form that contained consequence variables, premise variables, fuzzy sets, et. al. The algorithm of reasoning allowed the relations of piecewise linear to be reduced when compared with the traditional linear approximation method. Furthermore, the linguistic conditions can be presented into linear relations under the input space of the fuzzy partition. In the identification part, three steps are considered for the model that is consisting the implications of the previous format. The first one is to choose the premise variables, which contain a combination premise, an optimum premise, and the error between output values and output data. The next step is to identify the chosen premise variables, the step that searches for the optimum premise parameters to minimize the performance index. The final step is the consequent parameter's identifications, which finds the least performance index from the parameters given in the previous two steps. By considering the format of implication and the algorithm of identification, Tomohiro Takagi and Michio Sugeno gave a simple form that is able to represent the nonlinear system highly.

There has been widely study about the T-S fuzzy-model-based (FMB) control systems [9]-[14]. For its control design, a well-known option called "Parallel Distributed Compensation (PDC)" controller is usually a choice that contains T-S fuzzy system's membership functions [15]. PDC, a concept for fuzzy model control design, is a well-known approach for FMB controller design. The main idea is to set up the compensators with the rules corresponding to the T-S fuzzy model, respectively. Every fuzzy rule of the controller can be designed individually by using the linear control design technique. Note that T-S fuzzy mode shares

the fuzzy set to the controller. It is noted that the controller is nonlinear, generally. The controlling problem is how to select or determine the values of the local feedback gain. The selected feedback gain should satisfy the stabilization conditions to guarantee the quadratic stability for PDC controller and the model included in the closed-loop system. First of all, for the object that is needed to be controlled, present a T-S fuzzy model that can completely describe the target. Second, design the controllers, and each of them corresponds to one fuzzy rule. Third, use mathematical tools to test the stability by checking if the conditions are satisfied. If not, repeat the procedure until conditions are satisfied and find the control feedback gain. By applying the calculated feedback gain, the closed-loop system should be asymptotically stable.

The control design of such the model and its stability analysis can be presented in terms of LMI conditions [15], [16] based on the Lyapunov theory. A quadratic function which is composed of the multiplication of a matrix which is positive definite, the state vector, and its transpose is called quadratic Lyapunov function for deriving stability condition. From [19], the stability conditions are derived from a positive definite matrix, the system matrices, and each system matrix's transpose for T-S fuzzy system, which was multiplied together. If the system only contains one rule in the fuzzy set, the condition reduces to Lyapunov theorem. The method is to find that positive definite matrix which can prove the stability of the T-S fuzzy system from the chosen Lyapunov function. When the conditions are satisfied, the system would be in the quadratic stable situation. For the purpose of determining the positive definite matrix, an effective tool, a convex optimization technique called "Linear Matrices Inequalities (LMI) [21]", is introduced. LMI problems are one of the classes of numerical optimization problems. It takes polynomial-time to solve LMI problems' optimization issue. In fact, LMI problems can describe almost most of the control problems or systems. Therefore, LMI optimization became an important issue about solving the numerical optimization problem because the original control problems can be solved if it is transformed into LMI problems. An LMI constrain is a summation of the symmetric matrices multiplied with the corresponding variables, and the summation is positive definite. If the constraint is held, the fact that symmetric matrices are positive definite is also held. This constraint can present a large amount of convex constraints, including the inequalities of linears, matrix norms, Lyapunov and convex quadratics, et. al. Take stability conditions from Lyapunov approach, for example, the system matrices and its transform are known and need to determine the value

of the positive definite matrix which makes the inequality negative, the condition is cast into LMI problems, and the solution of the determined values are called feasible value.

The nonlinear control system was cast into a simple, natural, and effective form for design methodology by T-S fuzzy model in general. However, the double summation issue is the problem found in the PDC-based T-S fuzzy control system [22]. A descriptor approach which presents the T-S fuzzy system in the descriptor form has been raised by Tanaka et al. in 2007 [23], and the single fuzzy summation instead of the double fuzzy summation exists because of the redundancy of the descriptor form. The descriptor design methodology was presented via fuzzy Lyapunov function, and the stabilization conditions were cast in LMI terms. [24] shows that descriptor system has the advantage that it can deal the systems with the larger class when compared with the conventional state-space model. The other advantage is that it can represent independent parametric perturbations tighter than state-space design methodologies. As for stabilization, most of the studies [25]- [28] considered the piecewise Lyapunov functions or switched Lyapunov functions. Some studies [29]- [31], however, consider the fuzzy Lyapunov functions or piecewise Lyapunov functions, which made the stabilization conditions get the bilinear term. The bilinear LMI problems (i.e., BMI problems) can be transformed into LMI problems by considering completing the square technique [31] or using the path-following method, which will be discussed in the forward section. Nevertheless, at that time, such converting techniques like completing square contained the risk of conservative results. Thus [23] tried to use the new type of fuzzy Lyapunov function and controller for design methodology. The methodology successfully obtained the LMI conditions without BMI problems. The paper first converted T-S fuzzy model into the special form, then rewrote the equation into descriptor representation by defining some matrices containing the elements of the original system. The closed-loop system with descriptor representation is stabilized by applying a common Lyapunov function. Comparing it with state-space presenting T-S fuzzy model shows that the conditions of LMI were drastically reduced, which means the descriptor representation of T-S fuzzy model can handle a more complicated system. [23] then tried to stabilize the system with the fuzzy Lyapunov function. Two design methodologies were presented. The first one is the T-S fuzzy descriptor system with PDC control design. A new Lyapunov function that concerns the fuzzy summation's inverse matrix is called "Fuzzy Lyapunov function." The membership function's time derivatives were first time appeared in the stabilization process. By using the properties of the membership

function and some techniques that will be discussed later, it is able to do extractions of the differential of membership functions.

This term that ground the membership functions' time derivative would be a condition and was added to hold the stability of the fuzzy Lyapunov function. Nevertheless, it was difficult to select the value. The second control design introduced a new fuzzy controller by rewriting the feedback gain into the inverse of the fuzzy summation and LMI matrices. The stabilization was similar, but two series of LMI matrices determine the feedback gain. It is found out that the feasibility of common Lyapunov function's methodology is almost the same as the first fuzzy Lyapunov function's methodology. But the second fuzzy Lyapunov function's methodology obtained more relaxed results than the previous two methods, which means that the fuzzy Lyapunov function's methodology can obtain more relaxed results. Last but not least is that the third corollary in [23] contains the second corollary, which always obtain more relaxed results. This method has been widely applied [32] - [34]. One thing that should be noted is that this approach is different from the approach for fuzzy descriptor system design, which directly presented the nonlinear dynamic system into the fuzzy descriptor form [35], [36].

Although T-S fuzzy model, PDC controller, and LMI optimization obtained big success in the last two or three decades, the problems such as not every system can be represented in LMI problem or the results are too much conservative still be the issue for researchers. The study of the fuzzy control system gets a breakthrough in 2009. As the extension, [37] proposed a methodology which adds the T-S fuzzy system with the polynomials as subsystems and calls "Polynomial Fuzzy Model." Such polynomial FMB control system's stability analysis usually uses the Lyapunov candidate which also contains the polynomial terms. The stability analysis cannot be presented in LMI conditions since Lyapunov candidate contains polynomials. Instead, the stabilization criteria are presented in SOS conditions. Due to that polynomials are added into the fuzzy model, more control systems can be cast into fuzzy model, and more general and relaxed results than LMI optimization can be obtained since polynomial based Lyapunov function contains the previous quadratic Lyapunov function which the latter was taken as the special case. The polynomial FMB control system design has get a lot of attraction [42]- [48] because of the more extensive result than the LMI can be obtained [37], [66], [49]. Generally speaking, this kind of fuzzy model has the feature that rule consequence is consists of polynomials. Same as T-S fuzzy model, fuzzy's

IF-THEN rules are the way for representing the inputs and outputs of nonlinear systems. The difference is that the consequent part of the model contains polynomials. Besides, the column vector called "monomial vector" is defined and applied into the model instead of state vector-only. Polynomial fuzzy model provides the advantage of the fewer rules it generates when comparing with T-S fuzzy model. In the stabilization part, instead of the traditional type with only quadratics, a polynomial Lyapunov function is considered. Here, the positive definite matrix turns into the polynomial matrix, and the monomial vector replaces the state vector. As mentioned before, quadratics can be taken as a special case included in the polynomials. Hence, the more general status appears in the polynomial Lyapunov function. At the LMI period, the time derivative of the decision variable's matrices are zero matrices and was omitted in the differential of Lyapunov function's equation because that all the matrices are constant matrices. However, the decision variable's matrices' time derivatives have to be concerned when differentiating polynomial Lyapunov function since the polynomials are included. Using the concept of fuzzy Lyapunov function, this part can be rewritten as the summation of the decision variable's matrices which is partially differentiated by the state vector and multiplied by a system matrices without the controller. It is noted that the system matrices in the fuzzy summation only contain the elements corresponding to the row that the states' dynamic, which was not affected by the control inputs. The constrain that guarantees the system is stable is that the differential of the Lyapunov function is negative. Therefore, there usually is an identity matrix with the coefficients of a very small positive number which is added into the conditions.

SOSOPT [38], a MATLAB's third-party toolbox, is developed for finding SOS conditions' solutions. Let the conditions be the multivariate polynomials; the SOS conditions hold if a series of polynomial functions are found, and the condition is equal to the summation of those functions' square terms. Naturally, it points out that a property of SOS decomposition is that the equation of the condition should be positive for all values of state vectors. Recall the stabilization of the polynomial fuzzy model. Finding a positive semidefinite matrix that makes the polynomial Lyapunov function be sum of squares is necessary. Semidefinite programming is usually used for doing SOS decomposition of polynomial Lyapunov function, and the constraints come from its stabilization process. In general, if a condition is decomposed in SOS, it is also nonnegative. Thus, a polynomial-time computational relaxation obtained from decomposition with semidefinite programming proves the global nonnegativity of multivariate

polynomials [39] [40]. [41] also gives the fact that SOS and nonnegativity have little difference.

There are many options for polynomial FMB control design. Besides PDC controller, this thesis also concerns the rational controller. Nevertheless, the stability analysis of polynomial descriptor FMB controller causes the bilinear issue. In previous study, it used the particle swarm optimization (PSO) algorithm, the method that decides the coefficient of the rational function, to deal with the bilinear issue, but the solution may not be optimal. In contrast, this thesis uses the path-following method to solve the bilinear issue.

Path-following is an approach for solving nonconvex stabilization constraints. From [62], path-following can be applied to solve the bilinear matrix inequalities (BMI) problems, which means the LMI conditions that contain the bilinear terms. The approach first uses a first-order perturbation to linearize the BMI problems. By solving a semidefinite problem (SDP), the perturbation is computed, and the controller's performance is improved slightly. The program should repeat the process until the system achieves the desired performance or the performance cannot further be improved. In other words, to achieve the desired performance, the program solves a series of linearized problems. These problems improve the control results when each step that solves the linearized problems. The approach starts from an initial situation, and better and better designs the controller by modifying the design objects slowly. Because these objectives are closed in consecutive problems, the BMI can be converted into LMI constraints, which can be solved at each step. One thing that has to be noted is that this approach does not guarantee convergence, which means that the solution is not always acceptable.

Another technique considered in this thesis is the membership function's time derivatives [63]- [64]. Since it contains the differential of fuzzy summation in stabilization, the analysis of the membership functions' time derivatives must be concerned. This technique is applied for local stabilization analysis. That is, the membership function has the upper bound or lower bound. The differentials of membership function are divided into two parts, the differentials by the state vector and the differential of the time. In the differential of the time, it can be seen as a function of a fuzzy model. To deal with the problem, the equation of the differential of membership functions will be separated into two parts, common factors part and the rest. Because that the design methodology is locally constructed, this rest part contains the upper and lower bounds. Therefore, it can be extracted by sector nonlinearity techniques.

Polynomial FMB control system designs are applied by descriptor form methodologies in

this thesis. Transforming the closed-loop system of polynomial fuzzy model into descriptor representation is the initial step in this design methodology. Besides, by our greatest efforts of searching the information, no study applies such the descriptor form approach for designing polynomial FMB control system. As mentioned before, polynomial fuzzy descriptor systems [50]- [52] which directly uses the fuzzy descriptor system is different from the method this thesis proposed. The design methodology is similar to [35]. The so-called "Fuzzy Descriptor system" is presented as an extension of T-S fuzzy model. When the T-S fuzzy model got much attention in nonlinear control frameworks for its ability to present the nonlinear system into a general form, the descriptor system also got famous for the property that is different from the state-space expression. Similar to descriptor expression, the descriptor system [53] can describe the system more widely and tighter than state-space expression, too. The fuzzy descriptor system presents the rule consequence of descriptor, which means that the consequent part of T-S fuzzy system was represented as descriptor form. This methodology also contains T-S fuzzy model as the special case. In [35], the fuzzy descriptor system adds a summation of premise variables multiplied by nonsingular matrices to T-S fuzzy model. Note that the premise variables are independent of the control input for preventing the complicated defuzzification. Same as the work of the descriptor systems or the thesis, a new vector containing the state vector is defined, and the closed-loop system is rewritten. The control design was also extended. A modified PDC controller including premise variables and the vectors of feedback gain is applied, but the system needs to calculate the local feedback gains. The LMI approach was applied for stabilization in [35] with the common Lyapunov function. In the first method of the thesis, the closed-loop system and the decision variables' matrices' structures are similar to the stabilization analysis of [35]. However, the thesis considers a state-space expressed polynomial model with a different controller and "rewrites" it as descriptor form, which is different from this method. Fuzzy descriptor system contains two features when compared with state-space fuzzy models. It can describe a wider class of system or nondynamic constraints and is tighter for representing real independent parametric perturbations. The other is that the stabilization constraints can be reduced due to the redundancy of descriptor representation.

The descriptor approach has been widely used for T-S fuzzy model's system design [54]- [57]. No matter it is the fuzzy system, or other system's design [58]- [60], they can only deal with the systems which contain only constant matrices. In contrast, the thesis adds the

polynomial term to the consequent part of the system, which makes it possible to deal with the systems that contain the polynomial matrices.

This thesis presents a stabilization control design for descriptor's representing of polynomial FMB system. Four design methodologies are proposed, and all the design methodologies are constructed in the operation domain. Thus, their stabilization analysis is local stabilization, and the stabilization conditions are presented in SOS conditions.

At first, a rational control design is proposed. The thesis proposes a polynomial fuzzy model with the controller, which considers the rational functions. From [61], some cases show that the rational controller can have better performance than PDC-based controller. The closed-loop system, which polynomial FMB system applied by the rational controller, is represented in the descriptor form. A homogeneous functions' method is presented in this thesis for stabilization. The Lyapunov function candidate is chosen as the homogeneous Lyapunov function in which the matrix of decision variables is a homogeneous matrix. Considering the properties of Euler's homogeneity relation makes the differential of the Lyapunov function be able to be extracted. The rest part of the stabilization analysis is done by considering the stabilization method of the descriptor design methodology for T-S fuzzy model. However, the bilinear term appears in the stabilization conditions and makes it impossible to be solved by SOSTOOL directly. Therefore, this thesis applies the path-following approach [62] to solve the conditions. An example shows the comparison between the first proposed method and the polynomial fuzzy model without descriptor form [37]. The result has been proven that the proposed method obtains more relaxed results when in the same operation domain.

Based on the fact that the optimal result for solving the stabilization constraints may not be found by using path-following, the thesis has tried other descriptor form design methodology for polynomial FMB control design. The controller is chosen as PDC-based controller, which shares the same membership functions of the fuzzy model.

In the second method, a polynomial fuzzy model with PDC-based controller is presented and also be transformed into the descriptor form. The Lyapunov function candidate for stabilization is chosen as a Lyapunov function which is used in the research for polynomial fuzzy models commonly. The stabilization is extracted by considering some definition related to some vectors of the membership functions and state-space. Also, the properties of congruence transformation are considered for stabilization analysis. Compared with the polynomial fuzzy model without descriptor form, the matrices' dimension in the proposed method is

higher than the previous study in the same operation domain, though. The number of the SOS constraints from the proposed method is smaller than the previous study drastically. The contrast shows that the proposed method is more suitable with the cases that contain more rules, but the dimension of the state vector should be small. Moreover, compared with the method in the first method, the second method contains no bilinear or nonconvex terms, which means that the proposed method does not need to use the path-following algorithm to solve the conditions. Two examples are provided, including a numerical example and an application example. The numerical shows that the feasibility (relaxation) of the proposed approach is similar to the existing polynomial FMB control design approach [37], though. The smaller number of the constraints means that the proposed method still holds the advantage when compared with the existing polynomial FMB control design approach. The application example gives a bicycle dynamic system. The proposed method has been successfully made the system asymptotically stable by setting the operation domain of bicycle's angle and angular speed.

The rest two methods consider the same model, controller and Lyapunov function as the second method. The improvement is that it brings the matrices which contain fuzzy slack variables into the stabilization. The fuzzy slack matrices make the Lyapunov candidate be rewritten into a new form and produce a new stabilization analysis. The SOS conditions born from the stabilization analysis contain an issue like the co-positivity problem, the double fuzzy summation. Thus, applying the co-positive relaxation can be a way to deal with the double fuzzy summation. A numerical example is presented to make the comparison with [37], the descriptor design methodology for T-S fuzzy system [23], and the third method. It can be found out that the third method's results are more relaxed when making the comparison with [37] and [23] in some cases.

Furthermore, in some cases that membership functions have no relationship with the inputs states (i.e., the elements in the membership functions are the vectors which are corresponded to the system matrices' zero rows), the thesis proposes the novel fuzzy Lyapunov function for the fourth stabilization design approach. This kind of Lyapunov function contains an inverse of fuzzy summation matrix. Since the differential of the Lyapunov function contains membership function's differential terms in stabilization, the thesis also proposes a method to extract the time derivative of membership function. After extracting the differential of membership function, the state vectors' partially differential part would be extracted

by the techniques of sector nonlinearity. Because the third method can be seen as a special case included in the fourth proposed method, the fourth method is always more relaxed than the third method. The thesis presents two numerical examples to show the comparison with the fourth method, the previous method, and previous studies. The first one is a polynomial example, and the comparison is to compare with the fourth method, [37], and the third method. The results show that the fourth method gets the best relaxation from them. The second one is an example with constant matrices to compare with the fourth method, [37], the third method, and [23]. The fourth method has also gotten the best relaxation result. At the end of Chapter 5, three numerical examples are presented. The first two examples are the polynomial examples to show the comparison with all the proposed methods in this thesis and [37]. The final one is a constant example to show the comparison with [23], [37], and all the proposed methods in this thesis.

To summarize the contributions of the thesis, four points are presented as follows:

- The descriptor representation methodology for polynomial FMB design has no similar works by our greatest efforts of searching the information.
- More relaxed results are obtained from descriptor representation when comparing the state-space representation. The conditions of stabilization are also reduced.
- Taking the redundancy, this research brings fuzzy slack variables into stabilization control design.
- Novel fuzzy Lyapunov function for stabilization is applied as LMI based fuzzy descriptor systems' fuzzy Lyapunov function's extension.

The research also considers the differential of the membership functions and uses the technique of sector nonlinearity to extract the membership function's partially differentiated part by state vectors after the time differential process is extracted.

PRELIMINARIES

Chapter 2 is consists of some necessary mathematical tools and basic definitions which are applied in the research. Note that in the rest chapters (i.e. Chapter 3, 4, 5, and 6) of the thesis, the respect to time t will be dropped to simplify the notation. In addition, the Theorems proposed in the thesis are all constructed in the operation domain.

$$D_{op} = \{\mathbf{x}(t) : x_k^{\min} \leq x_k(t) \leq x_k^{\max}, k = 1, \dots, n\} \quad (2.1)$$

containing $\mathbf{x} = 0$.

2.1 Definitions

This section introduces the concepts, models, and matrices that will be used in the presented theorems in the thesis.

2.1.1 Positive Definiteness

A positive definite $\mathbf{A} \in \mathbb{R}^{n \times n}$'s definition is hold if and only if

$$\begin{aligned} \mathbf{x}^T \mathbf{A} \mathbf{x} &> 0, \forall \mathbf{x} \neq 0, \\ \lambda &> 0, \text{ when } \mathbf{A} \text{ is symmetric,} \\ \frac{\mathbf{A} + \mathbf{A}^T}{2} &\succ 0, \\ \mathbf{A} &= \mathbf{L} \mathbf{L}^T \end{aligned} \quad (2.2)$$

where λ is \mathbf{A} 's eigenvalue and \mathbf{L} is a nonsingular matrix. A positive definite matrix's determinant is always positive, which means that if a matrix is positive definite matrix, than it will also be "nonsingular".

Congruence Transformation

The relationship of congruent between two matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{S}^n$ is hold if a nonsingular matrix $\mathbf{L} \in \mathbb{R}^{n \times n}$ is found such that

$$\mathbf{Y} = \mathbf{L}^T \mathbf{X} \mathbf{L}.$$

Proof:

If \mathbf{X} is positive definite, $\mathbf{x}^T \mathbf{X} \mathbf{x} > 0, \forall \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0$. Because \mathbf{X} and \mathbf{Y} hold congruent relation, a nonsingular matrix \mathbf{L} which makes $\mathbf{Y} = \mathbf{L}^T \mathbf{X} \mathbf{L}$ is exist. Hence, for all $\mathbf{x} \neq 0$

$$\mathbf{y} = \mathbf{L}^{-1} \mathbf{x} \neq 0$$

$$\mathbf{X} \succ 0 \iff \mathbf{x}^T \mathbf{X} \mathbf{x} = \mathbf{y}^T \mathbf{L}^T \mathbf{X} \mathbf{L} \mathbf{y} = \mathbf{y}^T \mathbf{Y} \mathbf{y} \iff \mathbf{Y} \succ 0.$$

2.1.2 T-S Fuzzy model

The definition of Takagi-Sugeno fuzzy model is presented as:

Model rule i

If $z_1(t)$ is M_{i1} and \dots and $z_p(t)$ is M_{ip} (2.3)

then $\dot{\mathbf{x}}(t) = \mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t)$

$$i = 1, 2, \dots, r$$

where $\mathbf{A}_i \in \mathbb{R}^{n \times n}$ and $\mathbf{B}_i \in \mathbb{R}^{n \times m}$ are system matrices; r is the number of fuzzy rules; $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector; z_i is the known premise variable; and $\mathbf{u}(t) \in \mathbb{R}^m$ is the input vector. The polynomial fuzzy model (2.3) is inferred as

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^r h_i(\mathbf{z}(t)) \{ \mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t) \} \quad (2.4)$$

where $\mathbf{z}(t) = [z_1(t) \ \dots \ z_p(t)]$ and

$$h_i(\mathbf{z}(t)) = \frac{\prod_{j=1}^p M_{ij}(z_j(t))}{\sum_{k=1}^r \prod_{j=1}^p M_{kj}(z_j(t))}$$

with the following properties:

$$\begin{cases} \sum_{i=1}^r h_i(\mathbf{z}(t)) = 1, \\ h_i(\mathbf{z}(t)) > 0 \quad \forall i. \end{cases} \quad (2.5)$$

For $\mathbf{u} = 0$, use the quadratic Lyapunov function $\mathbf{x}^T \mathbf{P} \mathbf{x}$ can obtain the open loop system (2.4)'s stabilization criteria and are presented as follow:

$$\mathbf{P} > 0 \quad (2.6)$$

$$- \mathbf{A}_i^T \mathbf{P} - \mathbf{P} \mathbf{A}_i > 0 \quad (2.7)$$

which is shown as LMI problems.

2.1.3 Polynomial Fuzzy model

Consider the following polynomial fuzzy model:

Model rule i

If $z_1(t)$ is M_{i1} and \dots and $z_p(t)$ is M_{ip} (2.8)

then $\dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)$

$$i = 1, 2, \dots, r$$

where $\mathbf{A}_i(\mathbf{x}(t)) \in \mathbb{R}^{n \times N}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \mathbb{R}^{n \times m}$ are system matrices of polynomials; r is the number of fuzzy rules; $\mathbf{x}(t) \in \mathbb{R}^n$ denotes the state vector; $\hat{\mathbf{x}}(\mathbf{x}(t)) \in \mathbb{R}^N$ is a column vector consist of monomials in $\mathbf{x}(t)$ and has the property that $\hat{\mathbf{x}}(\mathbf{x}(t)) = 0$ iff $\mathbf{x}(t) = 0$; $z_i(t)$ is the known premise variable; and $\mathbf{u}(t) \in \mathbb{R}^m$ is the input vector. The polynomial fuzzy model (2.8) is inferred as

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^r h_i(\mathbf{z}(t)) \{ \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) \} \quad (2.9)$$

where $\mathbf{z}(t) = [z_1(t) \ \dots \ z_p(t)]$ and

$$h_i(\mathbf{z}(t)) = \frac{\prod_{j=1}^p M_{ij}(z_j(t))}{\sum_{k=1}^r \prod_{j=1}^p M_{kj}(z_j(t))}$$

with the following properties:

$$\begin{cases} \sum_{i=1}^r h_i(\mathbf{z}(t)) = 1, \\ h_i(\mathbf{z}(t)) > 0 \quad \forall i. \end{cases} \quad (2.10)$$

The stabilization criteria for the model (2.9) can be obtained by employing the polynomial Lyapunov function and are represented as SOS terms. Due to the fact that such the Lyapunov function contains the quadratic Lyapunov function employed by T-S fuzzy model, Polynomial fuzzy model has more general and relaxed stability and conditions than the T-S fuzzy model.

2.1.4 Sum of Squares Decomposition

For $\mathbf{x} \in \mathbb{R}^n$ and $i = 1, \dots, r$, a multivariate polynomial $p(\mathbf{x})$ is called "sum of squares" if and only if there exist polynomials $m_i(\mathbf{x})$ satisfying the following equation

$$p(\mathbf{x}) = \sum_{i=1}^r m_i^2(\mathbf{x}). \quad (2.11)$$

For the matrices' case, $p(\mathbf{x})$ is called sum of squares if and only if a monomials vector $\mathbf{U}(\mathbf{x})$ and a positive semidefinite matrix \mathbf{P} are exist such that

$$p(\mathbf{x}) = \mathbf{U}^T(\mathbf{x})\mathbf{P}\mathbf{U}(\mathbf{x}). \quad (2.12)$$

The sum of squares program tries to find the polynomial $p_i(\mathbf{x})$, $i = 1, \dots, \hat{r}$ and sum of squares $p_i(\mathbf{x})$, $i = (\hat{r} + 1), \dots, r$ such that

$$a_{0j}(\mathbf{x}) + \sum_{i=1}^r p_i(\mathbf{x})a_{ij}(\mathbf{x}) = 0 \quad i = j, \dots, \hat{r} \quad (2.13)$$

$$a_{0j}(\mathbf{x}) + \sum_{i=1}^r p_i(\mathbf{x})a_{ij}(\mathbf{x}) \text{ are SOS} \quad j = (\hat{r} + 1), \dots, r \quad (2.14)$$

where $a_{ij}(\mathbf{x})$ are some scalar constant coefficient polynomials.

2.1.5 The Transform Matrix

Because $\hat{\mathbf{x}}(\mathbf{x}(t))$ is a vector which is consist of monomials that have the property of $\hat{\mathbf{x}}(\mathbf{x}(t)) = 0$ iff $\mathbf{x}(t) = 0$, there always exists a transformation matrix $\mathbf{T}(\mathbf{x}(t))$ that makes

$\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{T}(\mathbf{x}(t))\mathbf{x}(t)$. When monomial vectors are equal to the state vector (i.e. $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{x}(t)$), we have $\mathbf{T}(\mathbf{x}(t)) = \mathbf{I}$. Moreover, consider that $\mathbf{x}(t) = [x_1(t) \ x_2(t)]^T$, the thesis gives the following example to show how $\mathbf{T}(\mathbf{x}(t))$ works:

$$\begin{aligned} \hat{\mathbf{x}}(\mathbf{x}(t)) &= \begin{bmatrix} x_1^2(t) \\ x_2^2(t) \end{bmatrix} & \text{with } \mathbf{T}(\mathbf{x}(t)) &= \begin{bmatrix} x_1(t) & 0 \\ 0 & x_2(t) \end{bmatrix} \\ \hat{\mathbf{x}}(\mathbf{x}(t)) &= \begin{bmatrix} x_1^4(t) \\ x_2^2(t) \end{bmatrix} & \text{with } \mathbf{T}(\mathbf{x}(t)) &= \begin{bmatrix} x_1^3(t) & 0 \\ 0 & x_2(t) \end{bmatrix} \\ \hat{\mathbf{x}}(\mathbf{x}(t)) &= \begin{bmatrix} x_1^4(t) \\ x_2^4(t) \end{bmatrix} & \text{with } \mathbf{T}(\mathbf{x}(t)) &= \begin{bmatrix} x_1^3(t) & 0 \\ 0 & x_2^3(t) \end{bmatrix}. \end{aligned}$$

2.2 Mathematical tools

This section introduces the algorithms and methods of relaxation that will be used in the presented theorems in the thesis.

2.2.1 Euler's homogeneity relation

Consider a function $V(\mathbf{y})$, which is define in $\mathbb{R}^n \rightarrow \mathbb{R}$. $V(\mathbf{y})$ is said to be a homogeneous function with degree $g \in I^+$ if and only if

$$gV(\mathbf{y}) = \mathbf{y}^T \nabla_{\mathbf{y}} V(\mathbf{y}) = \nabla_{\mathbf{y}} V(\mathbf{y})^T \mathbf{y}. \quad (2.15)$$

The proof of the relation above follows by differentiation of the homogeneous Lyapunov function

$$V(\gamma \mathbf{y}) = \gamma^g V(\mathbf{y}) \quad (2.16)$$

by setting $\gamma = 1$

2.2.2 Path-Following Algorithm

From [62], the purpose of the path-following algorithm is to deal with the bilinear terms in stability conditions. [62] presents the bilinear matrix inequality (BMI) case, which means

the LMI conditions containing the bilinear term. Consider the dynamic system

$$\dot{\mathbf{x}}(t) = \mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C} \mathbf{x}(t) \quad (2.17)$$

where $\mathbf{u}(t) = \delta \mathbf{F} \mathbf{y}(t)$ and there is the decay rate of $\alpha \in \mathbb{R}$ in the open-loop system. To find the feedback gain, the stability conditions are designed as

$$\mathbf{P} \succ 0 \quad (2.18)$$

$$|\delta \mathbf{F}_{ij}| \leq l_{ij} \quad (2.19)$$

$$(\mathbf{A} + \mathbf{BFC})^T \mathbf{P} + \mathbf{P}(\mathbf{A} + \mathbf{BFC}) \preceq -2(\alpha + \delta\alpha) \mathbf{P} \quad (2.20)$$

which contains the bilinear terms of \mathbf{P} and $\delta \mathbf{F}$. Since $\dot{\mathbf{x}}(t) = \mathbf{A}_i \mathbf{x}(t)$ has the decay rate, the following condition can be compute

$$\mathbf{A}^T \mathbf{P}_0 + \mathbf{P}_0 \mathbf{A} \preceq -2\alpha \mathbf{P}_0. \quad (2.21)$$

By writing $\delta \mathbf{P} = \mathbf{P} - \mathbf{P}_0$, we have

$$\mathbf{P}_0 + \delta \mathbf{P} \succ 0 \quad (2.22)$$

$$|\delta \mathbf{F}_{ij}| \leq l_{ij} \quad (2.23)$$

$$(\mathbf{A} + \mathbf{BFC})^T (\mathbf{P}_0 + \delta \mathbf{P}) + (\mathbf{P}_0 + \delta \mathbf{P})(\mathbf{A} + \mathbf{BFC}) \preceq -2(\alpha + \delta\alpha)(\mathbf{P}_0 + \delta \mathbf{P}). \quad (2.24)$$

Because $\delta \mathbf{P}$, $\delta\alpha$, and $\delta \mathbf{F}$ are very small, the third condition can be rewritten as

$$\mathbf{A}^T (\mathbf{P}_0 + \delta \mathbf{P}) + (\mathbf{P}_0 + \delta \mathbf{P}) \mathbf{A} + (\mathbf{BFC})^T \mathbf{P}_0 + \mathbf{P}_0 (\mathbf{BFC}) \preceq -2\alpha (\mathbf{P}_0 + \delta \mathbf{P}) - 2\delta\alpha \mathbf{P}_0 \quad (2.25)$$

which can be solved by LMI. Back to the dynamic system (2.17), if $\mathbf{u} = \mathbf{F} \mathbf{y}$, the path-following is shown as follows:

step 1: Decide the initial value of \mathbf{F} .

step 2: Find the minimum value of α by solving the problems

$$\mathbf{P} \succ 0 \quad (2.26)$$

$$\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} \preceq -2\alpha \mathbf{P}_0. \quad (2.27)$$

step 3: Apply the value of \mathbf{P} from *step 2*, and solve the following conditions

$$\mathbf{P} + \delta\mathbf{P} \succ 0 \quad (2.28)$$

$$\mathbf{A}^T(\mathbf{P} + \delta\mathbf{P}) + (\mathbf{P} + \delta\mathbf{P})\mathbf{A} + (\mathbf{BFC})^T\mathbf{P} + \mathbf{P}(\mathbf{BFC}) \preceq -2\alpha(\mathbf{P} + \delta\mathbf{P}) - 2\delta\alpha\mathbf{P} \quad (2.29)$$

to find the minimum value of α .

step 4: Set $\mathbf{F} = \mathbf{F} + \delta\mathbf{F}$ and $\mathbf{P} = \mathbf{P} + \delta\mathbf{P}$ and go back to *step 2*.

The system stops the loop until the α exceed the value that the user want or α cannot be further improved. The bilinear problem will turn into linear step by step in the stabilization conditions by applying path-following method, though. The solution of this algorithm cannot guarantee convergence.

2.2.3 Co-positive Relaxation

Consider a matrix $\mathbf{W} = [W_{ij}] \in \mathbb{R}^{r \times r}$. Checking the co-positivity of \mathbf{W} is to check if

$$\mathbf{q}^T \mathbf{W} \mathbf{q} = \sum_{i=1}^r \sum_{j=1}^r q_i q_j W_{ij} \geq 0 \quad (2.30)$$

for all $\mathbf{q} = [q_1, q_2, \dots, q_r]^T \in \mathbb{R}$, $q_i \geq 0$. Let $q_i = \hat{q}_i^2$, then the checking equation above means to check the condition

$$\mathbf{Z}^s(\hat{\mathbf{q}}) = \left(\sum_{k=1}^r \right)^s \sum_{i=1}^r \sum_{j=1}^r \hat{q}_i^2 \hat{q}_j^2 W_{ij} \text{ is SOS} \quad (2.31)$$

where $\hat{\mathbf{q}} = [\hat{q}_1, \hat{q}_2, \dots, \hat{q}_r]^T$, and s is a non-negative integer.

2.2.4 Membership function time derivative

When all membership functions are not related to the inputs' states (i.e. $h_\rho(\mathbf{z}(t)) = h_\rho(\tilde{\mathbf{x}}(t)) \forall \rho$), the membership function time derivative could be represented as

$$\begin{aligned} \dot{h}_\rho(\tilde{\mathbf{x}}(t)) &= \frac{\partial h_\rho(\tilde{\mathbf{x}}(t))}{\partial \mathbf{x}(t)} \dot{\mathbf{x}}(t) \\ &= \frac{\partial h_\rho(\tilde{\mathbf{x}}(t))}{\partial \mathbf{x}(t)} \sum_{i=1}^r h_i(\tilde{\mathbf{x}}(t)) \mathbf{A}_i(\mathbf{x}(t)) \hat{\mathbf{x}}(\mathbf{x}(t)). \end{aligned} \quad (2.32)$$

(2.32) are divided in two parts, a polynomial common factors part, which is denoted in $O_\rho(\mathbf{x}(t))$, and the rest part, which is denoted in $y_\rho(\mathbf{x}(t))$. Then (2.32) is simplified to the form

$$\dot{h}_\rho(\tilde{\mathbf{x}}(t)) = y_\rho(\mathbf{x}(t))O_\rho(\mathbf{x}(t)). \quad (2.33)$$

$y_\rho(\mathbf{x}(t))$ is the part that will be applied the technique of sector nonlinearity, after extracting with the technique, it can be rewritten as

$$y_\rho(\mathbf{x}(t)) = \sum_{m=1}^2 \omega_{\rho m}(\mathbf{x}(t))C_{\rho m} \quad (2.34)$$

where

$$C_{\rho 1} = \max_{\mathbf{x}(t) \in D_{op}} y_\rho(\mathbf{x}(t)), \quad C_{\rho 2} = \min_{\mathbf{x}(t) \in D_{op}} y_\rho(\mathbf{x}(t))$$

$$\omega_{\rho 1}(\mathbf{x}(t)) = \frac{y_\rho(\mathbf{x}(t)) - C_{\rho 2}}{C_{\rho 1} - C_{\rho 2}}, \quad \omega_{\rho 2}(\mathbf{x}(t)) = \frac{C_{\rho 1} - y_\rho(\mathbf{x}(t))}{C_{\rho 1} - C_{\rho 2}}$$

with the following properties:

$$\omega_{\rho m}(\mathbf{x}(t)) > 0, \quad \sum_{m=1}^2 \omega_{\rho m}(\mathbf{x}(t)) = 1.$$

By substituting (2.34) into (2.33), the membership function time derivative are represented as

$$\dot{h}_\rho(\tilde{\mathbf{x}}(t)) = \sum_{m=1}^2 \omega_{\rho m}(\mathbf{x}(t))\mu_{\rho m}(\mathbf{x}(t))$$

with $\mu_{\rho m}(\mathbf{x}(t)) = C_{\rho m}O_\rho(\mathbf{x}(t))$.

For example, for $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{x}(t) = [x_1(t) \ x_2(t)]^T$, consider a polynomial fuzzy model (2.8) with three rules, the system matrices, and membership functions as follows:

$$A_1 = \begin{bmatrix} x_1^2(t) + x_2^2(t) & x_1(t) + x_2(t) \\ 0.25 & 0.25 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3x_1(t)x_2(t) & -x_1(t) + x_2(t) \\ 0.25 & 0.25 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 + x_1(t)1 + x_1^2(t) & -4 \\ 0.25 & 0.25 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} x_2^2(t) \\ 0 \end{bmatrix}$$

$$h_1(x_2(t)) = \frac{1 + \sin(x_2(t))}{3}, \quad h_2(x_2(t)) = h_3(x_2(t)) = \frac{2 - \sin(x_2(t))}{6}$$

Then $\tilde{\mathbf{x}}(t) = x_2(t)$ and the time derivative of $h_1(x_2(t))$ can be obtained as

$$\begin{aligned}\dot{h}_1(x_2(t)) &= \frac{\partial h_1(x_2(t))}{\partial x_2}(t) \dot{x}_2(t) \\ &= \frac{\partial h_1(x_2(t))}{\partial x_2(t)} \sum_{i=1}^r h_i(x_2(t)) \mathbf{A}_i^2(\mathbf{x}(t)) \hat{\mathbf{x}}(\mathbf{x}(t)) \\ &= \frac{\cos(x_2(t))}{3} \times 0.25 \times (x_1(t) + x_2(t)).\end{aligned}$$

Rewrite $\dot{h}_1(x_2(t))$ into the form of (2.33), it can be obtained that

$$O_1(\mathbf{x}(t)) = \frac{x_1(t) + x_2(t)}{12}, \quad y_1(\mathbf{x}(t)) = \cos(x_2(t)).$$

This example is assumed to be constructed in the following operation domain:

$$D_{op} = \{\mathbf{x} : -\pi \leq x_k \leq \pi, \quad k = 1, 2\}.$$

Apply the technique of sector nonlinearity to $y_1(\mathbf{x}(t))$, and obtains the result that

$$y_1(\mathbf{x}(t)) = \sum_{m=1}^2 \omega_{1m}(\mathbf{x}(t)) C_{1m}$$

where

$$\begin{aligned}C_{11} &= \max_{\mathbf{x}(t) \in D_{op}} y_1(\mathbf{x}(t)) = 1, \quad C_{12} = \min_{\mathbf{x}(t) \in D_{op}} y_1(\mathbf{x}(t)) = -1 \\ \omega_{11}(\mathbf{x}(t)) &= \frac{y_1(\mathbf{x}(t)) - C_{12}}{C_{11} - C_{12}}, \quad \omega_{12}(\mathbf{x}(t)) = \frac{C_{11} - y_1(\mathbf{x}(t))}{C_{11} - C_{12}}.\end{aligned}$$

Finally, the decomposition of $\dot{h}_1(x_2(t))$ can be implemented as

$$\dot{h}_1(\tilde{\mathbf{x}}(t)) = \sum_{m=1}^2 \omega_{1m}(\mathbf{x}(t)) \mu_{1m}(\mathbf{x}(t))$$

with

$$\begin{aligned}\mu_{11}(\mathbf{x}(t)) &= C_{11} O_1(\mathbf{x}(t)) = \frac{x_1(t) + x_2(t)}{12} \\ \mu_{12}(\mathbf{x}(t)) &= C_{12} O_1(\mathbf{x}(t)) = -\frac{x_1(t) + x_2(t)}{12}.\end{aligned}$$

The above steps are also applied to the rest membership function to decompose their differentials.

A POLYNOMIAL FUZZY DESCRIPTOR SYSTEM APPROACH FOR RATIONAL FUZZY CONTROL DESIGN

This chapter proposed a rational method for polynomial FMB control design. This chapter first presents a polynomial fuzzy model. Second, a controller composed of the polynomial rational function is considered. The model and the controller form the closed-loop systems and are represented like a descriptor system. The stabilization analysis uses Lyapunov theory and homogeneous functions. Because of the polynomials, the stabilization conditions are represented in SOS instead of LMI terms. The stabilization analysis produces bilinear terms in the conditions. Thus, the path-following algorithm's technique is applied to solve the stabilization conditions.

3.1 Rational Controller and Closed-loop System

This chapter introduces a controller combined with rational functions. The rational functions contain the polynomial matrices and polynomial functions. Similar to the PDC controller, the elements in the rational functions share polynomial fuzzy model (2.9)'s membership functions. Applying the controller to the model (2.9) can obtain a closed-loop system. The structure of rational function makes it possible to rewrite the closed-loop system as descriptor form like [35].

Consider a rational controller which is shown as follow

$$\mathbf{u} = \frac{\sum_{i=1}^r h_i(\mathbf{z}) \mathbf{F}_i(\mathbf{x})}{\sum_{j=1}^r h_j(\mathbf{z}) n_j(\mathbf{x})} \hat{\mathbf{x}}(\mathbf{x}) \quad (3.1)$$

where the control feedback gain are $\mathbf{F}_i(\mathbf{x}) \in \mathbb{R}^{m \times N}$ for \mathbf{x} and $n_j(\mathbf{x})$ is the polynomial function in \mathbf{x} . By applying the controller (3.1) to (2.9), the closed-loop system will be

$$\dot{\mathbf{x}} = \sum_{i=1}^r h_i(\mathbf{z}) \left\{ \mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x}) \frac{\sum_{j=1}^r h_j(\mathbf{z}) \mathbf{F}_j(\mathbf{x})}{\sum_{k=1}^r h_k(\mathbf{z}) n_k(\mathbf{x})} \right\} \hat{\mathbf{x}}(\mathbf{x}). \quad (3.2)$$

Multiplying $\sum_{k=1}^r h_k(\mathbf{z}) n_k(\mathbf{x})$ on both side of (3.2) makes it be

$$\sum_{i=1}^r h_i(\mathbf{z}) n_i(\mathbf{x}) \dot{\mathbf{x}} = \sum_{i=1}^r \sum_{j=1}^r h_i(\mathbf{z}) h_j(\mathbf{z}) \{ n_j(\mathbf{x}) \mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x}) \mathbf{F}_j(\mathbf{x}) \} \hat{\mathbf{x}}(\mathbf{x}). \quad (3.3)$$

Define a vector $\mathbf{x}^\#(\mathbf{x}) = [\hat{\mathbf{x}}(\mathbf{x}) \ \dot{\mathbf{x}}]^T$ to simplify (3.3) and we have

$$\mathbf{E}^* \dot{\mathbf{x}}^\#(\mathbf{x}) = \sum_{i=1}^r \sum_{j=1}^r h_i(\mathbf{z}) h_j(\mathbf{z}) G_{ij}^\#(\mathbf{x}) \mathbf{x}^\#(\mathbf{x}) \quad (3.4)$$

where

$$\mathbf{E}^* = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{G}_{ij}^\#(\mathbf{x}) = \begin{bmatrix} 0 & \mathbf{I} \\ n_j(\mathbf{x}) \mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x}) \mathbf{F}_j(\mathbf{x}) & -n_j(\mathbf{x}) \mathbf{I} \end{bmatrix}.$$

The equation (3.4) is a closed-loop system written in descriptor form.

3.2 Main Result

This section shows the stabilization analysis for (3.4). The research uses a polynomial homogeneous Lyapunov function here. By using the homogeneous function's properties and the proving steps of the fuzzy descriptor system, stabilization can be achieved. The stabilization conditions is presented in SOS terms.

Theorem 1:

Consider a positive definite symmetric homogeneous polynomial matrix $\mathbf{Z}_1(\mathbf{x})$, polynomial matrices $\mathbf{Z}_3(\mathbf{x})$ and $\mathbf{M}_j(\mathbf{x})$, polynomials $\sigma_{ij\beta}(\mathbf{x})$ and $n_j(\mathbf{x})$ and a scalar $\alpha < 0$. The closed-loop system (3.3) is asymptotically stable if the following conditions are satisfied.

$$\mathbf{v}_{z_1}^T (\mathbf{Z}_1(\mathbf{x}) - \epsilon(\mathbf{x}) \mathbf{I}) \mathbf{v}_{z_1} \text{ is SOS} \quad (3.5)$$

$$- \mathbf{v}^T \mathbf{L}_{ii}^\alpha(\mathbf{x}) \mathbf{v} \text{ is SOS, } i = 1, \dots, r \quad (3.6)$$

$$-\mathbf{v}^T(\mathbf{L}_{ij}^\alpha(\mathbf{x}) + \mathbf{L}_{ji}^\alpha(\mathbf{x}))\mathbf{v} \text{ is SOS, } i < j \leq r \quad (3.7)$$

$$\sigma_{ij\beta}(\mathbf{x}) \text{ is SOS, } i = 1, \dots, r, j = 1, \dots, r, \beta = 1, \dots, n \quad (3.8)$$

where \mathbf{v} and \mathbf{v}_{z_1} are vectors which is independent from \mathbf{x} , $\epsilon(\mathbf{x}) > 0$ when $\mathbf{x} \neq 0$, and

$$\mathbf{L}_{ij}^\alpha(\mathbf{x}) = \begin{bmatrix} \mathbf{L}_1^\alpha(\mathbf{x}) & * \\ \mathbf{L}_{3ij}(\mathbf{x}) & \mathbf{L}_{4j}(\mathbf{x}) \end{bmatrix}$$

in which

$$\mathbf{L}_1^\alpha(\mathbf{x}) = \mathbf{Z}_3^T(\mathbf{x}) + \mathbf{Z}_3(\mathbf{x}) - \alpha\mathbf{Z}_1(\mathbf{x}) - \sum_{\beta=1}^n \sigma_{ij\beta}(\mathbf{x})\mathbf{Q}_\beta(\mathbf{x})\mathbf{I}$$

$$\mathbf{L}_{3ij}(\mathbf{x}) = n_j(\mathbf{x})\mathbf{A}_i(\mathbf{x})\mathbf{Z}_1(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{M}_j(\mathbf{x}) - n_j(\mathbf{x})\mathbf{Z}_3(\mathbf{x}) + \mathbf{Z}_1(\mathbf{x})$$

$$\mathbf{L}_{4j}(\mathbf{x}) = -2n_j(\mathbf{x})\mathbf{Z}_1(\mathbf{x})$$

$$\mathbf{Q}_\beta(\mathbf{x}) = (x_\beta - x_\beta^{max})(x_\beta - x_\beta^{min}).$$

Solving the conditions can obtain the feedback gain as

$$\mathbf{F}_j(\mathbf{x}) = \mathbf{M}_j(\mathbf{x})\mathbf{Z}_1^{-1}(\mathbf{x}).$$

Proof:

Consider the Lyapunov-based analysis utilizing the following homogeneous Lyapunov function:

$$V(\mathbf{x}) = \mathbf{x}^T \text{adj}(\mathbf{Z}_1(\mathbf{x}))\mathbf{x}. \quad (3.9)$$

As mention before, $\mathbf{Z}_1(\mathbf{x})$ has the properties as positive definite, symmetric, homogeneous, and polynomial. Consider the Euler's homogeneity relation introduced in preliminaries part, we have

$$gV(\mathbf{x}) = g\mathbf{x}^T \text{adj}(\mathbf{Z}_1(\mathbf{x}))\mathbf{x} = \mathbf{x}^T \nabla_x V(\mathbf{x}). \quad (3.10)$$

Therefore, it is obtained that

$$g\text{adj}(\mathbf{Z}_1(\mathbf{x}))\mathbf{x} = \nabla_x V(\mathbf{x}). \quad (3.11)$$

The next step is to use the homogeneous Lyapunov function's stabilization analysis raised

by [45] and the stabilization approach for fuzzy descriptor system raised by [35], to do the stabilization. The differential of the polynomial homogeneous Lyapunov function (3.9) is

$$\begin{aligned}
 \dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T \nabla_x V(\mathbf{x}) = g \dot{\mathbf{x}}^T \text{adj}(\mathbf{Z}_1(\mathbf{x})) \mathbf{x} \\
 &= g [\dot{\mathbf{x}} \ \ddot{\mathbf{x}}] \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \text{adj}(\mathbf{Z}_1(\mathbf{x})) & 0 \\ \mathbf{S}_3(\mathbf{x}) & \text{adj}(\mathbf{Z}_1(\mathbf{x})) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix} \\
 &= g \dot{\mathbf{x}}^{\#T}(\mathbf{x}) \mathbf{E}^{*T} \mathbf{X}(\mathbf{x}) \mathbf{x}^{\#}(\mathbf{x}) \\
 &= \frac{g}{2} [\dot{\mathbf{x}}^{\#T}(\mathbf{x}) \mathbf{E}^{*T} \mathbf{X}(\mathbf{x}) \mathbf{x}^{\#}(\mathbf{x}) + \mathbf{x}^{\#T}(\mathbf{x}) \mathbf{E}^{*T} \mathbf{X}(\mathbf{x}) \dot{\mathbf{x}}^{\#}(\mathbf{x})] \\
 &= \frac{g}{2} \sum_{i=1}^r \sum_{j=1}^r h_i(z) h_j(z) \mathbf{x}^{\#T}(\mathbf{x}) \{ \mathbf{G}_{ij}^{\#T}(\mathbf{x}) \mathbf{X}(\mathbf{x}) + \mathbf{X}^T(\mathbf{x}) \mathbf{G}_{ij}^{\#}(\mathbf{x}) \} \mathbf{x}^{\#}(\mathbf{x}). \tag{3.12}
 \end{aligned}$$

Because $\mathbf{Z}_1(\mathbf{x})$ is a homogeneous matrix, it has such the property

$$\mathbf{Z}_1^{-1}(\mathbf{x}) = \frac{\text{adj}(\mathbf{Z}_1(\mathbf{x}))}{\det(\mathbf{Z}_1(\mathbf{x}))} \tag{3.13}$$

which means that

$$\mathbf{Z}_1^{-1}(\mathbf{x}) \mathbf{Z}_1(\mathbf{x}) = \frac{\text{adj}(\mathbf{Z}_1(\mathbf{x})) \mathbf{Z}_1(\mathbf{x})}{\det(\mathbf{Z}_1(\mathbf{x}))} = \mathbf{I}. \tag{3.14}$$

According to (3.13) and (3.14), we have

$$\text{adj}(\mathbf{Z}_1(\mathbf{x})) \mathbf{Z}_1(\mathbf{x}) = \det(\mathbf{Z}_1(\mathbf{x})). \tag{3.15}$$

To obtain that (3.12) is negative, such a condition below should be satisfied

$$\mathbf{G}_{ij}^{\#T}(\mathbf{x}) \mathbf{X}(\mathbf{x}) + \mathbf{X}^T(\mathbf{x}) \mathbf{G}_{ij}^{\#}(\mathbf{x}) \prec 0. \tag{3.16}$$

Define a matrix

$$\mathbf{R}(\mathbf{x}) = \begin{bmatrix} \mathbf{Z}_1(\mathbf{x}) & 0 \\ \mathbf{Z}_3(\mathbf{x}) & \mathbf{Z}_1(\mathbf{x}) \end{bmatrix}$$

where

$$\mathbf{Z}_3(\mathbf{x}) = \frac{1}{\det(\mathbf{Z}_1(\mathbf{x}))} \mathbf{Z}_1(\mathbf{x}) \mathbf{S}_3(\mathbf{x}) \mathbf{Z}_1(\mathbf{x})$$

and multiply the left side of (3.16) with $\mathbf{R}^T(\mathbf{x})$ and the right side with $\mathbf{R}(\mathbf{x})$, we have

$$\mathbf{R}^T(\mathbf{x}) \mathbf{G}_{ij}^{\#T}(\mathbf{x}) \mathbf{X}(\mathbf{x}) \mathbf{R}(\mathbf{x}) + \mathbf{R}^T(\mathbf{x}) \mathbf{X}^T(\mathbf{x}) \mathbf{G}_{ij}^{\#}(\mathbf{x}) \mathbf{R}(\mathbf{x}) \prec 0. \tag{3.17}$$

Using the properties (3.13) to (3.15) can obtain that

$$\mathbf{X}(\mathbf{x})\mathbf{R}(\mathbf{x}) = \det(\mathbf{Z}_1(\mathbf{x}))\mathbf{I}. \quad (3.18)$$

Therefore, (3.17) can be transformed into

$$\mathbf{R}^T(\mathbf{x})\mathbf{G}_{ij}^{\#T}(\mathbf{x})\det(\mathbf{Z}_1(\mathbf{x}))\mathbf{I} + \det(\mathbf{Z}_1(\mathbf{x}))\mathbf{I}\mathbf{G}_{ij}^{\#}(\mathbf{x})\mathbf{R}(\mathbf{x}) \prec 0. \quad (3.19)$$

Divide (3.19) with $\det(\mathbf{Z}_1(\mathbf{x}))$ and define a matrix like the below to simplify it, we have

$$\mathbf{L}_{ij}(\mathbf{x}) = \begin{bmatrix} \mathbf{L}_1(\mathbf{x}) & \mathbf{L}_{3ij}^T(\mathbf{x}) \\ \mathbf{L}_{3ij}(\mathbf{x}) & \mathbf{L}_{4j}(\mathbf{x}) \end{bmatrix} \prec 0 \quad (3.20)$$

where

$$\mathbf{L}_1(\mathbf{x}) = \mathbf{Z}_3^T(\mathbf{x}) + \mathbf{Z}_3(\mathbf{x})$$

$$\mathbf{L}_{3ij}(\mathbf{x}) = n_j(\mathbf{x})\mathbf{A}_i(\mathbf{x})\mathbf{Z}_1(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{M}_j(\mathbf{x}) - n_j(\mathbf{x})\mathbf{Z}_3(\mathbf{x}) + \mathbf{Z}_1(\mathbf{x})$$

$$\mathbf{L}_{4j}(\mathbf{x}) = -2n_j(\mathbf{x})\mathbf{Z}_1(\mathbf{x}).$$

Furthermore, for

$$\mathbf{L}_{ij}(\mathbf{x}) \preceq \begin{bmatrix} \alpha\mathbf{Z}_1(\mathbf{x}) & 0 \\ 0 & 0 \end{bmatrix}$$

we have

$$\mathbf{L}_{ij}(\mathbf{x}) - \begin{bmatrix} \alpha\mathbf{Z}_1(\mathbf{x}) & 0 \\ 0 & 0 \end{bmatrix} \preceq 0 \quad (3.21)$$

Assume that there exist polynomials $\sigma_{ij\beta}(\mathbf{x})$ and they are positive definite, than the following inequality should hold in the operation domain

$$\psi(\mathbf{z}, \mathbf{x}) = - \sum_{i=1}^r \sum_{j=1}^r \sum_{\beta=1}^m h_i(z)h_j(z)\sigma_{ij\beta}(\mathbf{x})Q_\beta \geq 0.$$

Because $\psi(\mathbf{z}, \mathbf{x})$ is semi-positive definite, (-1)*(3.21) is also semi-positive definite can be hold

if the following condition is satisfy

$$-\left(\mathbf{L}_{ij}(\mathbf{x}) - \begin{bmatrix} \alpha \mathbf{Z}_1(\mathbf{x}) & 0 \\ 0 & 0 \end{bmatrix}\right) \succeq \mathbf{E}^* \psi(\mathbf{z}, \mathbf{x}) \succeq 0. \quad (3.22)$$

By moving the right part to the left, we have

$$-\left(\mathbf{L}_{ij}(\mathbf{x}) - \begin{bmatrix} \alpha \mathbf{Z}_1(\mathbf{x}) & 0 \\ 0 & 0 \end{bmatrix} + \mathbf{E}^* \psi(\mathbf{z}, \mathbf{x})\right) \succeq 0. \quad (3.23)$$

Then define matrices $\mathbf{L}_{ij}^\alpha(\mathbf{x})$ to simply (3.23)

$$\mathbf{L}_{ij}^\alpha(\mathbf{x}) = \begin{bmatrix} \mathbf{L}_1^\alpha(\mathbf{x}) & \mathbf{L}_{3ij}^T(\mathbf{x}) \\ \mathbf{L}_{3ij}(\mathbf{x}) & \mathbf{L}_{4j}(\mathbf{x}) \end{bmatrix}, \text{ and } -\mathbf{L}_{ij}^\alpha(\mathbf{x}) \succeq 0 \quad (3.24)$$

where

$$\begin{aligned} \mathbf{L}_1^\alpha(\mathbf{x}) &= \mathbf{Z}_3^T(\mathbf{x}) + \mathbf{Z}_3(\mathbf{x}) - \alpha \mathbf{Z}_1(\mathbf{x}) - \sum_{\beta=1}^n \sigma_{ij\beta}(\mathbf{x}) \mathbf{Q}_\beta(\mathbf{x}) \mathbf{I} \\ \mathbf{L}_{3ij}(\mathbf{x}) &= n_j(\mathbf{x}) \mathbf{A}_i(\mathbf{x}) \mathbf{Z}_1(\mathbf{x}) + \mathbf{B}_i(\mathbf{x}) \mathbf{M}_j(\mathbf{x}) - n_j(\mathbf{x}) \mathbf{Z}_3(\mathbf{x}) + \mathbf{Z}_1(\mathbf{x}) \\ \mathbf{L}_{4j}(\mathbf{x}) &= -2n_j(\mathbf{x}) \mathbf{Z}_1(\mathbf{x}). \end{aligned}$$

In addition,

$$\begin{aligned} & \sum_{i=1}^r \sum_{j=1}^r h_i(z) h_j(z) \mathbf{x}^{\#T}(\mathbf{x}) \mathbf{L}_{ij}^\alpha(\mathbf{x}) \mathbf{x}^\#(\mathbf{x}) \\ &= \sum_{i=1}^r h_i(z) \mathbf{x}^{\#T}(\mathbf{x}) \mathbf{L}_{ii}^\alpha(\mathbf{x}) \mathbf{x}^\#(\mathbf{x}) + \sum_{j=1}^r \sum_{i<j} h_i(z) h_j(z) \mathbf{x}^{\#T}(\mathbf{x}) (\mathbf{L}_{ij}^\alpha(\mathbf{x}) + \mathbf{L}_{ji}^\alpha(\mathbf{x})) \mathbf{x}^\#(\mathbf{x}) \end{aligned}$$

leads the conditions (3.6) and (3.7). \square

3.3 Path Following

Since there are the bilinear terms in the matrices $\mathbf{L}_{ij}^\alpha(\mathbf{x})$, SOSOPT cannot solve the stabilization criteria directly. Thus, the research applies path-following approach for solving the conditions (3.5)-(3.8). The path-following steps are shown as follow:

Step 1: Set a constant $\eta = 0$. The polynomials $n_{j0}(\mathbf{x})$ which is positive definite and its coefficients are randomly decided is defined.

Step 2: Apply SOSOPT to solve:

$$\min_{\mathbf{Z}_1(\mathbf{x}), \mathbf{Z}_3(\mathbf{x}), \mathbf{M}_j(\mathbf{x}), \sigma_{ij\beta}(\mathbf{x})} \alpha \text{ subject to (3.5) - (3.8)}$$

by setting $n_j(\mathbf{x}) = n_{j0}(\mathbf{x})$

Step 3: Apply the $\mathbf{Z}_1(\mathbf{x})$ and $\mathbf{Z}_3(\mathbf{x})$ obtained from *Step 2* and solve the following conditions by SOSOPT:

$$\min_{\delta \mathbf{Z}_1(\mathbf{x}), \delta \mathbf{Z}_3(\mathbf{x}), \delta n_j(\mathbf{x}), \mathbf{M}_j(\mathbf{x}), \sigma_{ij\beta}(\mathbf{x})} \alpha \text{ subject to (3.5) - (3.8)}$$

$$\mathbf{v}_{z1}^T (\mathbf{Z}_1(\mathbf{x}) + \delta \mathbf{Z}_1(\mathbf{x}) - \epsilon(\mathbf{x})\mathbf{I}) \mathbf{v}_{z1} \text{ is SOS}$$

$$- \mathbf{v}^T \mathbf{L}_{ii}^{\#\alpha}(\mathbf{x}) \mathbf{v} \text{ is SOS, } i = 1, \dots, r$$

$$- \mathbf{v}^T (\mathbf{L}_{ij}^{\#\alpha}(\mathbf{x}) + \mathbf{L}_{ji}^{\#\alpha}(\mathbf{x})) \mathbf{v} \text{ is SOS, } i < j \leq r$$

$$\sigma_{ij\beta}(\mathbf{x}) \text{ is SOS, } i = 1, \dots, r, j = 1, \dots, r, \beta = 1, \dots, n$$

$$\mathbf{v}_1^T \begin{bmatrix} \epsilon_n n_j^2(\mathbf{x}) & \delta n_j(\mathbf{x}) \\ \delta n_j(\mathbf{x}) & 1 \end{bmatrix} \mathbf{v}_1 \text{ is SOS } j = 1, \dots, r$$

$$\mathbf{v}_2^T \begin{bmatrix} \epsilon_{z1} \mathbf{Z}_1(\mathbf{x}) \mathbf{Z}_1^T(\mathbf{x}) & \delta \mathbf{Z}_1(\mathbf{x}) \\ \delta \mathbf{Z}_1^T(\mathbf{x}) & \mathbf{I} \end{bmatrix} \mathbf{v}_2 \text{ is SOS}$$

$$\mathbf{v}_3^T \begin{bmatrix} \epsilon_{z3} \mathbf{Z}_3(\mathbf{x}) \mathbf{Z}_3^T(\mathbf{x}) & \delta \mathbf{Z}_3(\mathbf{x}) \\ \delta \mathbf{Z}_3^T(\mathbf{x}) & \mathbf{I} \end{bmatrix} \mathbf{v}_3 \text{ is SOS}$$

where

$$\mathbf{L}_{ij}^{\#\alpha}(\mathbf{x}) = \begin{bmatrix} \mathbf{L}_1^{\#\alpha}(\mathbf{x}) & * \\ \mathbf{L}_{3ij}^{\#}(\mathbf{x}) & bm \mathbf{L}_{4j}^{\#}(\mathbf{x}) \end{bmatrix}$$

in which

$$\mathbf{L}_1^{\#\alpha}(\mathbf{x}) = (\mathbf{Z}_3(\mathbf{x}) + \delta \mathbf{Z}_3(\mathbf{x}))^T + (\mathbf{Z}_3(\mathbf{x}) + \delta \mathbf{Z}_3(\mathbf{x})) - \alpha(\mathbf{Z}_1(\mathbf{x}) + \delta \mathbf{Z}_1(\mathbf{x})) - \sum_{\beta=1}^n \sigma_{ij\beta}(\mathbf{x}) \mathbf{Q}_\beta(\mathbf{x}) \mathbf{I}$$

$$\mathbf{L}_{3ij}^{\#}(\mathbf{x}) = \mathbf{A}_i(\mathbf{x})(n_j(\mathbf{x})\mathbf{Z}_1(\mathbf{x}) + \delta \mathbf{Z}_1(\mathbf{x})n_j(\mathbf{x}) + \delta n_j(\mathbf{x})\mathbf{Z}_1(\mathbf{x})) + \mathbf{B}_i(\mathbf{x})\mathbf{M}_j(\mathbf{x})$$

$$- (n_j(\mathbf{x})\mathbf{Z}_3(\mathbf{x}) + \delta \mathbf{Z}_3(\mathbf{x})n_j(\mathbf{x}) + \delta n_j(\mathbf{x})\mathbf{Z}_3(\mathbf{x})) + (\mathbf{Z}_1(\mathbf{x}) + \delta \mathbf{Z}_1(\mathbf{x}))$$

$$\mathbf{L}_{4j}^\#(\mathbf{x}) = -2(n_j(\mathbf{x})\mathbf{Z}_1(\mathbf{x}) + \delta\mathbf{Z}_1(\mathbf{x})n_j(\mathbf{x}) + \delta n_j(\mathbf{x})\mathbf{Z}_1(\mathbf{x})).$$

Step 4: Set $n_{j(\eta+1)}(\mathbf{x}) = n_j(\mathbf{x}) + \delta n_j(\mathbf{x})$ with $\delta n_j(\mathbf{x})$ obtained from *Step 3*. After setting $\eta = \eta + 1$, go back to *Step 2*.

Repeat the iteration until $\alpha < 0$ which is found in *Step 2* or $\alpha < 0$ cannot decrease any more regarding to former iterations.

3.4 Designing Example

This section presents an example to compare the proposed Theorem 1 and previous studies. For $\mathbf{x} = [x_1 \ x_2]^T$, consider the polynomial fuzzy model (2.9) which has the system matrices and membership functions shown in [42] with $r = 3$

$$\begin{aligned} \mathbf{A}_1 &= \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ & -a & -6 \end{bmatrix} \\ \mathbf{A}_2 &= \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ & 0 & -6 \end{bmatrix} \\ \mathbf{A}_3 &= \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ & 0.2172a & -6 \end{bmatrix} \\ \mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}_3 &= \begin{bmatrix} x_1 \\ b \end{bmatrix} \\ h_1 &= \frac{1}{1 + e^{\frac{(x_1+4)}{2}}}, \quad h_3 = \frac{1}{1 + e^{\frac{-(x_1-4)}{2}}} \\ h_2 &= 1 - h_1 - h_3. \end{aligned} \tag{3.25}$$

The operation domain is set as $x_1 \in [-1 \ 1]$ and $x_2 \in [-1 \ 1]$ and the polynomial function is set as $n_j(\mathbf{x}) = n_{j0} + n_{j1}x_1^2 + n_{j2}x_2^2$. Under this situation and $a = 2.5$, the maximum feasible value obtained by the proposed method is $b = 8.5$. When $b = 8.5$, no solution can be found by Theorem 2 of [37] and [42]. This example proves that the extra polynomials provided by (3.1) and the homogeneous Lyapunov function, which removes the limitation of $\tilde{\mathbf{x}}$ can help the system (3.4) to obtain more relaxed results.

Remark 1. The purpose of the designing examples is to show the "relaxation" of our pro-

posed stabilization criteria, which means our proposed stabilization criteria can find feasible solutions that other studies cannot find. The importance of the maximum feasible value is the “relaxation” of the stabilization criterion. The quantity of the maximum feasible value of “b” itself has no meaning for the considering system. It is a methodology for presenting relaxation.

The solution of decision variable are presented as follows:

$$\mathbf{Z}_1(\mathbf{x}) = \begin{bmatrix} 3.351x_1^2 - 0.253x_1x_2 + 3.545x_2^2 & -3.483x_1^2 - 1.173x_1x_2 - 1.87x_2^2 \\ -3.483x_1^2 - 1.173x_1x_2 - 1.87x_2^2 & 8.763x_1^2 + 2.185x_1x_2 + 5.987x_2^2 \end{bmatrix}$$

$$\mathbf{Z}_3(\mathbf{x}) = \begin{bmatrix} -6.217x_1^2 + 2.661x_1x_2 - 8.856x_2^2 & 6.424x_1^2 - 1.138x_1x_2 + 7.82x_2^2 \\ -3.442x_1^2 + 0.719x_1x_2 - 4.755x_2^2 & -11.97x_1^2 + -1.048x_1x_2 - 11.132x_2^2 \end{bmatrix}$$

$$\mathbf{M}_1(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{l} -1.252x_1^4 - 0.595x_1^3x_2 - 1.871x_1^2x_2^2 \\ -0.595x_1x_2^3 - 0.49x_2^4 - 0.229x_1^3 \\ -0.026x_1^2x_2 - 0.28x_1x_2^2 - 0.029x_2^3 \\ -0.257x_1^2 - 0.085x_1x_2 - 0.14x_2^2 \end{array} \right) \\ \left(\begin{array}{l} 1.946x_1^4 + 0.577x_1^3x_2 + 3.223x_1^2x_2^2 \\ +0.491x_1x_2^3 + 1.257x_2^4 + 0.086x_1^3 \\ +0.104x_1^2x_2 - 0.011x_1x_2^2 - 0.008x_2^3 \\ -0.019x_1^2 + 0.131x_1x_2 - 0.173x_2^2 \end{array} \right) \end{array} \right]$$

$$\mathbf{M}_2(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{l} -2.014x_1^4 - 0.782x_1^3x_2 - 2.99x_1^2x_2^2 \\ -0.734x_1x_2^3 - 0.993x_2^4 - 0.158x_1^3 \\ +0.092x_1^2x_2 - 0.246x_1x_2^2 - 0.038x_2^3 \\ -2.267x_1^2 - 0.449x_1x_2 - 1.622x_2^2 \end{array} \right) \\ \left(\begin{array}{l} 2.613x_1^4 + 1.042x_1^3x_2 + 3.978x_1^2x_2^2 \\ +0.69x_1x_2^3 + 1.552x_2^4 - 0.087x_1^3 \\ +0.036x_1^2x_2 - 0.19x_1x_2^2 - 0.026x_2^3 \\ +2.828x_1^2 + 0.811x_1x_2 + 1.846x_2^2 \end{array} \right) \end{array} \right]$$

$$\mathbf{M}_3(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{l} -2.899x_1^4 - 0.851x_1^3x_2 - 3.134x_1^2x_2^2 \\ -0.406x_1x_2^3 - 0.79x_2^4 - 0.102x_1^3 \\ +0.083x_1^2x_2 - 0.144x_1x_2^2 - 0.011x_2^3 \\ -0.872x_1^2 - 0.141x_1x_2 - 0.654x_2^2 \end{array} \right) \\ \left(\begin{array}{l} 3.84x_1^4 + 1.185x_1^3x_2 + 4.092x_1^2x_2^2 \\ +0.477x_1x_2^3 + 1.053x_2^4 - 0.178x_1^3 \\ -0.059x_1^2x_2 - 0.102x_1x_2^2 - 0.015x_2^3 \\ +0.763x_1^2 + 0.356x_1x_2 + 0.324x_2^2 \end{array} \right) \end{array} \right]$$

$$n_1(\mathbf{x}) = 0.573x_1^2 + 0.564x_2^2 + 0.167$$

$$n_2(\mathbf{x}) = 0.502x_1^2 + 0.598x_2^2 + 0.875$$

$$n_3(\mathbf{x}) = 0.827x_1^2 + 0.363x_2^2 + 0.304$$

The simulation result is shown in Fig 3.1 and Fig 3.2.

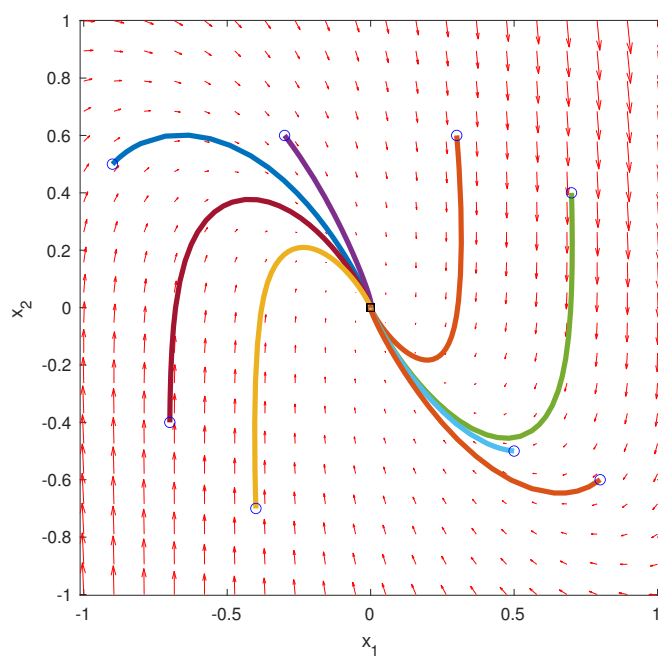


Figure 3.1: The simulation results of \mathbf{x}

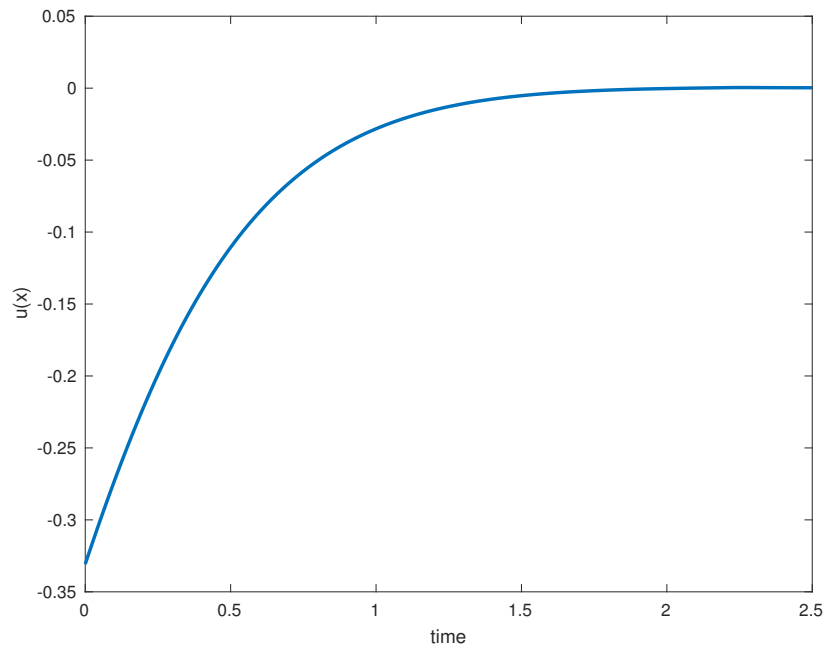


Figure 3.2: The simulation results of control input u

A DESCRIPTOR SYSTEM APPROACH FOR POLYNOMIAL FUZZY-MODEL-BASED CONTROL DESIGN

A polynomial FMB control design by using descriptor system approach is proposed in this chapter. A PDC-based controller is concerned in this chapter, and the closed-loop system is expressed in descriptor form. The stabilization is analyzed by applying a commonly used Lyapunov candidate. Two examples, including a numerical one used for comparison and the other one verifying the applicability, are provided.

4.1 A Model With PDC-Based Controller

Consider the polynomial fuzzy model (2.9). By applying the concept of PDC technique, the following controller is employed:

$$\mathbf{u}(\mathbf{x}) = \sum_{i=1}^r h_i(z) \mathbf{F}_i(\mathbf{x}) \mathbf{x}. \quad (4.1)$$

At First, the fuzzy controller (4.1) is converted into

$$\begin{aligned} 0 &= \sum_{i=1}^r h_i(z) \mathbf{F}_i(\mathbf{x}) \mathbf{x} - \mathbf{u}(\mathbf{x}) \\ &= \sum_{i=1}^r h_i(z) \{ \mathbf{F}_i(\mathbf{x}) \mathbf{x} - \mathbf{u}(\mathbf{x}) \}. \end{aligned} \quad (4.2)$$

Moreover, the transformation matrix $\mathbf{T}(\mathbf{x})$ which is introduced in preliminaries section is applied to represent the relationship between the state vector \mathbf{x} and its monomial vector $\hat{\mathbf{x}}(\mathbf{x})$, that is,

$$\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{x}.$$

Therefore, from (2.9) and (4.2), the closed-loop systems can be rewritten in descriptor form:

$$\mathbf{E}^* \dot{\mathbf{x}}^*(\mathbf{x}) = \sum_{i=1}^r h_i(z) \mathbf{A}_i^*(\mathbf{x}) \mathbf{x}^*(\mathbf{x}) \quad (4.3)$$

where

$$\mathbf{E}^* = \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{x}^* = \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}}(\mathbf{x}) \\ \mathbf{u}(\mathbf{x}) \end{bmatrix}, \mathbf{A}_i^* = \begin{bmatrix} 0 & \mathbf{A}_i(\mathbf{x}) & \mathbf{B}_i(\mathbf{x}) \\ \mathbf{T}(\mathbf{x}) & -\mathbf{I} & 0 \\ \mathbf{F}_i(\mathbf{x}) & 0 & -\mathbf{I} \end{bmatrix}.$$

4.2 Main Result

The stabilization analysis for (4.3) is presented in this section, which is presented in terms of SOS. Before introducing the Lyapunov function, the definitions of $\mathbf{A}_i^k(\mathbf{x})$ and $\tilde{\mathbf{x}}$ should be given [37]. Define $\mathbf{A}_i^k(\mathbf{x})$ which denotes system matrix $\mathbf{A}_i(\mathbf{x})$'s k th row, where $\mathbf{K} = \{k_1, k_2, \dots, k_m\}$ is corresponded to system matrix $\mathbf{B}_i(\mathbf{x})$'s row index in which contains the zero row, and define the vector

$$\tilde{\mathbf{x}}(t) = [x_{k_1}(t) \ \cdots \ x_{k_m}(t)]^T. \quad (4.4)$$

Moreover, the research defines the vector of membership functions

$$\mathbf{h} = [h_1(z) \ \cdots \ h_r(z)]^T.$$

Theorem 2:

If there exist a symmetric polynomial matrix $\mathbf{X}(\tilde{\mathbf{x}})$, polynomial matrices $\mathbf{X}_{21}(\mathbf{x})$, $\mathbf{X}_{22}(\mathbf{x})$, $\mathbf{X}_{23}(\mathbf{x})$, $\mathbf{X}_{31}(\mathbf{x})$, $\mathbf{X}_{32}(\mathbf{x})$, $\mathbf{X}_{33}(\mathbf{x})$, $\mathbf{M}_i(\mathbf{x})$, and polynomials $\sigma_{i\beta}(\mathbf{x})$, (4.3) is asymptotically stable when satisfying the following conditions,

$$\mathbf{v}^T (\hat{\mathbf{X}}(\mathbf{x}) - \epsilon_1(\mathbf{x})\mathbf{I}) \mathbf{v} \text{ is SOS} \quad (4.5)$$

$$- \mathbf{v}^T(\mathbf{H}_i(\mathbf{x}))\mathbf{v} \text{ is SOS, } i = 1, \dots, r \quad (4.6)$$

$$\sigma_{i\beta}(\mathbf{x}) \text{ is SOS, } i = 1, \dots, r, \beta = 1, \dots, n, \quad (4.7)$$

where

$$\hat{\mathbf{X}}(\mathbf{x}) = \begin{bmatrix} \mathbf{X}(\tilde{\mathbf{x}}) & 0 & 0 \\ \mathbf{X}_{21}(\mathbf{x}) & \mathbf{X}_{22}(\mathbf{x}) & \mathbf{X}_{23}(\mathbf{x}) \\ \mathbf{X}_{31}(\mathbf{x}) & \mathbf{X}_{32}(\mathbf{x}) & \mathbf{X}_{33}(\mathbf{x}) \end{bmatrix}.$$

$\mathbf{H}_i(\mathbf{x})$ is given as

$$\mathbf{H}_i(\mathbf{x}) = \begin{bmatrix} \left(\begin{array}{l} \mathbf{A}_i(\mathbf{x})\mathbf{X}_{21}(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{X}_{31}(\mathbf{x}) + \mathbf{X}_{21}^T(\mathbf{x})\mathbf{A}_i^T(\mathbf{x}) \\ + \mathbf{X}_{31}^T(\mathbf{x})\mathbf{B}_i^T(\mathbf{x}) - \sum_{k \in K} \frac{\partial \mathbf{X}}{\partial x_k}(\tilde{\mathbf{x}})\mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x}) \\ - \sum_{\beta=1}^n \sigma_{i\beta}(\mathbf{x})\mathbf{Q}_\beta(\mathbf{x})\mathbf{I} + \epsilon_{2i}(\mathbf{x})\mathbf{I} \end{array} \right) & * & * \\ \left(\begin{array}{l} \mathbf{T}(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) - \mathbf{X}_{21}(\mathbf{x}) + \\ \mathbf{X}_{22}^T(\mathbf{x})\mathbf{A}_i^T(\mathbf{x}) + \mathbf{X}_{32}^T(\mathbf{x})\mathbf{B}_i^T(\mathbf{x}) \end{array} \right) & (-\mathbf{X}_{22}(\mathbf{x}) - \mathbf{X}_{22}^T(\mathbf{x})) & * \\ \left(\begin{array}{l} \mathbf{M}_i(\mathbf{x}) - \mathbf{X}_{31}(\mathbf{x}) + \\ \mathbf{X}_{23}^T(\mathbf{x})\mathbf{A}_i^T(\mathbf{x}) + \mathbf{X}_{33}^T(\mathbf{x})\mathbf{B}_i^T(\mathbf{x}) \end{array} \right) & (-\mathbf{X}_{32}(\mathbf{x}) - \mathbf{X}_{23}^T(\mathbf{x})) & (-\mathbf{X}_{33}(\mathbf{x}) - \mathbf{X}_{33}^T(\mathbf{x})) \end{bmatrix} \quad (4.8)$$

in which

$$\mathbf{Q}_\beta(\mathbf{x}) = (x_\beta - x_\beta^{\min})(x_\beta - x_\beta^{\max})$$

polynomials $\epsilon_1(\mathbf{x}) > 0$ and $\epsilon_{2i}(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$. The feedback gain is obtained from

$$\mathbf{F}_i(\mathbf{x}) = \mathbf{M}_i(\mathbf{x})\mathbf{X}^{-1}(\tilde{\mathbf{x}}). \quad (4.9)$$

Proof:

A Lyapunov function candidate which is commonly used in the studies of polynomial

fuzzy model is considered:

$$V(\mathbf{x}) = \mathbf{x}^T \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \mathbf{x}. \quad (4.10)$$

From the definition of K , we have

$$\dot{\mathbf{x}}_k = \sum_{i=1}^r h_i(z) \mathbf{A}_i^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}) \quad (4.11)$$

for $k \in K$, and

$$\frac{\partial \mathbf{X}^{-1}}{\partial x_i}(\tilde{\mathbf{x}}) = 0 \quad (4.12)$$

for $i \notin K$. Then $V(\mathbf{x})$'s time derivative will be

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \mathbf{x} + \mathbf{x}^T \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \dot{\mathbf{x}} + \mathbf{x}^T \dot{\mathbf{X}}^{-1}(\tilde{\mathbf{x}}) \mathbf{x} \\ &= \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}}(\mathbf{x}) \\ \dot{\mathbf{u}}(\mathbf{x}) \end{bmatrix}^T \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}^{-1}(\tilde{\mathbf{x}}) & 0 & 0 \\ \mathbf{P}_{21}(\mathbf{x}) & \mathbf{P}_{22}(\mathbf{x}) & \mathbf{P}_{23}(\mathbf{x}) \\ \mathbf{P}_{31}(\mathbf{x}) & \mathbf{P}_{32}(\mathbf{x}) & \mathbf{P}_{33}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}}(\mathbf{x}) \\ \mathbf{u}(\mathbf{x}) \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}}(\mathbf{x}) \\ \mathbf{u}(\mathbf{x}) \end{bmatrix}^T \begin{bmatrix} \mathbf{X}^{-1}(\tilde{\mathbf{x}}) & \mathbf{P}_{21}^T(\mathbf{x}) & \mathbf{P}_{31}^T(\mathbf{x}) \\ 0 & \mathbf{P}_{22}^T(\mathbf{x}) & \mathbf{P}_{32}^T(\mathbf{x}) \\ 0 & \mathbf{P}_{23}^T(\mathbf{x}) & \mathbf{P}_{33}^T(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}}(\mathbf{x}) \\ \dot{\mathbf{u}}(\mathbf{x}) \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}}(\mathbf{x}) \\ \mathbf{u}(\mathbf{x}) \end{bmatrix}^T \begin{bmatrix} \sum_{k=1}^n \frac{\partial \mathbf{X}^{-1}}{\partial x_k}(\tilde{\mathbf{x}}) \mathbf{A}_i^k(\mathbf{x}) \mathbf{x} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}}(\mathbf{x}) \\ \mathbf{u}(\mathbf{x}) \end{bmatrix} \end{aligned} \quad (4.13)$$

Therefore, we have

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^{*T} \mathbf{E}^{*T} \hat{\mathbf{P}}(\mathbf{x}) \mathbf{x}^*(\mathbf{x}) + \mathbf{x}^{*T} \hat{\mathbf{P}}^T(\mathbf{x}) \mathbf{E}^* \dot{\mathbf{x}}^*(\mathbf{x}) \\ &\quad + \mathbf{x}^{*T} \mathbf{C}_i(\mathbf{x}) \mathbf{x}^*(\mathbf{x}) \\ &= \sum_{i=1}^r h_i(z) \mathbf{x}^{*T}(\mathbf{x}) \{ \mathbf{A}_i^{*T}(\mathbf{x}) \hat{\mathbf{P}}(\mathbf{x}) + \hat{\mathbf{P}}^T(\mathbf{x}) \mathbf{A}_i^*(\mathbf{x}) \\ &\quad + \mathbf{C}_i(\mathbf{x}) \} \mathbf{x}^*(\mathbf{x}) \end{aligned} \quad (4.14)$$

where

$$\hat{\mathbf{P}}(\mathbf{x}) = \begin{bmatrix} \mathbf{X}^{-1}(\tilde{\mathbf{x}}) & 0 & 0 \\ \mathbf{P}_{21}(\mathbf{x}) & \mathbf{P}_{22}(\mathbf{x}) & \mathbf{P}_{23}(\mathbf{x}) \\ \mathbf{P}_{31}(\mathbf{x}) & \mathbf{P}_{32}(\mathbf{x}) & \mathbf{P}_{33}(\mathbf{x}) \end{bmatrix}$$

and

$$\mathbf{C}_i(\mathbf{x}) = \begin{bmatrix} \sum_{k=1}^n \frac{\partial \mathbf{X}^{-1}}{\partial \mathbf{x}_k}(\tilde{\mathbf{x}}) \mathbf{A}_i^k(\mathbf{x}) \mathbf{x} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the operation domain D_{op} ,

$$\psi(z, \mathbf{x}) = - \sum_{i=1}^r \sum_{\beta=1}^n h_i(z) \sigma_{i\beta}(\mathbf{x}) (\mathbf{x}_\beta - \mathbf{x}_\beta^{min})(\mathbf{x}_\beta - \mathbf{x}_\beta^{max}) \geq 0$$

where $\sigma_{i\beta}(\mathbf{x}) \geq 0$ which is hold in (4.7). Therefore, $\dot{V}(\mathbf{x}) < 0$ for $D_{op} - \{0\}$ is satisfied if

$$\begin{aligned} \dot{V}(\mathbf{x}) \leq \\ \mathbf{x}^{*T} \hat{\mathbf{P}}^T(\mathbf{x}) \begin{bmatrix} -(\psi(z, \mathbf{x}) + \epsilon_{2i}(\mathbf{x})) \mathbf{I} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{\mathbf{P}}(\mathbf{x}) \mathbf{x}^* \end{aligned} \quad (4.15)$$

where polynomials $\epsilon_{2i}(\mathbf{x}) > 0$ in $\mathbf{x} \neq 0$. Let $\hat{\mathbf{X}}(\mathbf{x}) = \hat{\mathbf{P}}^{-1}(\mathbf{x})$. Condition (4.6) implies the truth that

$$\begin{aligned} - \{ \hat{\mathbf{X}}^T(\mathbf{x}) \mathbf{A}_i^{*T}(\mathbf{x}) + \mathbf{A}_i^*(\mathbf{x}) \hat{\mathbf{X}}(\mathbf{x}) - \mathbf{D}_i(\mathbf{x}) \\ - \left[\begin{array}{ccc} (\sum_{\beta=1}^n \sigma_{i\beta}(\mathbf{x}) \mathbf{Q}_\beta(\mathbf{x}) - \epsilon_{2i}(\mathbf{x})) \mathbf{I} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \} \geq 0 \end{aligned} \quad (4.16)$$

where

$$\mathbf{D}_i(\mathbf{x}) = \begin{bmatrix} \sum_{k \in K} \frac{\partial \mathbf{X}}{\partial \mathbf{x}_k}(\tilde{\mathbf{x}}) \mathbf{A}_i^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that $\mathbf{A}_i^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x})$ is a scalar. We have the inequality below by multiplying (4.16) from left side and right side by $\hat{\mathbf{P}}^T(\mathbf{x})$ and $\hat{\mathbf{P}}(\mathbf{x})$ respectively

$$\begin{aligned}
 & - \{ \mathbf{A}_i^{*T}(\mathbf{x}) \hat{\mathbf{P}}(\mathbf{x}) + \hat{\mathbf{P}}^T(\mathbf{x}) \mathbf{A}_i^*(\mathbf{x}) - \hat{\mathbf{P}}^T(\mathbf{x}) \left[\begin{array}{ccc} (\sum_{\beta=1}^n \sigma_{i\beta}(\mathbf{x}) Q_{\beta}(\mathbf{x}) - \epsilon_{2i}(\mathbf{x})) \mathbf{I} & 0 & 0 \\ 0 & & 0 \\ 0 & & 0 \end{array} \right] \hat{\mathbf{P}}(\mathbf{x}) \\
 & - \left[\begin{array}{ccc} \sum_{k \in K} \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \frac{\partial \mathbf{X}}{\partial x_k}(\tilde{\mathbf{x}}) \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \mathbf{A}_i^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}) & 0 & 0 \\ 0 & & 0 \\ 0 & & 0 \end{array} \right] \} \geq 0.
 \end{aligned}$$

The preliminaries' chapter gives that $\mathbf{X}(\tilde{\mathbf{x}})$'s invert matrix is exist. Therefore, $\mathbf{X}^{-1}(\tilde{\mathbf{x}}) \mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{I}$. By doing derivative on both side with respect to x_k , the following equation is obtained

$$\frac{\partial \mathbf{X}^{-1}}{\partial x_k}(\tilde{\mathbf{x}}) \mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \frac{\partial \mathbf{X}}{\partial x_k}(\tilde{\mathbf{x}}) = 0. \quad (4.17)$$

Therefore

$$\mathbf{X}^{-1}(\tilde{\mathbf{x}}) \frac{\partial \mathbf{X}}{\partial x_k}(\tilde{\mathbf{x}}) \mathbf{X}^{-1}(\tilde{\mathbf{x}}) = - \frac{\partial \mathbf{X}^{-1}}{\partial x_k}(\tilde{\mathbf{x}}). \quad (4.18)$$

From (4.18), the inequality (4.16) can be represented in

$$\begin{aligned}
 & - \{ \mathbf{A}_i^{*T}(\mathbf{x}) \hat{\mathbf{P}}(\mathbf{x}) + \hat{\mathbf{P}}^T(\mathbf{x}) \mathbf{A}_i^*(\mathbf{x}) + \mathbf{C}_i(\mathbf{x}) \\
 & - \hat{\mathbf{P}}^T(\mathbf{x}) \left[\begin{array}{ccc} (\sum_{\beta=1}^n \sigma_{i\beta}(\mathbf{x}) Q_{\beta}(\mathbf{x}) - \epsilon_{2i}(\mathbf{x})) \mathbf{I} & 0 & 0 \\ 0 & & 0 \\ 0 & & 0 \end{array} \right] \hat{\mathbf{P}}(\mathbf{x}) \} > 0. \quad (4.19)
 \end{aligned}$$

From (4.14), if (4.19) holds for $i = 1, \dots, r$, then (4.15) holds. Consequently, if the condition (4.6) holds, $\dot{V}(\mathbf{x}) < 0$ for $D_{op} - \{0\}$ is satisfied. \square

Remark 2. Compare the proposed Theorem 2 and the approach of [37] with considering the operation domain, Table 4.1 presents the differences. From the Table 4.1, the number of the SOS constrains of the approach of [37] is $r(r+1)/2 + r\beta + 1$, with considering the operation domain. In contrast, the number of SOS constraints of the proposed approach applying the descriptor form is only $r + r\beta + 1$. The value of β is n , and the term $r\beta$ is omitted since both the approach of [37] and proposed Theorem 2 contain this term. We have the fact that the descriptor representation's redundancy can decrease design conditions'

numbers drastically. The phenomenon goes evident with large r , that is, the nonlinear system with more non-polynomial nonlinear terms. In contrast, the dimension part, the stabilization matrix's dimension is n for the approach of [37] and is $2n + m$ for the proposed approach. Therefore, compared with the approach of [37], the proposed approach is more suitable for the polynomial fuzzy model with more rules and fewer states. Moreover, through our several trials, the feasibility (relaxation) of the proposed approach is similar to the approach of [37].

Table 4.1: Comparison Between the Proposed Approach and the Approach of [37] with Considering the Operation Domain.

	The Proposed Approach	Approach of [37]
Number of SOS Constraints	$r + 2$	$r(r + 1)/2 + 2$
Dimension of stabilization Matrix	$2n + m$	n

4.3 Design Examples

This section gives two examples. One example is to make a comparison with the existing polynomial FMB control design approach, and the other example verifies the applicability of the proposed method.

Example 1:

Given that $\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{x} = [x_1 \ x_2]^T$, consider the polynomial fuzzy model (2.9) which has the system matrices and membership functions shown in [42] with $r = 3$:

$$\mathbf{A}_1 = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ -a & -6 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ 0 & -6 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ 0.2172a & -6 \end{bmatrix}$$

$$\mathbf{B}_1 = \begin{bmatrix} x_1 \\ b \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} x_1 \\ b \end{bmatrix}, \mathbf{B}_3 = \begin{bmatrix} x_1 \\ b \end{bmatrix}$$

$$h_1 = \frac{1}{1 + e^{\frac{(x_1+4)}{2}}}, \quad h_3 = \frac{1}{1 + e^{-\frac{(x_1-4)}{2}}}$$

$$h_2 = 1 - h_1 - h_3.$$

The operation domain $x_1 \in [-2 \ 2]$ and $x_2 \in [-2 \ 2]$. Since no zero row are shown in $\mathbf{B}_i(\mathbf{x})$, the elements in matrix $\mathbf{X}(\tilde{\mathbf{x}})$ contain only constant. When a is set to be 2, the maximum values of b can be found for feasible solution in the proposed approach and the approach of [37] with considering the operation domain are both close to 8. It shows the feasibility of these two approaches is similar as mentioned in **Remark 2**. As **Remark 1** mentioned, the quantity of the maximum feasible value of “b” itself has no meaning for the considering system. It is just a methodology for presenting relaxation. When $a = 2$ and $b = 8$, the Fig 4.1 shows the simulation results, which is represented in the phase plot, and the solution matrices is

$$\mathbf{X}(\tilde{\mathbf{x}}) = \begin{bmatrix} 2.384 & -4.148 \\ -1.163 & 38.66 \end{bmatrix}$$

$$\mathbf{X}_{21}(\mathbf{x}) = \begin{bmatrix} 0.309x_1 - 0.072x_2 + 2.664 & 0.445x_1 + 0.323x_2 + 0.422 \\ -4.141x_1 - 1.628x_2 - 0.88 & -0.335x_1 - 0.696x_2 + 9.191 \end{bmatrix}$$

$$\mathbf{X}_{22}(\mathbf{x}) = \begin{bmatrix} \begin{pmatrix} 0.096x_1^2 + 0.007x_1x_2 + 0.078x_2^2 \\ +0.407x_1 - 0.275x_2 + 1.471 \end{pmatrix} & \begin{pmatrix} -0.068x_1^2 + 0.025x_1x_2 + 0.033x_2^2 \\ -1.036x_1 - 0.58x_2 + 0.383 \end{pmatrix} \\ \begin{pmatrix} -0.465x_1^2 - 0.024x_1x_2 + 0.328x_2^2 \\ -1.44x_1 - 1.397x_2 + 0.385 \end{pmatrix} & \begin{pmatrix} 2.336x_1^2 + 1.18x_1x_2 + 1.511x_2^2 \\ -1.802x_1 - 0.805x_2 + 3.656 \end{pmatrix} \end{bmatrix}$$

$$\mathbf{X}_{23}(\mathbf{x}) = \begin{bmatrix} \begin{pmatrix} -0.078x_1^2 + 0.014x_1x_2 - 0.015x_2^2 \\ -0.984x_1 - 0.513x_2 - 0.631 \end{pmatrix} \\ \begin{pmatrix} 1.886x_1^2 + 1.404x_1x_2 + 0.523x_2^2 \\ +0.058x_1 - 0.034x_2 + 5.127 \end{pmatrix} \end{bmatrix}$$

$$\mathbf{X}_{31}(\mathbf{x}) = \begin{bmatrix} -4.107x_1 - 1.199x_2 - 1.504 & -0.237x_1 - 0.523x_2 + 3.63 \end{bmatrix}$$

$$\mathbf{X}_{32}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} -0.328x_1^2 - 0.012x_1x_2 + 0.246x_2^2 \\ -1.023x_1 - 1.056x_2 + 0.784 \end{array} \right) \\ \left(\begin{array}{c} 1.649x_1^2 + 0.857x_1x_2 + 1.117x_2^2 \\ -1.425x_1 - 0.652x_2 - 3.693 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{33}(\mathbf{x}) = 1.967x_1^2 + 0.77x_1x_2 + 1.193x_2^2 - 0.019x_1 - 0.104x_2 + 4.909$$

$$\mathbf{M}_1(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} 0.187x_1^2 + 0.535x_1x_2 - 0.357x_2^2 \\ -7.872x_1 - 0.456x_2 - 5.301 \end{array} \right) \\ \left(\begin{array}{c} -3.998x_1^2 + 0.521x_1x_2 - 4.149x_2^2 \\ -0.081x_1 + 0.096x_2 - 0.796 \end{array} \right) \end{array} \right]$$

$$\mathbf{M}_2(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} 0.256x_1^2 + 0.528x_1x_2 - 0.287x_2^2 \\ -7.898x_1 - 0.408x_2 - 5.676 \end{array} \right) \\ \left(\begin{array}{c} -3.861x_1^2 + 0.489x_1x_2 - 3.943x_2^2 \\ -0.164x_1 - 0.039x_2 - 0.809 \end{array} \right) \end{array} \right]$$

$$\mathbf{M}_3(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} 0.28x_1^2 + 0.519x_1x_2 - 0.251x_2^2 \\ -7.857x_1 - 0.366x_2 - 5.817 \end{array} \right) \\ \left(\begin{array}{c} -3.811x_1^2 + 0.497x_1x_2 - 3.879x_2^2 \\ -0.202x_1 - 0.064x_2 - 0.847 \end{array} \right) \end{array} \right]$$

From Fig 4.1, it shows a locally asymptotically stable results for closed-loop FMB control system in the operation domain.

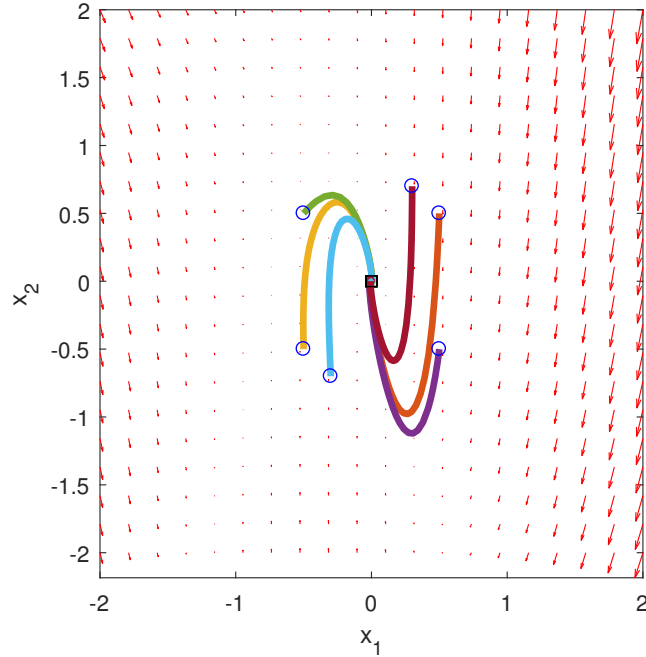


Figure 4.1: The phase plot of the simulation results

Table 4.2: Bicycle Dynamic's Parameters

Parameter	Value	Unit
M	25.5	[kg]
I_a	10.0	[kgm^2]
L	1.0	[m]
h	0.575	[m]
v	2.5	[m/s]
η	$\frac{\pi}{3}$	[rad]

Example 2:

Consider a bicycle dynamic system as Fig 4.2 and 4.3 shows. In Fig 4.2a L denotes the length of the wheels base, η denotes the steering angle, h denotes the height of the bicycle's center of gravity, ϕ denotes the steering angle, and v denotes the running velocity. In Fig 4.2b M denotes the whole mass of the bicycle, and θ is the camber angle of the bicycle. In Fig 4.3 β denotes the direction angle and R denotes the turning radius. Note that g denotes the gravitational acceleration and its value is $g = 9.81 \text{ m/s}^2$.

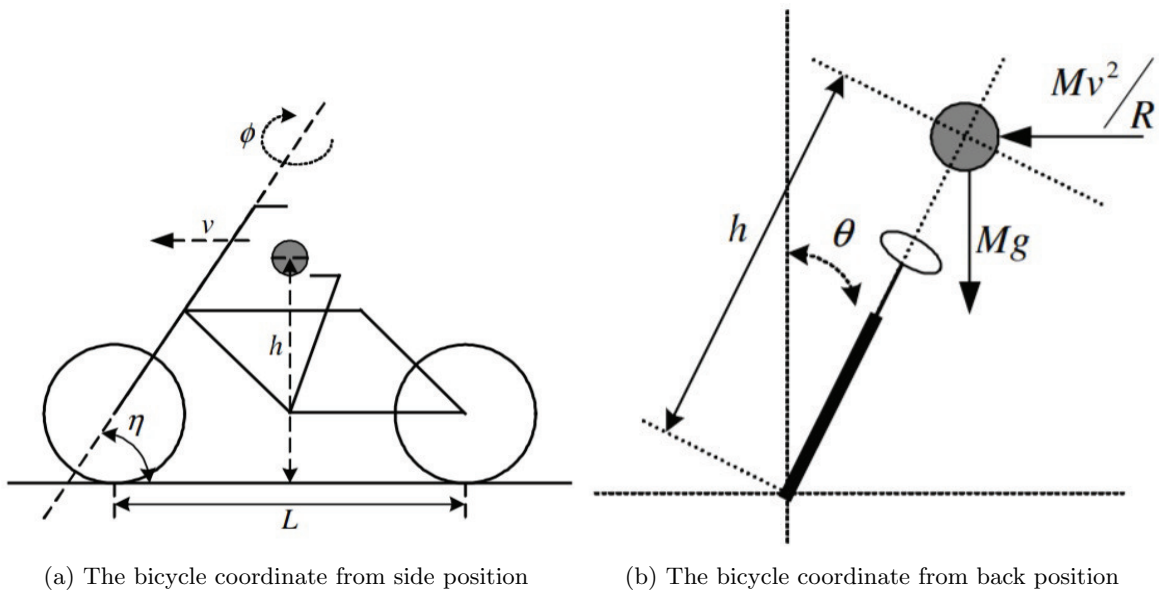


Figure 4.2: Two bicycle's coordinates

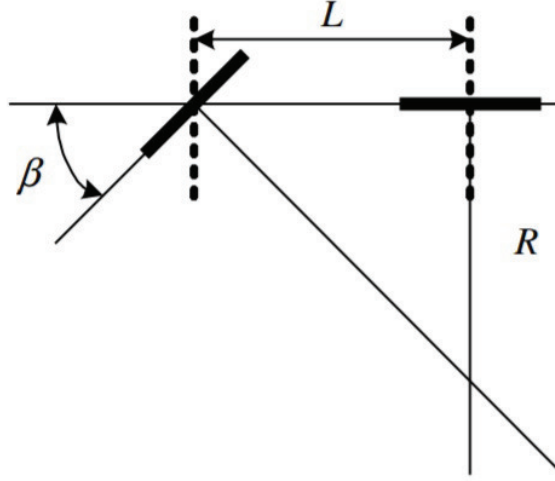


Figure 4.3: The bicycle coordinate from top position

The total dynamic of the bicycle is presented in the equation as

$$I_a \ddot{\theta} = Mgh \sin \theta - \frac{Mv^2 h}{L} \cdot \sin(\eta) \cdot \phi \quad (4.20)$$

where I_a is the moment of inertia. The dynamic equation (4.20) can be rewritten in matrices form as the following equation [65]:

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \frac{Mgh}{I_a} \sin \theta \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{Mv^2 h}{LI_a} \cdot \sin(\eta) \end{bmatrix} \phi. \quad (4.21)$$

By utilizing the Taylor series as technique proposed in [66], $\sin \theta$ can be represented as

$$\sin \theta = h_1 \theta + h_2 \left(\theta - \frac{\theta^3}{6} \right) \quad (4.22)$$

with the membership functions

$$h_1 = \begin{cases} \frac{6(\sin \theta - \theta)}{\theta^3} + 1, & \theta \neq 0 \\ 0 & \theta = 0 \end{cases}, h_2 = 1 - h_1.$$

Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Using (4.22), the dynamic (4.21) can be equal to the system matrices of polynomial fuzzy model (2.9) with the parameters $r = 2$ and $\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{x} = [x_1 \ x_2]^T$. Also,

the system matrices is represented as

$$\mathbf{A}_1(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ \frac{Mgh}{I_a} & 0 \end{bmatrix}, \mathbf{A}_2(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ \frac{Mgh}{I_a}(1 - \frac{x_1^2}{6}) & 0 \end{bmatrix}, \mathbf{B}_1(\mathbf{x}) = \mathbf{B}_2(\mathbf{x}) = \begin{bmatrix} 0 \\ -\frac{Mv^2h}{LI_a} \cdot \sin(\eta) \end{bmatrix}.$$

The operation domain is set to be $x_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $x_2 \in [-\pi, \pi]$, and the bicycle parameters are given in Table 4.2. By solving the SOS constraints (4.5)-(4.7), the solutions are shown as

$$\mathbf{X}(\tilde{\mathbf{x}}) = \begin{bmatrix} \begin{pmatrix} 1.791x_1^2 + 7.493 \times 10^{-7}x_1 \\ +5.268 \\ 2.98x_1^2 - 1.222 \times 10^{-6}x_1 \\ +0.869 \end{pmatrix} & \begin{pmatrix} -7.533x_1^2 + 6.857 \times 10^{-7}x_1 \\ -11.044 \\ 8.453x_1^2 + 4.635 \times 10^{-6}x_1 \\ +8.801 \end{pmatrix} \end{bmatrix}$$

$$\mathbf{X}_{21}(\mathbf{x}) =$$

$$\begin{bmatrix} \begin{pmatrix} 0.231x_1^4 + 0.001x_1^3x_2 + 0.022x_1^2x_2^2 \\ +4.804 \times 10^{-7}x_1^3 + 9.014 \times 10^{-8}x_1^2x_2 \\ -1.987 \times 10^{-8}x_1x_2^2 \\ +0.541x_1^2 - 0.016x_1x_2 + 0.094x_2^2 \\ +8.828 \times 10^{-7}x_1 + 4.169 \times 10^{-7}x_2 + 1.048 \end{pmatrix} & \begin{pmatrix} -0.512x_1^4 - 0.034x_1^3x_2 - 0.17x_1^2x_2^2 \\ +4.975 \times 10^{-8}x_1^3 + 8.595 \times 10^{-7}x_1^2x_2 \\ -3.34 \times 10^{-8}x_1x_2^2 \\ -0.855x_1^2 - 0.024x_1x_2 - 0.295x_2^2 \\ +8.936 \times 10^{-8}x_1 + 1.136 \times 10^{-6}x_2 - 1.129 \end{pmatrix} \\ \begin{pmatrix} -4.166x_1^4 + 0.154x_1^3x_2 - 0.506x_1^2x_2^2 \\ -4.73 \times 10^{-7}x_1^3 + 3.421 \times 10^{-7}x_1^2x_2 \\ -3.791 \times 10^{-7}x_1x_2^2 \\ -6.646x_1^2 + 0.157x_1x_2 - 2.103x_2^2 \\ -6.457 \times 10^{-7}x_1 + 1.562 \times 10^{-6}x_2 - 8.372 \end{pmatrix} & \begin{pmatrix} 3.169x_1^4 - 0.605x_1^3x_2 + 1.726x_1^2x_2^2 \\ -2.954 \times 10^{-7}x_1^3 - 1.241 \times 10^{-6}x_1^2x_2 \\ +6.351 \times 10^{-7}x_1x_2^2 \\ +3.341x_1^2 - 0.321x_1x_2 + 3.368x_2^2 \\ -3.505 \times 10^{-7}x_1 - 2.017 \times 10^{-6}x_2 + 5.373 \end{pmatrix} \end{bmatrix}$$

$$\mathbf{X}_{22}(\mathbf{x}) =$$

$$\left[\begin{array}{l} \left(\begin{array}{l} 0.478x_1^4 + 0.066x_1^3x_2 + 0.393x_1^2x_2^2 \\ +0.013x_1x_2^3 + 0.061x_2^4 + 2.021 \times 10^{-7}x_1^3 \\ +2.204 \times 10^{-7}x_1^2x_2 + 2.157 \times 10^{-7}x_1x_2^2 \\ +1.525 \times 10^{-7}x_2^3 \\ +0.558x_1^2 + 0.054x_1x_2 + 0.382x_2^2 \\ +2.448 \times 10^{-7}x_1 + 2.141 \times 10^{-7}x_2 \\ +0.655 \end{array} \right) \left(\begin{array}{l} -0.041x_1^4 + 0.003x_1^3x_2 - 0.07x_1^2x_2^2 \\ -0.007x_1x_2^3 - 0.048x_2^4 - 1.798 \times 10^{-9}x_1^3 \\ -2.391 \times 10^{-7}x_1^2x_2 + 1.166 \times 10^{-8}x_1x_2^2 \\ -1.528 \times 10^{-8}x_2^3 \\ +0.329x_1^2 - 0.007x_1x_2 + 0.189x_2^2 \\ +2.044 \times 10^{-7}x_1 - 1.85 \times 10^{-7}x_2 \\ +0.726 \end{array} \right) \\ \left(\begin{array}{l} -0.72x_1^4 + 0.039x_1^3x_2 - 0.172x_1^2x_2^2 \\ +0.002x_1x_2^3 - 0.008x_2^4 - 7.09 \times 10^{-7}x_1^3 \\ +2.284 \times 10^{-7}x_1^2x_2 - 2.844 \times 10^{-7}x_1x_2^2 \\ +2.394 \times 10^{-8}x_2^3 \\ -1.198x_1^2 + 0.116x_1x_2 - 0.501x_2^2 \\ -7.997 \times 10^{-7}x_1 + 3.686 \times 10^{-7}x_2 \\ -2.561 \end{array} \right) \left(\begin{array}{l} 5.951x_1^4 + 0.225x_1^3x_2 + 1.64x_1^2x_2^2 \\ +0.09x_1x_2^3 + 0.334x_2^4 + 4.387 \times 10^{-7}x_1^3 \\ -3.442 \times 10^{-8}x_1^2x_2 + 6.448 \times 10^{-7}x_1x_2^2 \\ +2.576 \times 10^{-7}x_2^3 \\ +5.1x_1^2 + 0.149x_1x_2 + 1.117x_2^2 \\ +5.792 \times 10^{-7}x_1 + 7.356 \times 10^{-8}x_2 \\ +7.597 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{23}(\mathbf{x}) = \left[\begin{array}{l} \left(\begin{array}{l} -0.32x_1^4 + 0.06x_1^3x_2 + 0.168x_1^2x_2^2 \\ +0.014x_1x_2^3 + 0.004x_2^4 - 3.581 \times 10^{-7}x_1^3 \\ -1.826 \times 10^{-7}x_1^2x_2 - 2.382 \times 10^{-7}x_1x_2^2 - 3.077 \times 10^{-7}x_2^3 \\ -0.175x_1^2 + 0.042x_1x_2 + 0.111x_2^2 \\ -4.839 \times 10^{-7}x_1 - 3.133 \times 10^{-7}x_2 - 0.489 \end{array} \right) \\ \left(\begin{array}{l} -1.951x_1^4 - 0.367x_1^3x_2 - 0.84x_1^2x_2^2 \\ -0.294x_1x_2^3 + 0.136x_2^4 - 1.364 \times 10^{-6}x_1^3 \\ -3.093 \times 10^{-7}x_1^2x_2 - 2.501 \times 10^{-7}x_1x_2^2 - 1.076 \times 10^{-7}x_2^3 \\ -2.35x_1^2 + 0.061x_1x_2 - 1.248x_2^2 \\ -1.48 \times 10^{-6}x_1 - 4.076 \times 10^{-7}x_2 - 3.971 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{31}(\mathbf{x}) =$$

$$\left[\begin{array}{c} \left(\begin{array}{l} 0.269x_1^4 + 0.436x_1^3x_2 + 0.174x_1^2x_2^2 \\ +7.399 \times 10^{-9}x_1^3 - 1.828 \times 10^{-8}x_1^2x_2 \\ +1.293 \times 10^{-7}x_1x_2^2 \\ +0.039x_1^2 + 0.647x_1x_2 + 0.443x_2^2 \\ +2.045 \times 10^{-7}x_1 \\ +1.677 \times 10^{-7}x_2 + 0.628 \end{array} \right) \\ \left(\begin{array}{l} 2.695x_1^4 - 1.721x_1^3x_2 + 3.064x_1^2x_2^2 \\ +1.068 \times 10^{-6}x_1^3 - 4.598 \times 10^{-7}x_1^2x_2 \\ +7.125 \times 10^{-7}x_1x_2^2 \\ +3.584x_1^2 - 1.08x_1x_2 + 4.127x_2^2 \\ +1.293 \times 10^{-6}x_1 \\ -2.099 \times 10^{-8}x_2 + 4.058 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{32}(\mathbf{x}) =$$

$$\left[\begin{array}{c} \left(\begin{array}{l} 1.183x_1^4 + 0.06x_1^3x_2 + 0.284x_1^2x_2^2 \\ +0.008x_1x_2^3 + 0.043x_2^4 + 1.739 \times 10^{-7}x_1^3 \\ +1.384 \times 10^{-7}x_1^2x_2 + 1.359 \times 10^{-7}x_1x_2^2 \\ +1.2 \times 10^{-7}x_2^3 \\ +1.201x_1^2 + 0.055x_1x_2 + 0.293x_2^2 \\ +1.369 \times 10^{-7}x_1 \\ +7.228 \times 10^{-8}x_2 + 1.587 \end{array} \right) \\ \left(\begin{array}{l} 0.821x_1^4 + 0.044x_1^3x_2 - 0.103x_1^2x_2^2 \\ +0.021x_1x_2^3 - 0.028x_2^4 + 7.371 \times 10^{-7}x_1^3 \\ +5.623 \times 10^{-8}x_1^2x_2 - 7.613 \times 10^{-8}x_1x_2^2 \\ -1.503 \times 10^{-9}x_2^3 \\ +1.091x_1^2 - 0.02x_1x_2 + 0.115x_2^2 \\ +8.365 \times 10^{-7}x_1 \\ -2.913 \times 10^{-7}x_2 + 1.637 \end{array} \right) \end{array} \right]$$

$$\begin{aligned} \mathbf{X}_{33}(\mathbf{x}) &= 1.538x_1^4 + 0.105x_1^3x_2 + 1.187x_1^2x_2^2 - 0.087x_1x_2^3 + 1.426x_2^4 \\ &+ 4.902 \times 10^{-7}x_1^3 + 2.796 \times 10^{-7}x_1^2x_2 + 4.12 \times 10^{-7}x_1x_2^2 + 2.808 \times 10^{-7}x_2^3 \\ &+ 1.286x_1^2 + 0.066x_1x_2 + 1.473x_2^2 + 5.504 \times 10^{-7}x_1 + 6.154 \times 10^{-7}x_2 + 1.76 \end{aligned}$$

$$\mathbf{M}_1(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{l} 4.299x_1^2 + 0.611x_1x_2 \\ +0.095x_2^2 + 3.03 \times 10^{-6}x_1 \\ +1.055 \times 10^{-6}x_2 + 9.47 \end{array} \right) \\ \left(\begin{array}{l} 9.458x_1^2 - 0.558x_1x_2 \\ +6.275x_2^2 + 2.318 \times 10^{-6}x_1 \\ +4.161 \times 10^{-6}x_2 + 8.526 \end{array} \right) \end{array} \right]$$

$$\mathbf{M}_2(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} 3.019x_1^2 + 0.684x_1x_2 \\ +0.121x_2^2 + 2.079 \times 10^{-6}x_1 \\ +8.995 \times 10^{-7}x_2 + 9.445 \end{array} \right) \\ \left(\begin{array}{c} 8.073x_1^2 - 0.588x_1x_2 \\ +6.254x_2^2 + 2.756 \times 10^{-6}x_1 \\ +4.144 \times 10^{-6}x_2 + 8.507 \end{array} \right) \end{array} \right]$$

By choosing the initial conditions as $x_1 = \frac{\pi}{6}$ and $x_2 = 0$, the simulation results are shown in Fig 4.4, 4.5, and 4.6. It is obtained that the bicycle system controlled by the proposed Theorem2 is asymptotically stable.

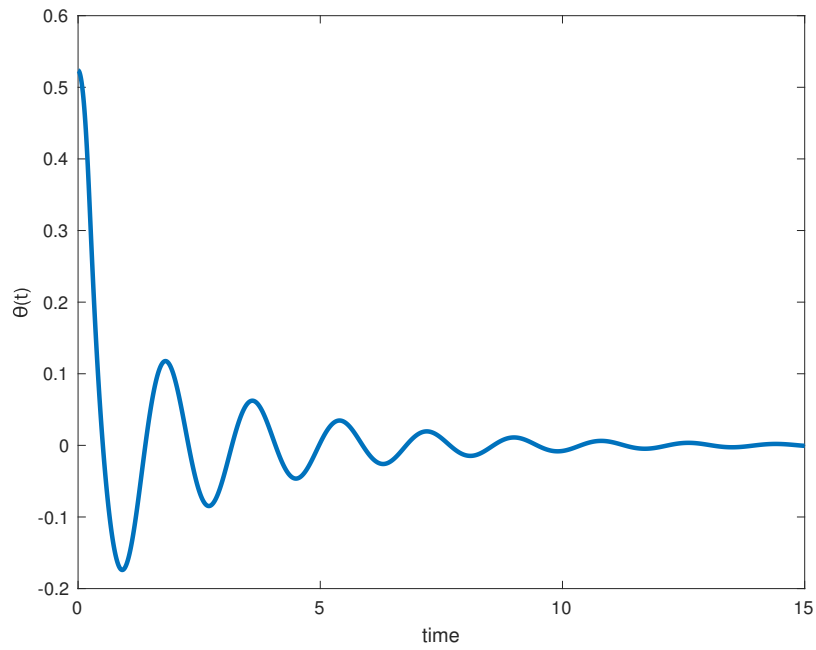


Figure 4.4: The bicycle's angle

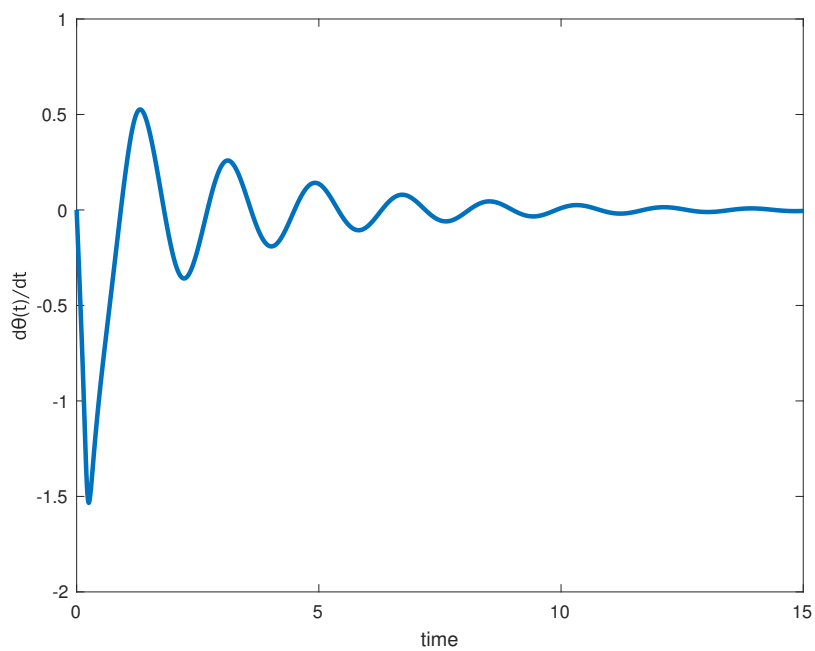


Figure 4.5: The bicycle's angle speed

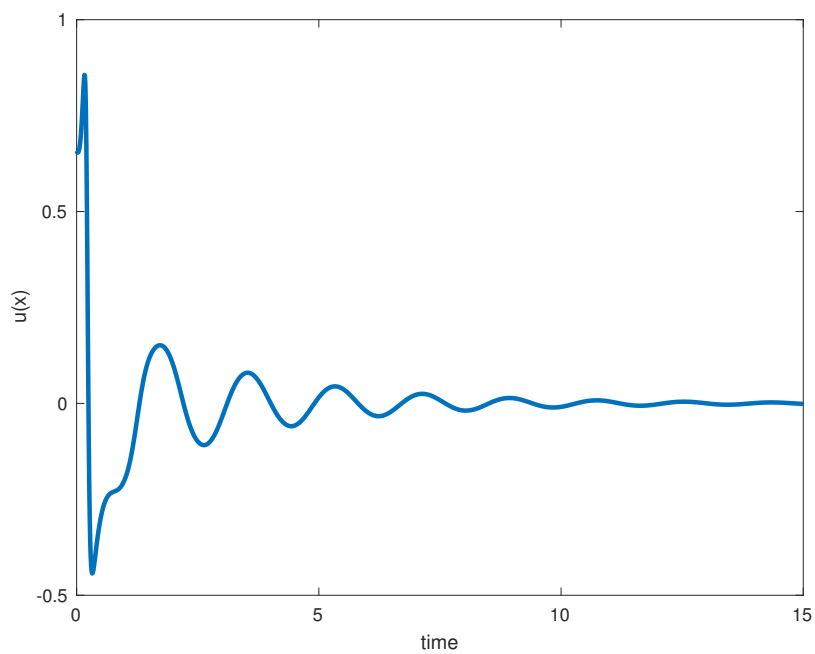


Figure 4.6: The control input

STABILIZATION BY FUZZY SLACK MATRICES AND NOVEL FUZZY LYAPUNOV CANDIDATE

Chapter 4 presents a control design of polynomial FMB system beyond descriptor form's methodology. In this chapter, the research tries different ways to do the previous chapter's stabilization. Same as Chapter 4, a polynomial fuzzy model with PDC-based controller forms a closed-loop system and is applied by descriptor representation. In the first proposed theorem in chapter 5, a common Lyapunov function like Chapter 4 is applied for stabilization. The difference is that chapter 5 brings polynomial fuzzy slack matrices into stabilization based on descriptor form representation's redundancy. The fuzzy summation problem, caused by the fuzzy slack matrices, is solved by co-positivity relaxation because it can be seen as the co-positivity problem. In the second proposed theorem in chapter 5, a novel fuzzy Lyapunov function is proposed to do the stabilization analysis. Note that the second proposed theorem is based on the situation that input vectors have no relation to membership functions. As the former is included in the novel fuzzy Lyapunov function and is taken as a special case, the fact that the second theorem obtains less conservative stabilization results than the first theorem is held. Because of the applying of novel fuzzy Lyapunov function, the membership functions' time derivatives are required to apply in the stabilization analysis. This extraction divides the differential function into common factors and the rest. Hence, to deal with this rest, the sector nonlinearity technique is applied. Finally, this chapter presents six examples to present the relaxation and comparison between these proposed theorems and previous studies.

5.1 Stabilization via fuzzy slack matrices

Consider a polynomial fuzzy model (2.9) applied with the PDC-based controller (4.1) and it is presented as the descriptor form (4.3). This section stabilizes the closed-loop system (4.3) by fuzzy slack matrices. The fuzzy slack matrix is defined as

$$\chi_{jk}(\mathbf{h}, \mathbf{x}) = \sum_{i=1}^r h_i(\mathbf{z}) \mathbf{X}_{jki}(\mathbf{x}), \quad j = 2, 3, k = 1, 2, 3 \quad (5.1)$$

where $\mathbf{X}_{jki}(\mathbf{x})$ are polynomial matrices in \mathbf{x} . Therefore, the Lyapunov function (4.10) can be rewritten as

$$V(\mathbf{x}) = \mathbf{x}^T \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \mathbf{x} = \mathbf{x}^{*T}(\mathbf{x}) \mathbf{E}^* \Theta^{-1}(\mathbf{h}, \mathbf{x}) \mathbf{x}^*(\mathbf{x}) \quad (5.2)$$

where

$$\Theta(\mathbf{h}, \mathbf{x}) = \begin{bmatrix} \mathbf{X}(\tilde{\mathbf{x}}) & 0 & 0 \\ \chi_{21}(\mathbf{h}, \mathbf{x}) & \chi_{22}(\mathbf{h}, \mathbf{x}) & \chi_{23}(\mathbf{h}, \mathbf{x}) \\ \chi_{31}(\mathbf{h}, \mathbf{x}) & \chi_{32}(\mathbf{h}, \mathbf{x}) & \chi_{33}(\mathbf{h}, \mathbf{x}) \end{bmatrix}. \quad (5.3)$$

By substituting the Lyapunov function (4.10) into the augmented form (5.2), the fuzzy slack matrices $\chi_{jk}(\mathbf{h}, \mathbf{x})$ approach's stabilization criterion are obtained.

Theorem 3:

Consider the operation domain (2.1). If there are polynomials $\sigma_{ij\beta}(\mathbf{x})$, a symmetric polynomial matrix $\mathbf{X}(\tilde{\mathbf{x}})$ and polynomial matrices $\mathbf{M}_i(\mathbf{x})$, $\mathbf{X}_{jki}(\mathbf{x})$, the system (2.9) is asymptotically stable when satisfying

$$\mathbf{v}^T (\hat{\mathbf{X}}_i(\mathbf{x}) - \epsilon_{1i}(\mathbf{x}) \mathbf{I}) \mathbf{v} \text{ is SOS}, \quad i = 1, \dots, r \quad (5.4)$$

$$- \left(\sum_{k=1}^r \hat{h}_k^2 \right)^s \sum_{i=1}^r \sum_{j=1}^r \hat{h}_i^2 \hat{h}_j^2 \mathbf{v}_1^T \mathbf{H}_{ij}(\mathbf{x}) \mathbf{v}_1 \text{ is SOS} \quad (5.5)$$

$$\sigma_{ij\beta}(\mathbf{x}) \text{ is SOS}, \quad i, j = 1, \dots, r, \beta = 1, \dots, n \quad (5.6)$$

where \mathbf{v} and \mathbf{v}_1 are the vectors which are independent from \mathbf{x} ; s is a non-negative integer;

$$\hat{\mathbf{X}}_i(\mathbf{x}) = \begin{bmatrix} \mathbf{X}(\tilde{\mathbf{x}}) & 0 & 0 \\ \mathbf{X}_{21i}(\mathbf{x}) & \mathbf{X}_{22i}(\mathbf{x}) & \mathbf{X}_{23i}(\mathbf{x}) \\ \mathbf{X}_{31i}(\mathbf{x}) & \mathbf{X}_{32i}(\mathbf{x}) & \mathbf{X}_{33i}(\mathbf{x}) \end{bmatrix}$$

and $\mathbf{H}_{ij}(\mathbf{x})$ are given as

$$\mathbf{H}_{ij}(\mathbf{x}) = \begin{bmatrix} \begin{pmatrix} \mathbf{A}_j(\mathbf{x})\mathbf{X}_{21i}(\mathbf{x}) + \mathbf{B}_j(\mathbf{x})\mathbf{X}_{31i}(\mathbf{x}) \\ + \mathbf{X}_{21i}^T(\mathbf{x})\mathbf{A}_j^T(\mathbf{x}) + \mathbf{X}_{31i}^T(\mathbf{x})\mathbf{B}_j^T(\mathbf{x}) \\ - \sum_{k \in \mathbf{K}} \frac{\partial \mathbf{X}}{\partial x_k}(\tilde{\mathbf{x}})\mathbf{A}_j^k(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x}) \\ - \sum_{\beta=1}^n \sigma_{ij\beta}(\mathbf{x})Q_\beta(\mathbf{x})\mathbf{I} + \epsilon_{2ij}(\mathbf{x})\mathbf{I} \end{pmatrix} & * & * \\ \begin{pmatrix} \mathbf{T}(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) - \mathbf{X}_{21i}(\mathbf{x}) + \\ \mathbf{X}_{22i}^T(\mathbf{x})\mathbf{A}_j^T(\mathbf{x}) + \mathbf{X}_{32i}^T(\mathbf{x})\mathbf{B}_j^T(\mathbf{x}) \end{pmatrix} & \begin{pmatrix} -\mathbf{X}_{22i}(\mathbf{x}) - \mathbf{X}_{22i}^T(\mathbf{x}) \end{pmatrix} & * \\ \begin{pmatrix} \mathbf{M}_j(\mathbf{x}) - \mathbf{X}_{31i}(\mathbf{x}) + \\ \mathbf{X}_{23i}^T(\mathbf{x})\mathbf{A}_j^T(\mathbf{x}) + \mathbf{X}_{33i}^T(\mathbf{x})\mathbf{B}_j^T(\mathbf{x}) \end{pmatrix} & \begin{pmatrix} -\mathbf{X}_{32i}(\mathbf{x}) - \mathbf{X}_{23i}^T(\mathbf{x}) \end{pmatrix} & \begin{pmatrix} -\mathbf{X}_{33i}(\mathbf{x}) - \mathbf{X}_{33i}^T(\mathbf{x}) \end{pmatrix} \end{bmatrix} \quad (5.7)$$

in which

$$Q_\beta(\mathbf{x}) = (x_\beta - x_\beta^{\max})(x_\beta - x_\beta^{\min}), \quad \beta = 1, \dots, n$$

polynomials $\epsilon_{1i}(\mathbf{x}) > 0$ and $\epsilon_{2ij}(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$. The following equation can be obtained the stabilizing feedback gain.

$$\mathbf{F}_j(\mathbf{x}) = \mathbf{M}_j(\mathbf{x})\mathbf{X}^{-1}(\tilde{\mathbf{x}}). \quad (5.8)$$

Proof:

Consider the Lyapunov function (4.10) in which $\mathbf{X}(\tilde{\mathbf{x}}) > 0$ is satisfied for $\mathbf{x} \neq 0$ if (5.4) holds. The Lyapunov function $V(\mathbf{x})$'s time derivative are

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T \mathbf{X}^{-1}(\tilde{\mathbf{x}})\mathbf{x} + \mathbf{x}^T \mathbf{X}^{-1}(\tilde{\mathbf{x}})\dot{\mathbf{x}} + \mathbf{x}^T \dot{\mathbf{X}}^{-1}(\tilde{\mathbf{x}})\mathbf{x} \\ &= \dot{\mathbf{x}}^T \mathbf{X}^{-1}(\tilde{\mathbf{x}})\mathbf{x} + \mathbf{x}^T \mathbf{X}^{-1}(\tilde{\mathbf{x}})\dot{\mathbf{x}} \\ &\quad + \mathbf{x}^T \sum_{k=1}^n \frac{\partial \mathbf{X}^{-1}}{\partial x_k}(\tilde{\mathbf{x}})\dot{x}_k. \end{aligned} \quad (5.9)$$

From the definition of \mathbf{K} , it is obtained that

$$\dot{\mathbf{x}}_k = \sum_{i=1}^r h_i(z) \mathbf{A}_i^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}) \quad (5.10)$$

for $k \in \mathbf{K}$, and

$$\frac{\partial \mathbf{X}^{-1}}{\partial x_i}(\tilde{\mathbf{x}}) = 0 \quad (5.11)$$

for $i \notin \mathbf{K}$. Moreover, $\mathbf{X}^{-1}(\tilde{\mathbf{x}})\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{I}$ holds the following relation [37]:

$$\frac{\partial \mathbf{X}^{-1}}{\partial x_k}(\tilde{\mathbf{x}}) = -\mathbf{X}^{-1}(\tilde{\mathbf{x}}) \frac{\partial \mathbf{X}}{\partial x_k}(\tilde{\mathbf{x}}) \mathbf{X}^{-1}(\tilde{\mathbf{x}}). \quad (5.12)$$

The derivative of Lyapunov function $V(\mathbf{x})$'s with respect of time can be represented as following by applying the augmented fuzzy matrix $\Theta(\mathbf{h}, \mathbf{x})$ of (5.3) with (5.9) - (5.12).

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^{*T}(\mathbf{x}) \mathbf{E}^* \Theta^{-1}(\mathbf{h}, \mathbf{x}) \mathbf{x}^*(\mathbf{x}) \\ &\quad + \mathbf{x}^{*T}(\mathbf{x}) \Theta^{-T}(\mathbf{h}, \mathbf{x}) \mathbf{E}^* \dot{\mathbf{x}}^*(\mathbf{x}) \\ &\quad - \mathbf{x}^{*T}(\mathbf{x}) \Theta^{-T}(\mathbf{h}, \mathbf{x}) \sum_{j=1}^r h_j(z) \\ &\quad \times \Gamma_j(\mathbf{x}) \Theta^{-1}(\mathbf{h}, \mathbf{x}) \mathbf{x}^*(\mathbf{x}) \\ &= \sum_{j=1}^r h_j(z) \mathbf{x}^{*T}(\mathbf{x}) \mathbf{A}_j^{*T}(\mathbf{x}) \Theta^{-1}(\mathbf{h}, \mathbf{x}) \mathbf{x}^*(\mathbf{x}) \\ &\quad + \mathbf{x}^{*T}(\mathbf{x}) \Theta^{-T}(\mathbf{h}, \mathbf{x}) \sum_{j=1}^r h_j(z) \mathbf{A}_j^*(\mathbf{x}) \mathbf{x}^*(\mathbf{x}) \\ &\quad - \mathbf{x}^{*T}(\mathbf{x}) \Theta^{-T}(\mathbf{h}, \mathbf{x}) \sum_{j=1}^r h_j(z) \\ &\quad \times \Gamma_j(\mathbf{x}) \Theta^{-1}(\mathbf{h}, \mathbf{x}) \mathbf{x}^*(\mathbf{x}) \end{aligned} \quad (5.13)$$

where

$$\Gamma_j(\mathbf{x}) = \begin{bmatrix} \sum_{k \in \mathbf{K}} \frac{\partial \mathbf{X}}{\partial x_k}(\tilde{\mathbf{x}}) \mathbf{A}_j^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Furthermore, in the operation domain D_{op} , we have the inequality:

$$\psi(\mathbf{h}, \mathbf{x}) = - \sum_{i=1}^r \sum_{j=1}^r \sum_{\beta=1}^n h_i(z) h_j(z) \sigma_{ij\beta}(\mathbf{x}) Q_\beta(\mathbf{x}) \geq 0 \quad (5.14)$$

where $\sigma_{ij\beta}(\mathbf{x}) \geq 0$ which is satisfied by (5.6). Define

$$\Psi(\mathbf{h}, \mathbf{x}) = \begin{bmatrix} (\psi(\mathbf{h}, \mathbf{x}) + \epsilon_2(\mathbf{h}, \mathbf{x}))\mathbf{I} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\epsilon_2(\mathbf{h}, \mathbf{x}) = \sum_{i=1}^r \sum_{j=1}^r h_j(\mathbf{z})h_i(\mathbf{z})\epsilon_{2ij}(\mathbf{x})$ being with polynomials $\epsilon_{2ij}(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$. According to (2.10), $\epsilon_2(\mathbf{h}, \mathbf{x}) > 0$ for $\mathbf{x} \neq 0$. In addition, (5.14) holds in D_{op} and $\mathbf{X}(\tilde{\mathbf{x}})$ is a symmetric positive definite matrix. From the introduction of congruence in preliminaries, the inequality

$$-\mathbf{X}^{-1}(\tilde{\mathbf{x}})(\psi(\mathbf{h}, \mathbf{x}) + \epsilon_2(\mathbf{h}, \mathbf{x}))\mathbf{I}\mathbf{X}^{-1}(\tilde{\mathbf{x}}) < 0 \quad (5.15)$$

holds for $D_{op} - \{0\}$. From (5.15), it can be obtained that

$$\begin{aligned} & -\mathbf{x}^T \mathbf{X}^{-1}(\tilde{\mathbf{x}})(\psi(\mathbf{h}, \mathbf{x}) + \epsilon_2(\mathbf{h}, \mathbf{x}))\mathbf{I}\mathbf{X}^{-1}(\tilde{\mathbf{x}})\mathbf{x} \\ & = -\mathbf{x}^{*T}(\mathbf{x})\Theta^{-T}(\mathbf{h}, \mathbf{x})\Psi(\mathbf{h}, \mathbf{x})\Theta^{-1}(\mathbf{h}, \mathbf{x})\mathbf{x}^*(\mathbf{x}) < 0 \end{aligned} \quad (5.16)$$

for $D_{op} - \{0\}$. Therefore, $\dot{V}(\mathbf{x}) < 0$ for $D_{op} - \{0\}$ is satisfied when the following condition hold

$$\dot{V}(\mathbf{x}) \leq -\mathbf{x}^{*T}(\mathbf{x})\Theta^{-T}(\mathbf{h}, \mathbf{x})\Psi(\mathbf{h}, \mathbf{x})\Theta^{-1}(\mathbf{h}, \mathbf{x})\mathbf{x}^*(\mathbf{x}). \quad (5.17)$$

Applying $\dot{V}(\mathbf{x})$ of (5.13) and letting (5.17)'s the right part move to the left, (5.17) is able to be rewritten as

$$\begin{aligned} & \sum_{j=1}^r h_j(\mathbf{z})\mathbf{x}^{*T}(\mathbf{x})\mathbf{A}_j^{*T}(\mathbf{x})\Theta^{-1}(\mathbf{h}, \mathbf{x})\mathbf{x}^*(\mathbf{x}) \\ & + \mathbf{x}^{*T}(\mathbf{x})\Theta^{-T}(\mathbf{h}, \mathbf{x})\sum_{j=1}^r h_j(\mathbf{z})\mathbf{A}_j^*(\mathbf{x})\mathbf{x}^*(\mathbf{x}) \\ & - \mathbf{x}^{*T}(\mathbf{x})\Theta^{-T}(\mathbf{h}, \mathbf{x})\sum_{j=1}^r h_j(\mathbf{z})\Gamma_j(\mathbf{x})\Theta^{-1}(\mathbf{h}, \mathbf{x})\mathbf{x}^*(\mathbf{x}) \\ & + \mathbf{x}^{*T}(\mathbf{x})\Theta^{-T}(\mathbf{h}, \mathbf{x})\Psi(\mathbf{h}, \mathbf{x})\Theta^{-1}(\mathbf{h}, \mathbf{x})\mathbf{x}^*(\mathbf{x}) \leq 0. \end{aligned} \quad (5.18)$$

Inequality (5.18) holds if

$$\begin{aligned} & \sum_{j=1}^r h_j(\mathbf{z})\{\mathbf{A}_j^{*T}(\mathbf{x})\Theta^{-1}(\mathbf{h}, \mathbf{x}) + \Theta^{-T}(\mathbf{h}, \mathbf{x})\mathbf{A}_j^*(\mathbf{x}) - \Theta^{-T}(\mathbf{h}, \mathbf{x})\Gamma_j(\mathbf{x})\Theta^{-1}(\mathbf{h}, \mathbf{x})\} \\ & + \Theta^{-T}(\mathbf{h}, \mathbf{x})\Psi(\mathbf{h}, \mathbf{x})\Theta^{-1}(\mathbf{h}, \mathbf{x}) \leq 0. \end{aligned} \quad (5.19)$$

Multiplying (5.19)'s left by $\Theta^T(\mathbf{h}, \mathbf{x})$ and right by $\Theta(\mathbf{h}, \mathbf{x})$ respectively to obtained simplify the inequality above

$$\sum_{i=1}^r \sum_{j=1}^r h_i(\mathbf{z})h_j(\mathbf{z})\{\hat{\mathbf{X}}_i^T(\mathbf{x})\mathbf{A}_j^{*T}(\mathbf{x}) + \mathbf{A}_j^*(\mathbf{x})\hat{\mathbf{X}}_i(\mathbf{x}) - \mathbf{\Gamma}_j(\mathbf{x}) + \begin{bmatrix} (-\sum_{\beta=1}^n \sigma_{ij\beta}(\mathbf{x})Q_\beta(\mathbf{x}) + \epsilon_{2ij}(\mathbf{x}))\mathbf{I} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\} \leq 0. \quad (5.20)$$

Letting $\mathbf{M}_j(\mathbf{x}) = \mathbf{F}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})$, if the following condition:

$$\sum_{i=1}^r \sum_{j=1}^r h_i(\mathbf{z})h_j(\mathbf{z})\mathbf{v}_1^T(-\mathbf{H}_{ij}(\mathbf{x}))\mathbf{v}_1 \geq 0. \quad (5.21)$$

holds for all $\mathbf{v}_1 \in \mathbb{R}^{2n+m}$, inequality (5.20) holds

By applying the technique of co-positive relaxation in preliminaries' chapter, the condition (5.5) guarantees (5.21) being satisfied that means $\dot{V}(\mathbf{x}) < 0$ for $D_{op} - \{0\}$. \square

Remark 3. In [37], the polynomial fuzzy model (2.9) is stabilized by the following commonly used Lyapunov function:

$$V(\mathbf{x}) = \hat{\mathbf{x}}^T(\mathbf{x})\mathbf{X}^{-1}(\tilde{\mathbf{x}})\hat{\mathbf{x}}(\mathbf{x}). \quad (5.22)$$

In contrast, the Lyapunov function (4.10) in chapter 4 is seen as a slightly modified version of (5.22) which changes $\hat{\mathbf{x}}(\mathbf{x})$ to \mathbf{x} . Another thing is that the descriptor representation (4.3)'s redundancy allows fuzzy slack matrices $\chi_{jk}(\mathbf{h}, \mathbf{x})$ be introduced in the designing of stabilization. These slack matrices make Theorem 3 get more relaxed results than [37] in some cases. Example 1 shows a numerical case to prove this remark. One thing that has to be noted is that this Theorem does not guarantee all the cases to have more relaxed results than [37]. The reason is the fact that the stabilization criterion of [37] cannot be taken as Theorem 3's special case.

5.2 Stabilization via the novel fuzzy Lyapunov function

The following design methodology for stabilization is based on the cases that the inputs vectors don't relate to membership functions (i.e. $h_i(\mathbf{z}) = h_i(\tilde{\mathbf{x}}) \forall i$) in this subsection. A

function called "Novel Fuzzy" is proposed for obtaining more relaxed stabilization results. Furthermore, extracting membership functions' time derivatives should be an issue when doing the stabilization analysis since the novel fuzzy Lyapunov function is concerned. The preliminaries have introduced membership functions' differential extraction, which is presented as

$$\dot{h}_i(\tilde{\mathbf{x}}) = \sum_{m=1}^2 \omega_{im}(\mathbf{x}) \mu_{im}(\mathbf{x}) \quad (5.23)$$

with the techniques of sector nonlinearity. Where $\mu_{im}(\mathbf{x})$ are polynomial functions in \mathbf{x} and

$$\sum_{m=1}^2 \omega_{im}(\mathbf{x}) = 1 \quad (5.24)$$

(For more details, see the preliminaries' Chapter). The membership functions' differential extraction (5.23) and novel fuzzy Lyapunov function produce the stabilization criterion, which will be introduced in Theorem 4.

Theorem 4:

Consider the operation domain (2.1) and for $h_i(\mathbf{z}) = h_i(\tilde{\mathbf{x}}) \forall i$ satisfying (5.23), if there exist polynomials $\sigma_{ij\beta}(\mathbf{x})$, symmetric polynomial matrices $\mathbf{X}_i(\tilde{\mathbf{x}})$ and polynomial matrices $\mathbf{M}_i(\mathbf{x})$, $\mathbf{X}_{jki}(\mathbf{x})$, the transformed polynomial fuzzy descriptor system (4.3) is asymptotically stable when the conditions below are satisfied.

$$\mathbf{v}^T (\hat{\mathbf{X}}_i^*(\mathbf{x}) - \epsilon_{1i}(\mathbf{x}) \mathbf{I}) \mathbf{v} \text{ is SOS } \quad i = 1, \dots, r \quad (5.25)$$

$$- \left(\sum_{k=1}^r \hat{h}_k^2 \right)^s \sum_{i=1}^r \sum_{j=1}^r \hat{h}_i^2 \hat{h}_j^2 \mathbf{v}_1^T \bar{\mathbf{H}}_{ij\rho m}(\mathbf{x}) \mathbf{v}_1 \text{ is SOS} \quad (5.26)$$

$$\rho = 1, \dots, r-1, \quad m = 1, 2$$

$$\sigma_{ij\beta}(\mathbf{x}) \text{ is SOS } \quad i, j = 1, \dots, r, \quad \beta = 1, \dots, n \quad (5.27)$$

where \mathbf{v} and \mathbf{v}_1 are the vectors being independent from \mathbf{x} ; s is a non-negative integer;

$$\hat{\mathbf{X}}_i^*(\mathbf{x}) = \begin{bmatrix} \mathbf{X}_i(\tilde{\mathbf{x}}) & 0 & 0 \\ \mathbf{X}_{21i}(\mathbf{x}) & \mathbf{X}_{22i}(\mathbf{x}) & \mathbf{X}_{23i}(\mathbf{x}) \\ \mathbf{X}_{31i}(\mathbf{x}) & \mathbf{X}_{32i}(\mathbf{x}) & \mathbf{X}_{33i}(\mathbf{x}) \end{bmatrix}$$

and $\bar{\mathbf{H}}_{ij\rho m}(\mathbf{x})$ are

$$\bar{\mathbf{H}}_{ij\rho m}(\mathbf{x}) = \left[\begin{array}{l} \left(\begin{array}{l} \mathbf{A}_j(\mathbf{x})\mathbf{X}_{21i}(\mathbf{x}) + \mathbf{B}_j(\mathbf{x})\mathbf{X}_{31i}(\mathbf{x}) \\ + \mathbf{X}_{21i}^T(\mathbf{x})\mathbf{A}_j^T(\mathbf{x}) + \mathbf{X}_{31i}^T(\mathbf{x})\mathbf{B}_j^T(\mathbf{x}) \\ - \sum_{k \in K} \frac{\partial \mathbf{X}_i}{\partial x_k}(\tilde{\mathbf{x}})\mathbf{A}_j^k(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x}) \\ - (r-1)\mu_{\rho m}(\mathbf{x})(\mathbf{X}_\rho(\tilde{\mathbf{x}}) - \mathbf{X}_r(\tilde{\mathbf{x}})) \\ - \sum_{\beta=1}^n \sigma_{ij\beta}(\mathbf{x})Q_\beta(\mathbf{x})\mathbf{I} + \epsilon_{2ij}(\mathbf{x})\mathbf{I} \end{array} \right) \quad * \quad * \\ \left(\begin{array}{l} \mathbf{T}(\mathbf{x})\mathbf{X}_i(\tilde{\mathbf{x}}) - \mathbf{X}_{21i}(\mathbf{x}) + \\ \mathbf{X}_{22i}^T(\mathbf{x})\mathbf{A}_j^T(\mathbf{x}) + \mathbf{X}_{32i}^T(\mathbf{x})\mathbf{B}_j^T(\mathbf{x}) \end{array} \right) \quad -\mathbf{X}_{22i}(\mathbf{x}) - \mathbf{X}_{22i}^T(\mathbf{x}) \quad * \\ \left(\begin{array}{l} \mathbf{M}_j(\mathbf{x}) - \mathbf{X}_{31i}(\mathbf{x}) + \\ \mathbf{X}_{23i}^T(\mathbf{x})\mathbf{A}_j^T(\mathbf{x}) + \mathbf{X}_{33i}^T(\mathbf{x})\mathbf{B}_j^T(\mathbf{x}) \end{array} \right) \quad -\mathbf{X}_{32i}(\mathbf{x}) - \mathbf{X}_{23i}^T(\mathbf{x}) \quad -(\mathbf{X}_{33i}(\mathbf{x}) + \mathbf{X}_{33i}^T(\mathbf{x})) \end{array} \right] \quad (5.28)$$

in which

$$Q_\beta(\mathbf{x}) = (x_\beta - x_\beta^{\max})(x_\beta - x_\beta^{\min}), \quad \beta = 1, \dots, n$$

polynomials $\epsilon_{1i}(\mathbf{x}) > 0$ and $\epsilon_{2ij}(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$. It can be obtained that the stabilizing feedback gain is

$$\mathbf{F}_j(\mathbf{x}) = \mathbf{M}_j(\mathbf{x})\bar{\mathbf{X}}^{-1}(\mathbf{h}, \tilde{\mathbf{x}}). \quad (5.29)$$

Proof:

Choose a novel fuzzy Lyapunov function as the candidate

$$V(\mathbf{x}) = \mathbf{x}^T \bar{\mathbf{X}}^{-1}(\mathbf{h}, \tilde{\mathbf{x}})\mathbf{x} \quad (5.30)$$

where

$$\bar{\mathbf{X}}(\mathbf{h}, \tilde{\mathbf{x}}) = \sum_{i=1}^r h_i(\tilde{\mathbf{x}})\mathbf{X}_i(\tilde{\mathbf{x}})$$

and $\mathbf{X}_i(\tilde{\mathbf{x}})$ are symmetric polynomial in $\tilde{\mathbf{x}}$ of (4.4). If (5.25) holds, the inequality $\bar{\mathbf{X}}(\mathbf{h}, \tilde{\mathbf{x}}) > 0$ is satisfied for $\mathbf{x} \neq 0$. The novel fuzzy Lyapunov function $V(\mathbf{x})$'s time derivatives are

$$\dot{V}(\mathbf{x}) = \dot{\mathbf{x}}^T \bar{\mathbf{X}}^{-1}(\mathbf{h}, \tilde{\mathbf{x}}) \mathbf{x} + \mathbf{x}^T \bar{\mathbf{X}}^{-1}(\mathbf{h}, \tilde{\mathbf{x}}) \dot{\mathbf{x}} + \mathbf{x}^T \dot{\bar{\mathbf{X}}}^{-1}(\mathbf{h}, \tilde{\mathbf{x}}) \mathbf{x}. \quad (5.31)$$

Since $\bar{\mathbf{X}}^{-1}(\mathbf{h}, \tilde{\mathbf{x}}) \bar{\mathbf{X}}(\mathbf{h}, \tilde{\mathbf{x}}) = \mathbf{I}$, it is obtained that

$$\dot{\bar{\mathbf{X}}}^{-1}(\mathbf{h}, \tilde{\mathbf{x}}) \bar{\mathbf{X}}(\mathbf{h}, \tilde{\mathbf{x}}) + \bar{\mathbf{X}}^{-1}(\mathbf{h}, \tilde{\mathbf{x}}) \dot{\bar{\mathbf{X}}}(\mathbf{h}, \tilde{\mathbf{x}}) = 0$$

which means

$$\dot{\bar{\mathbf{X}}}^{-1}(\mathbf{h}, \tilde{\mathbf{x}}) = -\bar{\mathbf{X}}^{-1}(\mathbf{h}, \tilde{\mathbf{x}}) \dot{\bar{\mathbf{X}}}(\mathbf{h}, \tilde{\mathbf{x}}) \bar{\mathbf{X}}^{-1}(\mathbf{h}, \tilde{\mathbf{x}}). \quad (5.32)$$

Furthermore,

$$\begin{aligned} \dot{\bar{\mathbf{X}}}(\mathbf{h}, \tilde{\mathbf{x}}) &= \sum_{i=1}^r \dot{h}_i(\tilde{\mathbf{x}}) \mathbf{X}_i(\tilde{\mathbf{x}}) + \sum_{i=1}^r h_i(\tilde{\mathbf{x}}) \dot{\mathbf{X}}_i(\tilde{\mathbf{x}}) \\ &= \sum_{i=1}^r \dot{h}_i(\tilde{\mathbf{x}}) \mathbf{X}_i(\tilde{\mathbf{x}}) + \sum_{i=1}^r h_i(\tilde{\mathbf{x}}) \sum_{k=1}^n \frac{\partial \mathbf{X}_i}{\partial x_k}(\tilde{\mathbf{x}}) \dot{x}_k. \end{aligned} \quad (5.33)$$

From the definition of \mathbf{K} , (5.10) is satisfied for $k \in \mathbf{K}$, and

$$\frac{\partial \mathbf{X}_i}{\partial x_j}(\tilde{\mathbf{x}}) = 0 \quad (5.34)$$

for $j \notin \mathbf{K}$. Therefore, (5.33) is transformed to

$$\begin{aligned} \dot{\bar{\mathbf{X}}}(\mathbf{h}, \tilde{\mathbf{x}}) &= \sum_{i=1}^r \dot{h}_i(\tilde{\mathbf{x}}) \mathbf{X}_i(\tilde{\mathbf{x}}) + \sum_{i=1}^r \sum_{j=1}^r h_i(\tilde{\mathbf{x}}) h_j(\tilde{\mathbf{x}}) \\ &\quad \times \sum_{k \in \mathbf{K}} \frac{\partial \mathbf{X}_i}{\partial x_k}(\tilde{\mathbf{x}}) \mathbf{A}_j^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}). \end{aligned} \quad (5.35)$$

Taking (5.32) with (5.35) into (5.31), it is obtained that

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T \bar{\mathbf{X}}^{-1}(\mathbf{h}, \tilde{\mathbf{x}}) \mathbf{x} + \mathbf{x}^T \bar{\mathbf{X}}^{-1}(\mathbf{h}, \tilde{\mathbf{x}}) \dot{\mathbf{x}} - \mathbf{x}^T \bar{\mathbf{X}}^{-1}(\mathbf{h}, \tilde{\mathbf{x}}) \\ &\quad \times \left(\sum_{i=1}^r \dot{h}_i(\tilde{\mathbf{x}}) \mathbf{X}_i(\tilde{\mathbf{x}}) + \sum_{i=1}^r \sum_{j=1}^r h_i(\tilde{\mathbf{x}}) h_j(\tilde{\mathbf{x}}) \right. \\ &\quad \left. \times \sum_{k \in \mathbf{K}} \frac{\partial \mathbf{X}_i}{\partial x_k}(\tilde{\mathbf{x}}) \mathbf{A}_j^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}) \right) \bar{\mathbf{X}}^{-1}(\mathbf{h}, \tilde{\mathbf{x}}) \mathbf{x}. \end{aligned} \quad (5.36)$$

By defining

$$\begin{aligned}\bar{\Theta}(\mathbf{h}, \mathbf{x}) &= \begin{bmatrix} \bar{\mathbf{X}}(\mathbf{h}, \tilde{\mathbf{x}}) & 0 & 0 \\ \chi_{21}(\mathbf{h}, \mathbf{x}) & \chi_{22}(\mathbf{h}, \mathbf{x}) & \chi_{23}(\mathbf{h}, \mathbf{x}) \\ \chi_{31}(\mathbf{h}, \mathbf{x}) & \chi_{32}(\mathbf{h}, \mathbf{x}) & \chi_{33}(\mathbf{h}, \mathbf{x}) \end{bmatrix} \\ \hat{\Gamma}_{ij}(\mathbf{x}) &= \begin{bmatrix} \sum_{k \in K} \frac{\partial X_i}{\partial x_k}(\tilde{\mathbf{x}}) \mathbf{A}_j^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \bar{\Gamma}(\mathbf{h}, \mathbf{x}) &= \sum_{i=1}^r \sum_{j=1}^r h_i(\tilde{\mathbf{x}}) h_j(\tilde{\mathbf{x}}) \hat{\Gamma}_{ij}(\mathbf{x}) \\ \varphi_i(\tilde{\mathbf{x}}) &= \begin{bmatrix} \mathbf{X}_i(\tilde{\mathbf{x}}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

then it is able to rewrite (5.36) as

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^{*T}(\mathbf{x}) \mathbf{E}^* \bar{\Theta}^{-1}(\mathbf{h}, \mathbf{x}) \mathbf{x}^*(\mathbf{x}) \\ &\quad + \mathbf{x}^{*T}(\mathbf{x}) \bar{\Theta}^{-T}(\mathbf{h}, \mathbf{x}) \mathbf{E}^* \dot{\mathbf{x}}^*(\mathbf{x}) \\ &\quad - \mathbf{x}^{*T}(\mathbf{x}) \bar{\Theta}^{-T}(\mathbf{h}, \mathbf{x}) \left\{ \sum_{i=1}^r \dot{h}_i(\tilde{\mathbf{x}}) \varphi_i(\tilde{\mathbf{x}}) \right. \\ &\quad \left. + \bar{\Gamma}(\mathbf{h}, \mathbf{x}) \right\} \bar{\Theta}^{-1}(\mathbf{h}, \mathbf{x}) \mathbf{x}^*(\mathbf{x}) \\ &= \sum_{j=1}^r h_j(\tilde{\mathbf{x}}) \mathbf{x}^{*T}(\mathbf{x}) \mathbf{A}_j^{*T}(\mathbf{x}) \bar{\Theta}^{-1}(\mathbf{h}, \mathbf{x}) \mathbf{x}^*(\mathbf{x}) \\ &\quad + \mathbf{x}^{*T}(\mathbf{x}) \bar{\Theta}^{-T}(\mathbf{h}, \mathbf{x}) \sum_{j=1}^r h_j(\tilde{\mathbf{x}}) \mathbf{A}_j^*(\mathbf{x}) \mathbf{x}^*(\mathbf{x}) \\ &\quad - \mathbf{x}^{*T}(\mathbf{x}) \bar{\Theta}^{-T}(\mathbf{h}, \mathbf{x}) \left\{ \sum_{i=1}^r \dot{h}_i(\tilde{\mathbf{x}}) \varphi_i(\tilde{\mathbf{x}}) \right. \\ &\quad \left. + \bar{\Gamma}(\mathbf{h}, \mathbf{x}) \right\} \bar{\Theta}^{-1}(\mathbf{h}, \mathbf{x}) \mathbf{x}^*(\mathbf{x}).\end{aligned}\tag{5.37}$$

Furthermore, inequality (5.14) holds for D_{op} . Therefore, $\dot{V}(\mathbf{x}) < 0$ in the region $D_{op} - \{0\}$ is satisfied if

$$\dot{V}(\mathbf{x}) \leq -\mathbf{x}^{*T}(\mathbf{x}) \bar{\Theta}^{-T}(\mathbf{h}, \mathbf{x}) \Psi(\mathbf{h}, \mathbf{x}) \bar{\Theta}^{-1}(\mathbf{h}, \mathbf{x}) \mathbf{x}^*(\mathbf{x}).\tag{5.38}$$

From (5.37), inequality (5.38) can be rewritten as

$$\begin{aligned}
 & \mathbf{x}^{*T}(\mathbf{x}) \left\{ \sum_{j=1}^r h_j(\tilde{\mathbf{x}}) (\mathbf{A}_j^{*T}(\mathbf{x}) \bar{\Theta}^{-1}(\mathbf{h}, \mathbf{x}) \right. \\
 & \quad + \bar{\Theta}^{-T}(\mathbf{h}, \mathbf{x}) \mathbf{A}_j^*(\mathbf{x})) - \bar{\Theta}^{-T}(\mathbf{h}, \mathbf{x}) \left(\sum_{i=1}^r \dot{h}_i(\tilde{\mathbf{x}}) \varphi_i(\tilde{\mathbf{x}}) \right. \\
 & \quad \left. \left. + \bar{\Gamma}(\mathbf{h}, \mathbf{x}) \right) \bar{\Theta}^{-1}(\mathbf{h}, \mathbf{x}) \right. \\
 & \quad \left. + \bar{\Theta}^{-T}(\mathbf{h}, \mathbf{x}) \Psi(\mathbf{h}, \mathbf{x}) \bar{\Theta}^{-1}(\mathbf{h}, \mathbf{x}) \right\} \mathbf{x}^*(\mathbf{x}) \leq 0.
 \end{aligned} \tag{5.39}$$

Inequality (5.39) holds if

$$\begin{aligned}
 & \sum_{j=1}^r h_j(\tilde{\mathbf{x}}) \{ \mathbf{A}_j^{*T}(\mathbf{x}) \bar{\Theta}^{-1}(\mathbf{h}, \mathbf{x}) + \bar{\Theta}^{-T}(\mathbf{h}, \mathbf{x}) \mathbf{A}_j^*(\mathbf{x}) \} \\
 & \quad - \bar{\Theta}^{-T}(\mathbf{h}, \mathbf{x}) \left\{ \sum_{i=1}^r \dot{h}_i(\tilde{\mathbf{x}}) \varphi_i(\tilde{\mathbf{x}}) + \bar{\Gamma}(\mathbf{h}, \mathbf{x}) \right\} \bar{\Theta}^{-1}(\mathbf{h}, \mathbf{x}) \\
 & \quad + \bar{\Theta}^{-T}(\mathbf{h}, \mathbf{x}) \Psi(\mathbf{h}, \mathbf{x}) \bar{\Theta}^{-1}(\mathbf{h}, \mathbf{x}) \leq 0.
 \end{aligned} \tag{5.40}$$

Multiply (5.40)'s right by $\bar{\Theta}(\mathbf{h}, \mathbf{x})$ and its left by $\bar{\Theta}^T(\mathbf{h}, \mathbf{x})$ respectively, we have

$$\begin{aligned}
 & \sum_{j=1}^r h_j(\tilde{\mathbf{x}}) \{ \bar{\Theta}^T(\mathbf{h}, \mathbf{x}) \mathbf{A}_j^{*T}(\mathbf{x}) + \mathbf{A}_j^*(\mathbf{x}) \bar{\Theta}(\mathbf{h}, \mathbf{x}) \} \\
 & \quad - \sum_{\rho=1}^r \dot{h}_\rho(\tilde{\mathbf{x}}) \varphi_\rho(\tilde{\mathbf{x}}) - \bar{\Gamma}(\mathbf{h}, \mathbf{x}) + \Psi(\mathbf{h}, \mathbf{x}) \leq 0.
 \end{aligned} \tag{5.41}$$

According to (2.10), $\sum_{\rho=1}^r \dot{h}_\rho(\tilde{\mathbf{x}}) = 0$ holds such that

$$\dot{h}_r(\tilde{\mathbf{x}}) = - \sum_{\rho=1}^{r-1} \dot{h}_\rho(\tilde{\mathbf{x}}). \tag{5.42}$$

Then inequality (5.41) can be written as

$$\begin{aligned}
 & \sum_{j=1}^r h_j(\tilde{\mathbf{x}}) \{ \bar{\Theta}^T(\mathbf{h}, \mathbf{x}) \mathbf{A}_j^{*T}(\mathbf{x}) + \mathbf{A}_j^*(\mathbf{x}) \bar{\Theta}(\mathbf{h}, \mathbf{x}) \} \\
 & \quad - \sum_{\rho=1}^{r-1} \dot{h}_\rho(\tilde{\mathbf{x}}) (\varphi_\rho(\tilde{\mathbf{x}}) - \varphi_r(\tilde{\mathbf{x}})) \\
 & \quad - \bar{\Gamma}(\mathbf{h}, \mathbf{x}) + \Psi(\mathbf{h}, \mathbf{x}) \leq 0.
 \end{aligned} \tag{5.43}$$

Applying (5.23) with the property(5.24), inequality (5.43) is equivalent to

$$\begin{aligned} & \frac{1}{r-1} \sum_{\rho=1}^{r-1} \sum_{m=1}^2 \omega_{\rho m}(\mathbf{x}) \left\{ \sum_{j=1}^r h_j(\tilde{\mathbf{x}}) \{ \bar{\Theta}^T(\mathbf{h}, \mathbf{x}) \mathbf{A}_j^{*T}(\mathbf{x}) \right. \\ & \quad + \mathbf{A}_j^*(\mathbf{x}) \bar{\Theta}(\mathbf{h}, \mathbf{x}) \} - (r-1) \mu_{\rho m}(\mathbf{x}) (\varphi_{\rho}(\tilde{\mathbf{x}}) - \varphi_r(\tilde{\mathbf{x}})) \\ & \quad \left. - \bar{\Gamma}(\mathbf{h}, \mathbf{x}) + \Psi(\mathbf{h}, \mathbf{x}) \right\} \leq 0. \end{aligned} \quad (5.44)$$

Letting $\mathbf{M}_j(\mathbf{x}) = \mathbf{F}_j(\mathbf{x}) \bar{\mathbf{X}}(\mathbf{h}, \tilde{\mathbf{x}})$, inequality (5.44) holds if the following conditions hold for all $\mathbf{v}_1 \in \mathbb{R}^{2n+m}$:

$$\begin{aligned} & \sum_{i=1}^r \sum_{j=1}^r h_i(\mathbf{z}) h_j(\mathbf{z}) \mathbf{v}_1^T (-\bar{\mathbf{H}}_{ij\rho m}(\mathbf{x})) \mathbf{v}_1 \geq 0 \\ & \quad \rho = 1, \dots, r-1, \quad m = 1, 2. \end{aligned} \quad (5.45)$$

Also applies the technique of co-positive relaxation, the condition (5.26) guarantees (5.45) being satisfied that means $\dot{V}(\mathbf{x}) < 0$ for $D_{op} - \{0\}$. \square

Remark 4. If $\mathbf{X}_i(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}}) \forall i$, Theorem 4 will become Theorem 3. The more relaxed results are always obtained by Theorem 4 when compared to Theorem 3, though. There is the limit that only the case no relation between membership functions and the inputs (i.e. $h_i(\mathbf{z}) = h_i(\tilde{\mathbf{x}}) \forall i$) can apply Theorem 4, while no constrain is included in Theorem 3. Beyond the situation that the system's inputs is not related to membership functions, the membership function's time derivatives can be represented as

$$\begin{aligned} \dot{h}_i(\mathbf{z}) &= \frac{\partial h_i(\mathbf{z})}{\partial \mathbf{x}} \dot{\mathbf{x}} \\ &= \frac{\partial h_i(\mathbf{z})}{\partial \mathbf{x}} \sum_{i=1}^r h_i(\mathbf{z}) \{ \mathbf{A}_i(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}) + \mathbf{B}_i(\mathbf{x}) \mathbf{u} \} \end{aligned}$$

which shows that the equation above is held with the control input \mathbf{u} , which is necessary. However, the fact that if the stabilization progress, which includes membership function's time derivative decomposition (5.23) (i.e. $\dot{h}_i(\mathbf{z})$), the control input \mathbf{u} cannot be obtained. That is why the control input \mathbf{u} is not able to appear in the equation above from Theorem 4.

Remark 5. Consider an open-loop descriptor systems presented in [4]

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (5.46)$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{A}, \mathbf{E} \in \mathbb{R}^{n \times n}$. From theorem 2.2 of [4], (5.46) is regular, impulse-free, and asymptotically stable if and only if there exist a matrix \mathbf{V} such that

$$\begin{cases} \mathbf{V}^T \mathbf{A} + \mathbf{A}^T \mathbf{V} < -\mathbf{W} \\ \mathbf{E}^T \mathbf{V} = \mathbf{V}^T \mathbf{E} \geq \mathbf{0} \end{cases} \quad (5.47)$$

for any positive definite matrix \mathbf{W} . Conditions (5.47) are locally hold by,

- (3.5), (3.6), and (3.7) in Theorem 1 in Chapter 3.
- (4.5) and (4.6) in Theorem 2 in Chapter 4.
- (5.4) and (5.5) in Theorem 3 in Chapter 5.
- (5.25) and (5.26) in Theorem 3 in Chapter 5.

Therefore, if the proposed Theorems can find feasible solutions of designing examples in Chapter 3, 4, and 5, the systems in designing examples are "impulse-free".

Remark 6. As Remark 2 mentioned, the descriptor form increases the size of the matrices. Thus, the computational time increases when the structure of the system goes complicated. The proposed Theorem 2 has the lowest computational time, which is almost the same as the state-space polynomial model of [37] since the results of stabilization (4.5), (4.6), and (4.7) are convex, and it has the fewest slack variables from Theorem 2 - 4. Theorem 3 and 4's computational time are decided by the parameter of co-positive s from (5.5) and (5.26). The bigger s is, the longer it takes. Theorem 1 also contains low computational time in SOS decomposition, though. Its nonconvex stabilization results require path-following algorithm, which does SOS decomposition many times make a long computational time. The conclusion is, all the proposed Theorems have a big computational time to obtain the solution except Theorem 2. However, the design examples show that the proposed Theorems can achieve the real time control.

5.3 Design Examples

In this section, to show the relaxation and effectiveness, six examples are given. In the first example, the result gives that the proposed Theorem 3 obtains the best result. The example uses the existing corollaries including the LMI-based descriptor form approach with quadratic Lyapunov function and SOS-based non-descriptor form approach. The second and third examples utilize the case for no relation between input vectors and membership functions. The second example is a polynomial FMB example with the comparison of Theorem 3, Theorem 4, and SOS-based non-descriptor form approach. The results show that Theorem 4 has the best stability. The third example presents a LMI case (i.e. only constant system matrices are considered in polynomial fuzzy models). Previous corollaries such as LMI-based descriptor form approach with quadratic Lyapunov function and SOS-based non-descriptor form approach, Theorem 3, and Theorem 4 are considered for the comparison. Consequently, Theorem 4, which applies the novel fuzzy Lyapunov function (5.30) can obtain the best result. The last three examples show the comparison between Theorem 1-4 and previous studies, including two polynomial examples and one constant example.

Example 1:

Consider the polynomial fuzzy model (2.9) with parameters $r = 3$, $\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{x} = [x_1 \ x_2]^T$, and the constant system matrices [71]:

$$\begin{aligned} \mathbf{A}_1 &= \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix} \\ \mathbf{A}_3 &= \begin{bmatrix} -a & -4.33 \\ 0 & 0.05 \end{bmatrix} \\ \mathbf{B}_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \quad \mathbf{B}_3 = \begin{bmatrix} -b + 6 \\ -1 \end{bmatrix}. \end{aligned}$$

To compare the stability, a is set as $a = 2$ and operation domain is set as $x_1 \in [-1 \ 1]$ and $x_2 \in [-1 \ 1]$ Table 5.1 shows the stabilization results, which is represented in the maximum values of b that each corollary can find, including [23], [37] and the proposed Theorem 3. As **Remark 1** mentioned, the quantity of the maximum feasible value of “ b ” itself has no meaning for the considering system. It is just a methodology for presenting relaxation.

Table 5.1: Comparison of the Results

Studies	Max feasible value of b
Theorem 2 of [37]	None
Corollary 2 of [23]	6.34
The proposed Theorem 3	6.38

Since $\hat{\boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{x}$, the Lyapunov candidates for corollary 2 of [37] and Theorem 3 are the same. As mentioned in **Remark 3**, the fuzzy slack matrices are introduced into Theorem 3's stabilization analysis under the redundancy of descriptor representation. The results reveal that Theorem 3 has the best stability. Moreover, $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ is a constant matrix since there is no zero row in \boldsymbol{B}_3 . Hence, the Lyapunov candidates for the proposed Theorem 3 and [23] are also the same. Nevertheless, in this example, Theorem 3 has fuzzy slack matrices and local feedback gains, which are polynomials, while [23] only contains constant matrices. Moreover, co-positive technique is applied by the proposed Theorem 3 applies with SOS-based to deal with the double fuzzy summation problems from the stabilization analysis. Thus, Theorem 3 can obtain more relaxed results than [23] in this example. Fig. 5.1 shows the simulation result presented in phase plot. The membership functions are chosen as:

$$h_1(\boldsymbol{z}) = \frac{1 + \sin(x_2)}{3}, \quad h_2(\boldsymbol{z}) = h_3(\boldsymbol{z}) = \frac{2 - \sin(x_2)}{6}.$$

and phase plot are consider the parameters $a = 2$ and $b = 6.38$ for Theorem 3 and shows that the system is asymptotically stable. The solution of decision variables are shown as follows:

$$\boldsymbol{X}_{211}(\boldsymbol{x}) = \begin{bmatrix} \begin{pmatrix} 6.767 \times 10^{-10}x_1 \\ -3.706 \times 10^{-9}x_2 - 4.684 \\ -9.521 \times 10^{-12}x_1 \\ -8.705 \times 10^{-11}x_2 + 1.577 \end{pmatrix} & \begin{pmatrix} 7.92 \times 10^{-11}x_1 \\ +1.449 \times 10^{-11}x_2 - 0.549 \\ 1.132 \times 10^{-12}x_1 \\ -1.498 \times 10^{-12}x_2 + 0.95 \end{pmatrix} \end{bmatrix}$$

$$\mathbf{X}_{212}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} 1.283 \times 10^{-8}x_1 \\ -7.687 \times 10^{-9}x_2 + 15.218 \end{array} \right) \\ \left(\begin{array}{c} -3.238 \times 10^{-9}x_1 \\ +6.51 \times 10^{-10}x_2 + 2.264 \end{array} \right) \end{array} \right] \left[\begin{array}{c} \left(\begin{array}{c} -2.135 \times 10^{-9}x_1 \\ +8.329 \times 10^{-13}x_2 - 11.377 \end{array} \right) \\ \left(\begin{array}{c} -3.926 \times 10^{-11}x_1 \\ -9.209 \times 10^{-11}x_2 + 1.908 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{213}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} -1.083 \times 10^{-8}x_1 \\ -6.925 \times 10^{-9}x_2 + 13.972 \end{array} \right) \\ \left(\begin{array}{c} 1.426 \times 10^{-9}x_1 \\ +2.949 \times 10^{-10}x_2 + 1.718 \end{array} \right) \end{array} \right] \left[\begin{array}{c} \left(\begin{array}{c} -7.279 \times 10^{-11}x_1 \\ +1.362 \times 10^{-9}x_2 - 5.768 \end{array} \right) \\ \left(\begin{array}{c} -9.013 \times 10^{-11}x_1 \\ -4.013 \times 10^{-11}x_2 + 0.889 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{221}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} 0.045x_1^2 - 3.92 \times 10^{-11}x_1x_2 \\ +0.045x_2^2 + 2.321 \times 10^{-9}x_1 \\ -4.202 \times 10^{-10}x_2 + 4.425 \end{array} \right) \\ \left(\begin{array}{c} 0.006x_1^2 - 5.525 \times 10^{-11}x_1x_2 \\ +0.006x_2^2 - 3.298 \times 10^{-11}x_1 \\ +3.026 \times 10^{-10}x_2 - 0.409 \end{array} \right) \end{array} \right] \left[\begin{array}{c} \left(\begin{array}{c} -0.006x_1^2 + 8.644 \times 10^{-11}x_1x_2 \\ -0.006x_2^2 - 6.182 \times 10^{-10}x_1 \\ -5.612 \times 10^{-10}x_2 - 0.663 \end{array} \right) \\ \left(\begin{array}{c} 0.002x_1^2 + 1.879 \times 10^{-12}x_1x_2 \\ +0.002x_2^2 - 5.088 \times 10^{-12}x_1 \\ -1.322 \times 10^{-11}x_2 + 0.432 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{222}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} 0.103x_1^2 - 3.209 \times 10^{-10}x_1x_2 \\ +0.103x_2^2 + 7.251 \times 10^{-9}x_1 \\ +6.711 \times 10^{-10}x_2 + 43.373 \end{array} \right) \\ \left(\begin{array}{c} 0.025x_1^2 + 8.852 \times 10^{-11}x_1x_2 \\ +0.025x_2^2 + 8.119 \times 10^{-10}x_1 \\ +1.434 \times 10^{-10}x_2 - 4.533 \end{array} \right) \end{array} \right] \left[\begin{array}{c} \left(\begin{array}{c} -0.045x_1^2 + 1.471 \times 10^{-10}x_1x_2 \\ -0.045x_2^2 - 5.367 \times 10^{-10}x_1 \\ -7.572 \times 10^{-10}x_2 - 4.441 \end{array} \right) \\ \left(\begin{array}{c} 0.015x_1^2 + 1.798 \times 10^{-11}x_1x_2 \\ +0.015x_2^2 - 9.032 \times 10^{-10}x_1 \\ +7.904 \times 10^{-11}x_2 + 1.47 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{223}(\mathbf{x}) = \left[\begin{array}{l} \left(\begin{array}{l} 0.385x_1^2 + 2.515 \times 10^{-9}x_1x_2 \\ +0.385x_2^2 - 4.525 \times 10^{-9}x_1 \\ -2.012 \times 10^{-9}x_2 + 23.438 \end{array} \right) \\ \left(\begin{array}{l} -0.06x_1^2 - 3.041 \times 10^{-10}x_1x_2 \\ -0.06x_2^2 - 2.028 \times 10^{-10}x_1 \\ +8.995 \times 10^{-11}x_2 + 0.175 \end{array} \right) \end{array} \right] \left[\begin{array}{l} \left(\begin{array}{l} 0.015x_1^2 - 8.749 \times 10^{-11}x_1x_2 \\ +0.015x_2^2 - 9.279 \times 10^{-10}x_1 \\ -1.033 \times 10^{-9}x_2 - 2.417 \end{array} \right) \\ \left(\begin{array}{l} 0.003x_1^2 + 1.174 \times 10^{-11}x_1x_2 \\ +0.003x_2^2 + 2.109 \times 10^{-10}x_1 \\ +5.424 \times 10^{-11}x_2 + 0.451 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{231}(\mathbf{x}) = \left[\begin{array}{l} \left(\begin{array}{l} -0.142x_1^2 + 2.65 \times 10^{-8}x_1x_2 - 0.142x_2^2 \\ +4.396 \times 10^{-8}x_1 + 2.383 \times 10^{-8}x_2 + 29.183 \end{array} \right) \\ \left(\begin{array}{l} -0.046x_1^2 - 1.566 \times 10^{-9}x_1x_2 - 0.046x_2^2 + 4.774 \times 10^{-10}x_1 + 1.171 \times 10^{-9}x_2 + 0.432 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{232}(\mathbf{x}) = \left[\begin{array}{l} \left(\begin{array}{l} 1.128x_1^2 - 6.952 \times 10^{-10}x_1x_2 + 1.128x_2^2 \\ +1.143 \times 10^{-9}x_1 + 7.085 \times 10^{-9}x_2 + 46.309 \end{array} \right) \\ \left(\begin{array}{l} -0.178x_1^2 + 5.992 \times 10^{-9}x_1x_2 - 0.178x_2^2 - 6.052 \times 10^{-10}x_1 - 1.044 \times 10^{-9}x_2 - 1.731 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{233}(\mathbf{x}) = \left[\begin{array}{l} \left(\begin{array}{l} 0.87x_1^2 + 4.674 \times 10^{-8}x_1x_2 + 0.87x_2^2 \\ +8.177 \times 10^{-9}x_1 - 3.884 \times 10^{-9}x_2 + 36.735 \end{array} \right) \\ \left(\begin{array}{l} 0.026x_1^2 - 6.593 \times 10^{-9}x_1x_2 + 0.026x_2^2 - 1.088 \times 10^{-9}x_1 - 6.346 \times 10^{-10}x_2 - 3.186 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{311}(\mathbf{x}) = \left[\begin{array}{l} \left(\begin{array}{l} 1.137 \times 10^{-8}x_1 \\ +5.593 \times 10^{-9}x_2 - 11.084 \end{array} \right) \\ \left(\begin{array}{l} 2.443 \times 10^{-11}x_1 \\ +5.315 \times 10^{-11}x_2 + 7.413 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{312}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} -9.556 \times 10^{-9}x_1 \\ -6.468 \times 10^{-9}x_2 - 2.704 \end{array} \right) \\ \left(\begin{array}{c} -1.854 \times 10^{-9}x_1 \\ -3.093 \times 10^{-10}x_2 - 1.192 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{313}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} 8.586 \times 10^{-9}x_1 \\ +7.531 \times 10^{-9}x_2 - 10.162 \end{array} \right) \\ \left(\begin{array}{c} 1.297 \times 10^{-9}x_1 \\ -4.035 \times 10^{-10}x_2 + 3.414 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{321}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} 0.026x_1^2 - 1.365 \times 10^{-10}x_1x_2 \\ +0.026x_2^2 + 1.929 \times 10^{-9}x_1 \\ -8.651 \times 10^{-10}x_2 - 26.339 \end{array} \right) \\ \left(\begin{array}{c} 0.037x_1^2 - 1.195 \times 10^{-10}x_1x_2 \\ +0.037x_2^2 + 1.633 \times 10^{-9}x_1 \\ +1.538 \times 10^{-10}x_2 + 1.834 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{322}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} -0.06x_1^2 + 3.742 \times 10^{-10}x_1x_2 \\ -0.06x_2^2 + 6.532 \times 10^{-9}x_1 \\ +1.201 \times 10^{-9}x_2 + 3.894 \end{array} \right) \\ \left(\begin{array}{c} 0.046x_1^2 - 3.319 \times 10^{-10}x_1x_2 \\ +0.046x_2^2 - 7.555 \times 10^{-10}x_1 \\ +7.834 \times 10^{-11}x_2 + 0.779 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{323}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} -0.06x_1^2 - 1.494 \times 10^{-9}x_1x_2 \\ -0.06x_2^2 - 1.003 \times 10^{-9}x_1 \\ -1.107 \times 10^{-9}x_2 - 11.882 \end{array} \right) \\ \left(\begin{array}{c} -0.052x_1^2 + 1.295 \times 10^{-10}x_1x_2 \\ -0.052x_2^2 + 2.034 \times 10^{-9}x_1 \\ +2.318 \times 10^{-9}x_2 + 1.402 \end{array} \right) \end{array} \right]$$

$$\begin{aligned} \mathbf{X}_{331}(\mathbf{x}) &= 8.594x_1^2 + 8.927 \times 10^{-8}x_1x_2 + 8.594x_2^2 \\ &\quad + 5.225 \times 10^{-8}x_1 - 3.337 \times 10^{-8}x_2 + 28.576 \end{aligned}$$

$$\begin{aligned} \mathbf{X}_{332}(\mathbf{x}) &= 8.42x_1^2 + 1.089 \times 10^{-7}x_1x_2 + 8.42x_2^2 \\ &\quad + 8.877 \times 10^{-8}x_1 - 2.491 \times 10^{-10}x_2 + 55.065 \end{aligned}$$

$$\begin{aligned} \mathbf{X}_{333}(\mathbf{x}) &= 8.932x_1^2 + 6.438 \times 10^{-8}x_1x_2 + 8.932x_2^2 \\ &\quad + 4.657 \times 10^{-8}x_1 - 3.457 \times 10^{-8}x_2 + 39.605 \end{aligned}$$

$$\mathbf{M}_1(\mathbf{x}) =$$

$$\left[\begin{array}{l} \left(\begin{array}{l} 0.012x_1^4 - 5.064 \times 10^{-9}x_1^3x_2 + 0.023x_1^2x_2^2 \\ -3.145 \times 10^{-9}x_1x_2^3 + 0.012x_2^4 \\ -4.247 \times 10^{-9}x_1^3 \\ -5.561 \times 10^{-9}x_1^2x_2 + 1.237 \times 10^{-9}x_1x_2^2 \\ -4.25 \times 10^{-9}x_2^3 \\ -8.579x_1^2 - 7.937 \times 10^{-8}x_1x_2 - 8.579x_2^2 \\ -6.088 \times 10^{-8}x_1 \\ +5.076 \times 10^{-9}x_2 + 0.655 \end{array} \right) \\ \left(\begin{array}{l} -0.015x_1^4 - 1.425 \times 10^{-10}x_1^3x_2 + 0.0004x_1^2x_2^2 \\ -1.321 \times 10^{-10}x_1x_2^3 - 0.015x_2^4 \\ -8.436 \times 10^{-11}x_1^3 \\ +1.881 \times 10^{-11}x_1^2x_2 - 5.415 \times 10^{-11}x_1x_2^2 \\ +6.053 \times 10^{-11}x_2^3 \\ +0.015x_1^2 + 3.673 \times 10^{-10}x_1x_2 + 0.015x_2^2 \\ +8.558 \times 10^{-11}x_1 \\ -8.489 \times 10^{-11}x_2 + 6.859 \end{array} \right) \end{array} \right]$$

$$\mathbf{M}_2(\mathbf{x}) =$$

$$\left[\begin{array}{l} \left(\begin{array}{l} 0.339x_1^4 + 7.12 \times 10^{-9}x_1^3x_2 + 0.059x_1^2x_2^2 \\ +2.979 \times 10^{-9}x_1x_2^3 + 0.339x_2^4 \\ -7.495 \times 10^{-9}x_1^3 \\ +3.462 \times 10^{-9}x_1^2x_2 - 1.046 \times 10^{-10}x_1x_2^2 \\ +1.245 \times 10^{-9}x_2^3 \\ -71.616x_1^2 - 8.738 \times 10^{-7}x_1x_2 - 71.616x_2^2 \\ -5.464 \times 10^{-7}x_1 \\ +8.797 \times 10^{-8}x_2 - 378.656 \end{array} \right) \\ \left(\begin{array}{l} 0.034x_1^4 + 3.684 \times 10^{-10}x_1^3x_2 + 0.039x_1^2x_2^2 \\ +4.566 \times 10^{-10}x_1x_2^3 + 0.034x_2^4 \\ +2.192 \times 10^{-10}x_1^3 \\ +3.691 \times 10^{-11}x_1^2x_2 + 1.589 \times 10^{-10}x_1x_2^2 \\ +2.366 \times 10^{-11}x_2^3 \\ -0.015x_1^2 - 1.739 \times 10^{-9}x_1x_2 - 0.015x_2^2 \\ -4.48 \times 10^{-10}x_1 \\ +1.174 \times 10^{-10}x_2 - 2.517 \end{array} \right) \end{array} \right]$$

$$\mathbf{M}_3(\mathbf{x}) =$$

$$\left[\begin{array}{l} \left(\begin{array}{l} -0.631x_1^4 + 8.629 \times 10^{-9}x_1^3x_2 - 0.28x_1^2x_2^2 \\ +1.318 \times 10^{-8}x_1x_2^3 - 0.631x_2^4 \\ +9.107 \times 10^{-9}x_1^3 \\ -5.55 \times 10^{-9}x_1^2x_2 + 5.189 \times 10^{-9}x_1x_2^2 \\ -2.36 \times 10^{-9}x_2^3 \\ +3.301x_1^2 - 1.112 \times 10^{-8}x_1x_2 + 3.301x_2^2 \\ +8.481 \times 10^{-9}x_1 \\ +7.803 \times 10^{-10}x_2 + 52.225 \end{array} \right) \\ \left(\begin{array}{l} -0.16x_1^4 - 3.302 \times 10^{-9}x_1^3x_2 - 0.101x_1^2x_2^2 \\ -7.699 \times 10^{-9}x_1x_2^3 - 0.16x_2^4 \\ -3.045 \times 10^{-9}x_1^3 \\ -1.229 \times 10^{-9}x_1^2x_2 - 2.641 \times 10^{-9}x_1x_2^2 \\ +7.307 \times 10^{-10}x_2^3 \\ +9.447x_1^2 + 1.25 \times 10^{-7}x_1x_2 + 9.447x_2^2 \\ +7.527 \times 10^{-8}x_1 \\ -1.187 \times 10^{-8}x_2 + 49.727 \end{array} \right) \end{array} \right]$$

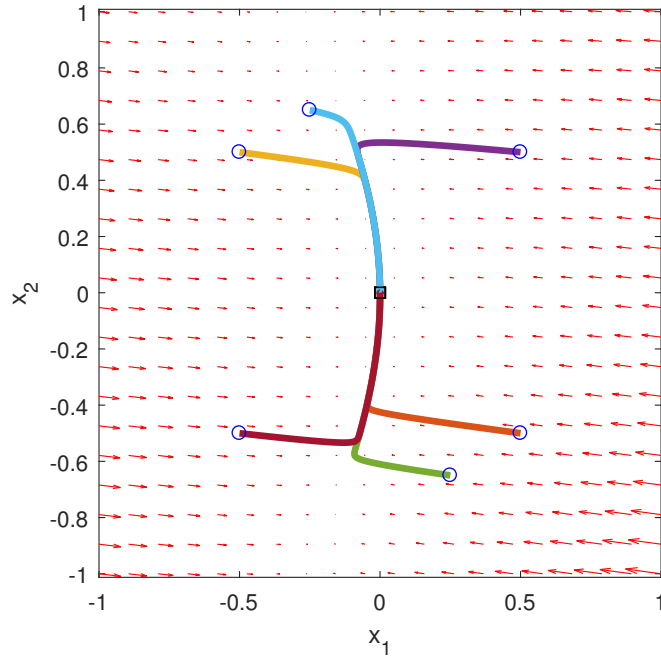


Figure 5.1: The phase plot of the simulation results

Example 2:

Consider the polynomial fuzzy model (2.9) with the parameters $r = 2$ and $\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{x} = [x_1 \ x_2]^T$. The example gives the following system matrices and membership functions:

$$\mathbf{A}_1(\mathbf{x}) = \begin{bmatrix} 1.59 - 1.66x_1x_2 & \begin{pmatrix} -8 + 0.22x_2 \\ -1.68x_2^2 - 1.45x_1^2 \end{pmatrix} \\ 0 & -0.36 \end{bmatrix}$$

$$\mathbf{A}_2(\mathbf{x}) = \begin{bmatrix} -a - x_1 + x_2 & -4 - 0.5x_1x_2^2 - 3.35x_1^2 \\ 0 & -0.04 \end{bmatrix}$$

$$\mathbf{B}_1(\mathbf{x}) = \begin{bmatrix} 1 + x_1^2 + x_2^2 \\ 0 \end{bmatrix}, \quad \mathbf{B}_2(\mathbf{x}) = \begin{bmatrix} -b + 6 + 9x_1^2 + 6x_2^2 \\ 0 \end{bmatrix}$$

$$h_1(x_2) = \frac{1 + \sin(x_2)}{2}, \quad h_2(x_2) = \frac{1 - \sin(x_2)}{2}.$$

To show the comparison, a is set as $a = 2$ and operation domain is set as $x_1 \in [-2 \ 2]$ and $x_2 \in [-2 \ 2]$.

By applying the technique described preliminaries which is about the decomposition (5.23)

Table 5.2: Comparison of the Results

Studies	Max feasible value of b
The proposed Theorem 1	7.02
Theorem 2 of [37]	7.06
The proposed Theorem 2	7.1

of $\dot{h}_1(\mathbf{x}_2)$, it can be obtained that

$$\mu_{11} = 0.1589, \mu_{12} = -0.1002.$$

In preliminaries, the property (5.42) points that the numbers of membership function's differentials are $r - 1$. In this case, the decomposition of $\dot{h}_2(\mathbf{x}_2)$ is not needed for Theorem 4. Table 5.2 shows the maximum value of b in which [37], Theorem 3, and Theorem 4 can find. As Remark 1 mentioned, the quantity of the maximum feasible value of "b" itself has no meaning for the considering system. It is just a methodology for presenting relaxation.

As Remark 3 says, Theorem 3 cannot always obtain better results than [37] since its Lyapunov function is not a special case. In this example, Table 5.2 reveals that better stability is seen in [37] when compared to Theorem 3. The table also shows that the stabilization analysis done by novel fuzzy Lyapunov function (5.30) and the decomposition (5.23) of time derivatives of membership function can obtain a more relaxed result than [37]. Moreover, it is no doubt that more relaxed results than Theorem 3 are obtained in Theorem 4, as mentioned in Remark 4. Fig. 5.2 shows the simulation results presented in phase plot of Theorem 4 with the parameters $a = 2$ and $b = 7.1$, and it can be seen that the system is asymptotically stable.

The solution of the decision variables are shown as follows:

$$\mathbf{X}_1(\tilde{\mathbf{x}}) = \begin{bmatrix} (6.19x_2^2 - 1.044x_2 + 21.677) & (-0.03x_2^2 + 0.006x_2 + 0.327) \\ (-0.03x_2^2 + 0.006x_2 + 0.327) & (0.002x_2^2 - 0.004x_2 + 0.046) \end{bmatrix}$$

$$\mathbf{X}_2(\tilde{\mathbf{x}}) = \begin{bmatrix} (15.825x_2^2 - 5.629x_2 + 14.396) & (-0.025x_2^2 - 0.008x_2 + 0.046) \\ (-0.025x_2^2 - 0.008x_2 + 0.046) & (0.004x_2^2 - 0.004x_2 + 0.039) \end{bmatrix}$$

$$\mathbf{X}_{211}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} 0.515x_1x_2 - 0.413x_2^2 \\ -0.121x_1 + 0.82x_2 + 9.178 \end{array} \right) \\ \left(\begin{array}{c} 0.063x_1x_2 + 0.01x_2^2 \\ -0.391x_1 - 0.108x_2 + 1.532 \end{array} \right) \end{array} \right] \left[\begin{array}{c} \left(\begin{array}{c} -0.018x_1x_2 - 0.001x_2^2 \\ +0.008x_1 + 0.003x_2 - 0.037 \end{array} \right) \\ \left(\begin{array}{c} -0.001x_1x_2 + 0.005x_2^2 \\ +0.001x_1 - 0.003x_2 + 0.112 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{212}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} 3.025x_1x_2 - 0.011x_2^2 \\ -1.792x_1 + 2.977x_2 + 14.313 \end{array} \right) \\ \left(\begin{array}{c} -0.026x_1x_2 - 0.025x_2^2 \\ -0.083x_1 - 0.968x_2 + 1.474 \end{array} \right) \end{array} \right] \left[\begin{array}{c} \left(\begin{array}{c} 0.003x_1x_2 - 0.001x_2^2 \\ -0.002x_1 - 0.025x_2 - 0.068 \end{array} \right) \\ \left(\begin{array}{c} 3.264 \times 10^{-5}x_1x_2 + 0.002x_2^2 \\ +0.001x_1 - 0.004x_2 + 0.054 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{221}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} 0.24x_1^2 + 0.099x_1x_2 + 0.2x_2^2 \\ -0.033x_1 + 0.172x_2 + 3.336 \end{array} \right) \\ \left(\begin{array}{c} -0.045x_1^2 - 0.079x_1x_2 - 0.005x_2^2 \\ -0.134x_1 - 0.044x_2 + 0.723 \end{array} \right) \end{array} \right] \left[\begin{array}{c} \left(\begin{array}{c} 0.013x_1^2 - 0.042x_1x_2 + 0.001x_2^2 \\ -0.011x_1 + 0.005x_2 - 0.761 \end{array} \right) \\ \left(\begin{array}{c} 0.019x_1^2 + 0.003x_1x_2 + 0.012x_2^2 \\ +0.003x_1 + 0.007x_2 + 0.186 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{222}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} 1.519x_1^2 + 0.054x_1x_2 + 0.875x_2^2 \\ -0.366x_1 + 0.195x_2 + 7.632 \end{array} \right) \\ \left(\begin{array}{c} -0.014x_1^2 - 0.004x_1x_2 + 0.009x_2^2 \\ -0.174x_1 + 0.112x_2 + 0.201 \end{array} \right) \end{array} \right] \left[\begin{array}{c} \left(\begin{array}{c} 0.005x_1^2 + 0.003x_1x_2 - 0.136x_2^2 \\ -0.034x_1 - 0.216x_2 - 0.872 \end{array} \right) \\ \left(\begin{array}{c} 0.01x_1^2 - 4.685 \times 10^{-5}x_1x_2 \\ +0.001x_2^2 + 0.004x_1 + 0.001x_2 + 0.268 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{231}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} -0.029x_1^2 - 0.139x_1x_2 + 0.036x_2^2 \\ +0.072x_1 - 0.37x_2 - 2.167 \end{array} \right) \\ \left(\begin{array}{c} 0.008x_1^2 - 0.003x_1x_2 + 0.0001x_2^2 \\ +0.263x_1 + 0.018x_2 - 0.836 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{232}(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} 9.147x_1^2 + 0.073x_1x_2 + 0.007x_2^2 \\ +0.201x_1 + 0.168x_2 - 4.557 \end{array} \right) \\ \left(\begin{array}{c} 0.0004x_1^2 + 1.606 \times 10^{-5}x_1x_2 \\ +0.0002x_2^2 + 0.083x_1 + 0.001x_2 - 0.369 \end{array} \right) \end{array} \right]$$

$$\mathbf{X}_{311}(\mathbf{x}) = \begin{bmatrix} \begin{pmatrix} -0.023x_1x_2 - 0.384x_2^2 \\ -0.054x_1 + 0.191x_2 - 25.383 \end{pmatrix} & \begin{pmatrix} -6.541x_1x_2 + 5.295 \times 10^{-5}x_2^2 \\ +0.01x_1 - 0.009x_2 + 0.197 \end{pmatrix} \\ \mathbf{X}_{312}(\mathbf{x}) = \begin{bmatrix} \begin{pmatrix} -0.021x_1x_2 - 0.314x_2^2 \\ -0.004x_1 - 0.074x_2 - 14.821 \end{pmatrix} & \begin{pmatrix} -4.131x_1x_2 + 0.0002x_2^2 \\ +0.0009x_1 - 0.001x_2 + 0.02 \end{pmatrix} \\ \mathbf{X}_{321}(\mathbf{x}) = \begin{bmatrix} \begin{pmatrix} 0.034x_1^2 + 0.209x_1x_2 - 0.015x_2^2 \\ -0.056x_1 - 0.084x_2 - 3.989 \end{pmatrix} & \begin{pmatrix} 0.003x_1^2 - 0.001x_1x_2 + 0.001x_2^2 \\ +0.037x_1 + 0.001x_2 + 0.641 \end{pmatrix} \\ \mathbf{X}_{322}(\mathbf{x}) = \begin{bmatrix} \begin{pmatrix} 0.011x_1^2 + 0.008x_1x_2 - 0.006x_2^2 \\ +0.123x_1 - 0.142x_2 - 2.025 \end{pmatrix} & \begin{pmatrix} 0.0002x_1^2 + 5.925 \times 10^{-6}x_1x_2 \\ +0.0002x_2^2 + 0.009x_1 + 0.001x_2 + 0.12 \end{pmatrix} \end{bmatrix}$$

$$\mathbf{X}_{331}(\mathbf{x}) = 0.009x_1^2 + 0.001x_1x_2 + 0.014x_2^2 + 0.115x_1 + 0.016x_2 + 9.564$$

$$\mathbf{X}_{332}(\mathbf{x}) = 0.002x_1^2 + 0.0004x_1x_2 + 0.003x_2^2 + 0.017x_1 + 0.004x_2 + 6.296$$

$$\mathbf{M}_1(\mathbf{x}) =$$

$$\begin{bmatrix} \begin{pmatrix} -13.896x_1^2 - 3.044x_1x_2 - 15.848x_2^2 \\ +2.078x_1 + 2.565x_2 - 36.743 \end{pmatrix} & \begin{pmatrix} -0.019x_1^2 - 0.023x_1x_2 \\ +0.011x_2^2 + 0.016x_1 - 0.006x_2 - 0.025 \end{pmatrix} \end{bmatrix}$$

$$\mathbf{M}_2(\mathbf{x}) =$$

$$\begin{bmatrix} \begin{pmatrix} -59.709x_1^2 - 1.136x_1x_2 - 41.135x_2^2 \\ -0.622x_1 + 1.277x_2 - 9.791 \end{pmatrix} & \begin{pmatrix} 0.001x_1^2 + 0.002x_1x_2 \\ +0.01x_2^2 + 0.0005x_1 - 0.002x_2 - 0.01 \end{pmatrix} \end{bmatrix}$$

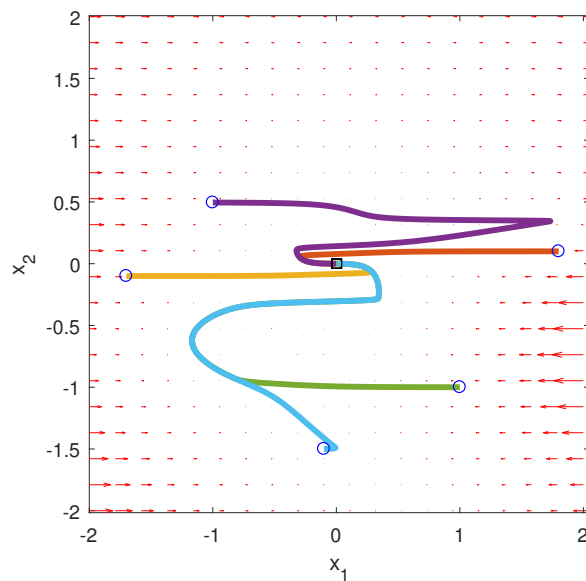


Figure 5.2: The phase plot of the simulation results

Table 5.3: Comparison of the Results

Studies	Max feasible value of b
Theorem 2 of [37]	52.13
The proposed Theorem 3	52.13
Corollary 3 of [23]	52.14
The proposed Theorem 4	52.16

Example 3:

Consider the polynomial fuzzy model (2.9) with the parameters $r = 2$ and $\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{x} = [x_1 \ x_2]^T$ and give the following constant system matrices and membership functions:

$$\mathbf{A}_1(\mathbf{x}) = \begin{bmatrix} 0.20 & -3.22 \\ 0.35 & 0.12 \end{bmatrix}, \quad \mathbf{A}_2(\mathbf{x}) = \begin{bmatrix} -a & -6.63 \\ 0.45 & 0.15 \end{bmatrix}$$

$$\mathbf{B}_1(\mathbf{x}) = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \quad \mathbf{B}_2(\mathbf{x}) = \begin{bmatrix} -b + 6 \\ 0 \end{bmatrix}$$

$$h_1(x_2) = \frac{1 + \sin(x_2)}{2}, \quad h_2(x_2) = \frac{1 - \sin(x_2)}{2}.$$

To show the comparison, a is set as $a = 2$ and operation domain is set as $x_1 \in [-1 \ 1]$ and $x_2 \in [-1 \ 1]$.

By applying the technique described preliminaries for the decomposition (5.23) of $\dot{h}_1(\mathbf{x}_2)$, it can be obtained that

$$\mu_{11} = 0.2041, \quad \mu_{12} = -0.2191.$$

Same as Example 2, the decomposition of $\dot{h}_2(\mathbf{x}_2)$ is not needed. Table 5.3 shows the maximum values of b in which [37], [23], Theorem 3 and Theorem 4 can find. As **Remark 1** mentioned, the quantity of the maximum feasible value of “b” itself has no meaning for the considering system. It is just a methodology for presenting relaxation.

From table 5.3, it gives the information that LMI-based descriptor form design approach stabilized by quadratic Lyapunov function as the approach presented in [23] sometimes acquires better stability than polynomial method as Theorem 3 and [37]. Here, Theorem 4 chooses $\tilde{\mathbf{x}} = x_2$ and apply to the novel fuzzy Lyapunov function (5.30)’s stabilization because that input matrices \mathbf{B}_i ’ second rows are all zeros. This Lyapunov function is more flexible than [23] which applies quadratic fuzzy Lyapunov function. Besides, citeTanaka2007

just considers the lower bound of membership function time derivative, while the proposed Theorem 4 considers both the upper and lower bounds. Hence, Theorem 4 can have more relaxed stability than [23] in this example. Fig. 5.3 shows the simulation result represented in phase plot of Theorem 4 with the parameters $a = 2$ and $b = 52.16$, and it can be seen that the system is asymptotically stable. The solution of the decision values are shown as follows:

$$\begin{aligned}
\mathbf{X}_1(\tilde{\mathbf{x}}) &= \begin{bmatrix} (0.003x_2^2 - 1.002 \times 10^{-12}x_2 + 87.025) & (-0.0002x_2^2 - 6.902 \times 10^{-14}x_2 - 2.924) \\ (-0.0002x_2^2 - 6.902 \times 10^{-14}x_2 - 2.924) & (0.001x_2^2 - 1.274 \times 10^{-13}x_2 + 8.524) \end{bmatrix} \\
\mathbf{X}_2(\tilde{\mathbf{x}}) &= \begin{bmatrix} (0.018x_2^2 - 1.575 \times 10^{-12}x_2 + 86.997) & (0.001x_2^2 + 9.637 \times 10^{-13}x_2 - 2.885) \\ (0.001x_2^2 + 9.637 \times 10^{-13}x_2 - 2.885) & (0.001x_2^2 - 7.181 \times 10^{-14}x_2 + 8.524) \end{bmatrix} \\
\mathbf{X}_{211}(\mathbf{x}) &= \begin{bmatrix} \begin{pmatrix} 2.66 \times 10^{-6}x_1x_2 \\ +0.001x_2^2 + 15.722 \\ -3.724 \times 10^{-9}x_1 + 2.893 \times 10^{-8}x_2 \end{pmatrix} & \begin{pmatrix} 0.0003x_1x_2 - 0.01x_2^2 - 3.959 \\ -7.291 \times 10^{-10}x_1 \\ -5.191 \times 10^{-10}x_2 \end{pmatrix} \\ \begin{pmatrix} 7.285 \times 10^{-5}x_1x_2 \\ +0.005x_2^2 + 11.338 \\ +2.089 \times 10^{-9}x_1 - 6.633 \times 10^{-9}x_2 \end{pmatrix} & \begin{pmatrix} -1.247 \times 10^{-5}x_1x_2 \\ +0.0002x_2^2 + 1.996 \\ -6.52 \times 10^{-10}x_1 + 1.127 \times 10^{-9}x_2 \end{pmatrix} \end{bmatrix} \\
\mathbf{X}_{212}(\mathbf{x}) &= \begin{bmatrix} \begin{pmatrix} 0.0006x_1x_2 - 0.013x_2^2 - 0.979 \\ +1.226 \times 10^{-8}x_1 + 3.699 \times 10^{-8}x_2 \end{pmatrix} & \begin{pmatrix} 0.002x_1x_2 - 0.022x_2^2 - 4.619 \\ -2.581 \times 10^{-9}x_1 - 4.422 \times 10^{-9}x_2 \end{pmatrix} \\ \begin{pmatrix} -0.001x_1x_2 + 0.025x_2^2 + 1.974 \\ -6.256 \times 10^{-9}x_1 + 1.52 \times 10^{-8}x_2 \end{pmatrix} & \begin{pmatrix} -9.985 \times 10^{-5}x_1x_2 + 0.0002x_2^2 + 0.229 \\ -2.811 \times 10^{-9}x_1 + 1.706 \times 10^{-9}x_2 \end{pmatrix} \end{bmatrix} \\
\mathbf{X}_{221}(\mathbf{x}) &= \begin{bmatrix} \begin{pmatrix} 0.118x_1^2 + 0.043x_1x_2 + 0.134x_2^2 + 9.903 \\ +2.299 \times 10^{-9}x_1 - 2.139 \times 10^{-9}x_2 \end{pmatrix} & \begin{pmatrix} -0.0001x_1^2 - 0.001x_1x_2 - 0.004x_2^2 - 20.535 \\ -4.558 \times 10^{-10}x_1 + 1.076 \times 10^{-8}x_2 \end{pmatrix} \\ \begin{pmatrix} -0.014x_1^2 - 0.004x_1x_2 - 0.011x_2^2 + 17.737 \\ +2.04 \times 10^{-9}x_1 - 2.636 \times 10^{-9}x_2 \end{pmatrix} & \begin{pmatrix} 0.016x_1^2 + 0.005x_1x_2 + 0.017x_2^2 + 2.683 \\ -4.92 \times 10^{-10}x_1 - 1.692 \times 10^{-9}x_2 \end{pmatrix} \end{bmatrix} \\
\mathbf{X}_{222}(\mathbf{x}) &= \begin{bmatrix} \begin{pmatrix} 0.84x_1^2 + 0.014x_1x_2 + 0.817x_2^2 + 9.035 \\ +6.186 \times 10^{-9}x_1 + 1.309 \times 10^{-8}x_2 \end{pmatrix} & \begin{pmatrix} -0.109x_1^2 - 0.02x_1x_2 - 0.101x_2^2 - 17.791 \\ -4.461 \times 10^{-10}x_1 - 7.298 \times 10^{-9}x_2 \end{pmatrix} \\ \begin{pmatrix} 0.001x_1^2 - 0.002x_1x_2 - 0.013x_2^2 + 17.146 \\ +5.606 \times 10^{-9}x_1 - 5.265 \times 10^{-9}x_2 \end{pmatrix} & \begin{pmatrix} 0.127x_1^2 + 0.036x_1x_2 + 0.129x_2^2 + 3.704 \\ +3.799 \times 10^{-9}x_1 + 4.24 \times 10^{-9}x_2 \end{pmatrix} \end{bmatrix}
\end{aligned}$$

$$\mathbf{X}_{231}(\mathbf{x}) = \begin{bmatrix} \begin{pmatrix} -0.002x_1^2 - 0.001x_1x_2 + 0.001x_2^2 \\ +8.42 \times 10^{-9}x_1 + 1.133 \times 10^{-8}x_2 + 0.431 \end{pmatrix} \\ \begin{pmatrix} -4.902x_1^2 + 0.001x_1x_2 - 0.008x_2^2 \\ -1.036 \times 10^{-8}x_1 - 4.237 \times 10^{-9}x_2 - 1.106 \end{pmatrix} \end{bmatrix}$$

$$\mathbf{X}_{232}(\mathbf{x}) = \begin{bmatrix} \begin{pmatrix} -0.021x_1^2 + 0.005x_1x_2 - 0.025x_2^2 \\ -3.878 \times 10^{-9}x_1 + 2.133 \times 10^{-8}x_2 + 0.302 \end{pmatrix} \\ \begin{pmatrix} -0.005x_1^2 - 0.008x_1x_2 + 0.033x_2^2 \\ -4.844 \times 10^{-9}x_1 + 1.54 \times 10^{-9}x_2 - 1.114 \end{pmatrix} \end{bmatrix}$$

$$\mathbf{X}_{311}(\mathbf{x}) = \begin{bmatrix} \begin{pmatrix} 0.002x_1x_2 - 0.015x_2^2 - 1.393 \\ +8.652 \times 10^{-9}x_1 + 1.119 \times 10^{-8}x_2 \end{pmatrix} \\ \begin{pmatrix} 6.304x_1x_2 - 0.0003x_2^2 + 0.056 \\ +1.968 \times 10^{-10}x_1 + 3.133 \times 10^{-10}x_2 \end{pmatrix} \end{bmatrix}$$

$$\mathbf{X}_{312}(\mathbf{x}) = \begin{bmatrix} \begin{pmatrix} -0.004x_1x_2 + 0.164x_2^2 + 0.722 \\ +7.391 \times 10^{-9}x_1 + 2.437 \times 10^{-9}x_2 \end{pmatrix} \\ \begin{pmatrix} -0.0004x_1x_2 + 0.002x_2^2 + 0.162 \\ +9.235 \times 10^{-10}x_1 + 8.437 \times 10^{-10}x_2 \end{pmatrix} \end{bmatrix}$$

$$\mathbf{X}_{321}(\mathbf{x}) = \begin{bmatrix} \begin{pmatrix} -0.006x_1^2 - 0.002x_1x_2 - 0.007x_2^2 - 1.146 \\ -1.539 \times 10^{-9}x_1 - 1.892 \times 10^{-9}x_2 \end{pmatrix} \\ \begin{pmatrix} 0.001x_1^2 - 5.211 \times 10^{-5}x_1x_2 + 0.005x_2^2 \\ +2.82 \times 10^{-9}x_1 + 2.886 \times 10^{-9}x_2 + 0.728 \end{pmatrix} \end{bmatrix}$$

$$\mathbf{X}_{322}(\mathbf{x}) = \begin{bmatrix} \begin{pmatrix} -0.033x_1^2 - 0.0002x_1x_2 - 0.031x_2^2 - 0.968 \\ -2.014 \times 10^{-9}x_1 - 2.562 \times 10^{-9}x_2 \end{pmatrix} \\ \begin{pmatrix} -0.025x_1^2 - 0.003x_1x_2 - 0.025x_2^2 + 0.205 \\ -4.784 \times 10^{-10}x_1 + 1.846 \times 10^{-9}x_2 \end{pmatrix} \end{bmatrix}$$

$$\mathbf{X}_{331}(\mathbf{x}) = 0.599x_1^2 - 0.0001x_1x_2 + 0.63x_2^2 - 6.345 \times 10^{-10}x_1 + 8.16 \times 10^{-10}x_2 + 0.689$$

$$\mathbf{X}_{332}(\mathbf{x}) = 0.599x_1^2 - 0.001x_1x_2 + 0.63x_2^2 + 2.201 \times 10^{-9}x_1 - 1.64 \times 10^{-9}x_2 + 0.709$$

$$\mathbf{M}_1(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} -4.777x_1^2 + 0.015x_1x_2 - 5.03x_2^2 \\ -1.142 \times 10^{-9}x_1 \\ +6.902 \times 10^{-10}x_2 - 8.327 \end{array} \right) \\ \left(\begin{array}{c} -0.0002x_1^2 + 7.374 \times 10^{-5}x_1x_2 - 0.0003x_2^2 \\ -4.023 \times 10^{-12}x_1 \\ +6.902 \times 10^{-10}x_2 - 0.259 \end{array} \right) \end{array} \right]$$

$$\mathbf{M}_2(\mathbf{x}) = \left[\begin{array}{c} \left(\begin{array}{c} 27.568x_1^2 - 0.083x_1x_2 + 29.028x_2^2 \\ +6.28 \times 10^{-9}x_1 \\ -4.154 \times 10^{-9}x_2 - 25.334 \end{array} \right) \\ \left(\begin{array}{c} -0.001x_1^2 - 0.0004x_1x_2 - 0.001x_2^2 \\ +3.405 \times 10^{-11}x_1 \\ -8.366 \times 10^{-11}x_2 - 0.26 \end{array} \right) \end{array} \right]$$

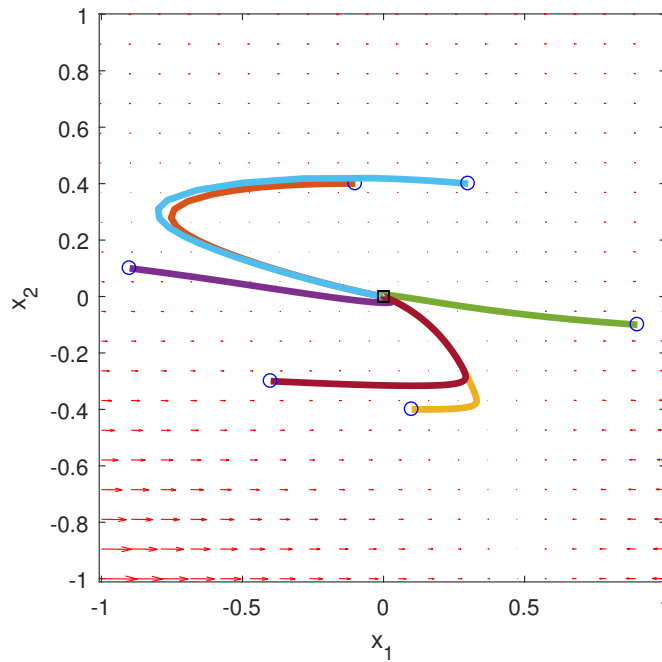


Figure 5.3: The phase plot of the simulation results

Example 4:

Consider the polynomial fuzzy model (2.9) with parameters $r = 2$, $\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{x} = [x_1 \ x_2]^T$, and the polynomial system matrices:

$$\begin{aligned}
\mathbf{A}_1(\mathbf{x}) &= \begin{bmatrix} 1.59 - 1.66x_1x_2 & \begin{pmatrix} -8 + 0.22x_2 \\ -1.68x_2^2 - 1.45x_1^2 \end{pmatrix} \\ 0 & -0.36 \end{bmatrix} \\
\mathbf{A}_2(\mathbf{x}) &= \begin{bmatrix} -a - x_1 + x_2 & -6 - 0.5x_1x_2^2 - 3.35x_1^2 \\ 0 & -0.01 \end{bmatrix} \\
\mathbf{B}_1(\mathbf{x}) &= \begin{bmatrix} 1 + x_1^2 + x_2^2 \\ 0 \end{bmatrix}, \quad \mathbf{B}_2(\mathbf{x}) = \begin{bmatrix} -b + 6 + 9x_1^2 + 6x_2^2 \\ 0 \end{bmatrix} \\
h_1 &= \frac{1 + \sin(x_2)}{2}, \quad h_2 = 1 - h_1.
\end{aligned}$$

By applying the technique described preliminaries which is about the decomposition (5.23) of $\dot{h}_1(\mathbf{x}_2)$, it can be obtained that

$$\mu_{11} = 0.0031, \quad \mu_{12} = -0.0401.$$

In preliminaries, the property (5.42) points that the numbers of membership function's differentials are $r - 1$. In this case, the decomposition of $\dot{h}_2(\mathbf{x}_2)$ is not needed for Theorem 4. Table 5.4 shows the maximum value of b in which [37] and all Theorems in the thesis can find. As **Remark 1** mentioned, the quantity of the maximum feasible value of “b” itself has no meaning for the considering system. It is just a methodology for presenting relaxation.

The results reveal that the proposed Theorem 1 and 2 doesn't obtain any feasible solution while [37] does. As mentioned in **Remark 3**, Theorem 2 cannot always obtain better result than [37] since its Lyapunov function is not a special case, neither does Theorem 3. Furthermore, Theorem 1 is almost an independent system from Theorem 2 - 4 and [37]. Thus, it is not guaranteed that Theorem 1 - 3 can always obtain more relaxed results than [37]. The only thing that can be guaranteed is that Theorem 3, which brings polynomial fuzzy slack matrices into stabilization, is more general and relaxed than Theorem 2 since it has more slack variables. The other thing is that this example proves that novel fuzzy Lyapunov function is better than quadratic Lyapunov function as **Remark 4** mentioned. Fig. 5.4 shows the simulation result of Theorem 4 presented in phase plot, and it brings out that the system is asymptotically stable.

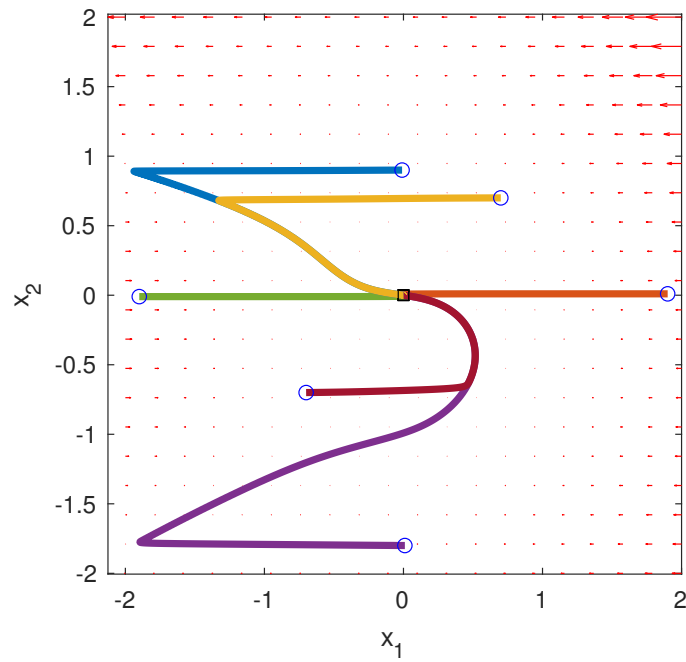


Figure 5.4: The phase plot of the simulation results

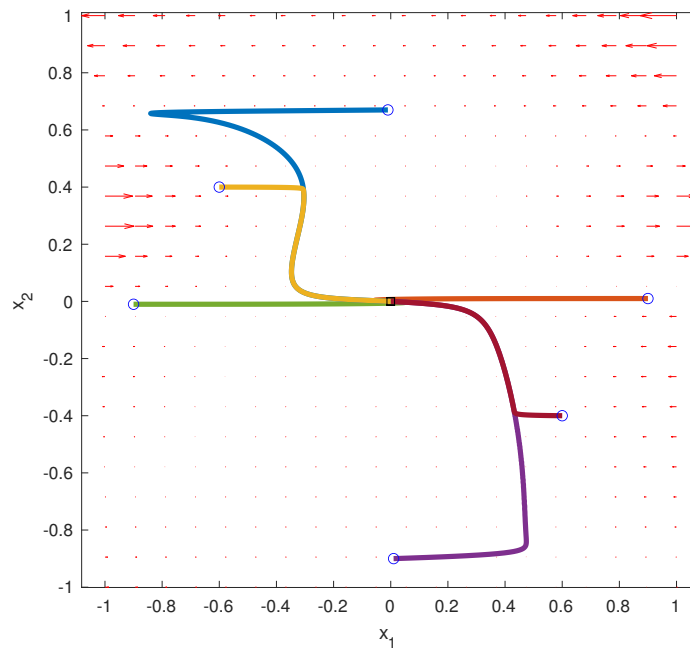


Figure 5.5: The phase plot of the simulation results

Example 5:

Consider the polynomial fuzzy model (2.9) with the parameters $r = 2$ and $\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{x} =$

Table 5.4: Comparison of the Results

Studies	Max feasible value of b
The proposed Theorem 1	None
The proposed Theorem 2	None
Corollary 2 of [37]	5.06
The proposed Theorem 3	5.7
The proposed Theorem 4	6.15

$[x_1 \ x_2]^T$. The example gives the following system matrices and membership functions:

$$\begin{aligned}
\mathbf{A}_1(\mathbf{x}) &= \begin{bmatrix} 1.59 - 1.66x_1x_2 & \begin{pmatrix} -8 + 0.22x_2 \\ -1.68x_2^2 - 1.45x_1^2 \end{pmatrix} \\ 0 & -0.36 \end{bmatrix} \\
\mathbf{A}_2(\mathbf{x}) &= \begin{bmatrix} -a - x_1 + x_2 & -6 - 0.5x_2^2 - 3.35x_1^2 \\ 0 & -0.01 \end{bmatrix} \\
\mathbf{B}_1(\mathbf{x}) &= \begin{bmatrix} 1 + x_1^2 + x_2^2 \\ 0 \end{bmatrix}, \quad \mathbf{B}_2(\mathbf{x}) = \begin{bmatrix} -b + 6 + 9x_1^2 + 6x_2^2 \\ 0 \end{bmatrix} \\
h_1 &= 1 - \frac{1}{1 + 0.5e^{-\frac{x_2}{2}}}, \quad h_2 = 1 - h_1.
\end{aligned}$$

To show the comparison, a is set as $a = 2$ and operation domain is set as $x_1 \in [-1 \ 1]$ and $x_2 \in [-1 \ 1]$.

By applying the technique described preliminaries which is about the decomposition (5.23) of $\dot{h}_1(\mathbf{x}_2)$, it can be obtained that

$$\mu_{11} = 0.003, \quad \mu_{12} = -0.0119.$$

In preliminaries, the property (5.42) points that the numbers of membership function's differentials are $r - 1$. In this case, the decomposition of $\dot{h}_2(\mathbf{x}_2)$ is not needed for Theorem 4. Table 5.6 shows the maximum value of b in which [37] and all Theorems in the thesis can find. As **Remark 1** mentioned, the quantity of the maximum feasible value of “b” itself has no meaning for the considering system. It is just a methodology for presenting relaxation.

Theorem 1 is almost an independent system from Theorem 2 - 4 and [37]. Thus, it is not guaranteed that Theorem 1 can always obtain more relaxed results than [37]. Theorem 3,

Table 5.5: Comparison of the Results

Studies	Max feasible value of b
The proposed Theorem 1	41.25
Corollary 2 of [37]	41.49
The proposed Theorem 2	41.51
The proposed Theorem 3	41.51
Corollary 3 of [23]	45.45
The proposed Theorem 4	45.84

which brings polynomial fuzzy slack matrices into stabilization, is more general and relaxed than Theorem 2 since it has more slack variables. Theorem 4 obtains the best result, proving that the novel fuzzy Lyapunov function is better than the quadratic Lyapunov function as **Remark 4** mentioned. Fig. 5.5 shows the simulation results presented in phase plot of Theorem 4. It can be seen that the system is asymptotically stable.

Example 6:

Consider the polynomial fuzzy model (2.9) with the parameters $r = 2$ and $\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{x} = [x_1 \ x_2]^T$ and give the following constant system matrices and membership functions:

$$\mathbf{A}_1(\mathbf{x}) = \begin{bmatrix} 0.25 & -3.12 \\ 0.35 & 0.1 \end{bmatrix}, \quad \mathbf{A}_2(\mathbf{x}) = \begin{bmatrix} -a & -5.63 \\ 0.45 & 0.2 \end{bmatrix}$$

$$\mathbf{B}_1(\mathbf{x}) = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \quad \mathbf{B}_2(\mathbf{x}) = \begin{bmatrix} -b + 6 \\ 0 \end{bmatrix}$$

$$h_1(x_2) = \frac{1 + \sin(x_2)}{2}, \quad h_2(x_2) = \frac{1 - \sin(x_2)}{2}.$$

To show the comparison, a is set as $a = 2$ and operation domain is set as $x_1 \in [-1 \ 1]$ and $x_2 \in [-1 \ 1]$.

By applying the technique described preliminaries for the decomposition (5.23) of $\dot{h}_1(\mathbf{x}_2)$, it can be obtained that

$$\mu_{11} = 0.2049, \quad \mu_{12} = -0.2247.$$

Same as Example 2, the decomposition of $\dot{h}_2(\mathbf{x}_2)$ is not needed. Table 5.5 shows the maximum values of b in which [37], [23], and all Theorems in the thesis can find. As **Remark 1** mentioned, the quantity of the maximum feasible value of “ b ” itself has no meaning for the considering system. It is just a methodology for presenting relaxation.

Table 5.6: Comparison of the Results

Studies	Max feasible value of b
The proposed Theorem 1	None
Corollary 2 of [37]	7.12
The proposed Theorem 2	7.13
The proposed Theorem 3	7.14
The proposed Theorem 4	7.25

Here, Theorem 4 chooses $\tilde{x} = x_2$ and apply to the novel fuzzy Lyapunov function (5.30)'s stabilization because that input matrices B_i 's second rows are all zeros. The first thing is that sometimes polynomial fuzzy slack matrices don't greatly affect the stabilization results beyond the same Lyapunov function. This is shown by Theorem 2 and 3, which have almost the same results. The next thing is that novel fuzzy Lyapunov function in Theorem 4 and fuzzy Lyapunov function in [23] perform better than quadratic Lyapunov function. Especially novel fuzzy Lyapunov function in Theorem 4 is more flexible than [23]. Besides, [23] just considers the lower bound of membership function time derivative, while the proposed Theorem 4 considers both the upper and lower bounds. Hence, Theorem 4 can have the most relaxed stability than [23] in this example. Fig. 5.6 shows the simulation result represented in phase plot of Theorem 4, and it can be seen that the system is asymptotically stable.

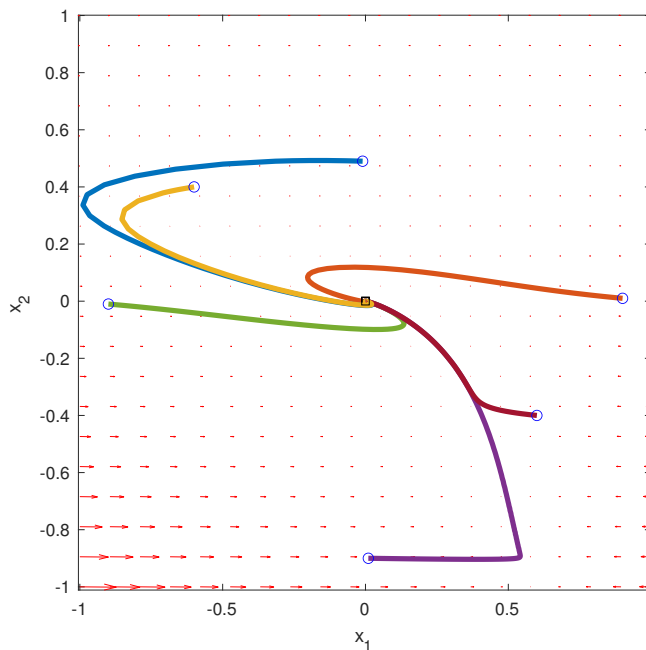


Figure 5.6: The phase plot of the simulation results

CONCLUSIONS AND FUTURE WORK

6.1 Conclusion

A descriptor form design methodology for polynomial FMB control systems has been presented in this thesis. The polynomial fuzzy model has been represented in descriptor form. Four design methodologies have been proposed, and all of them are constructed in the operation domain. Thus, their stabilization analysis is local stabilization, and the stabilization conditions are presented in SOS terms.

At first, in Chapter 3, a rational control design has been proposed. The thesis proposed a polynomial fuzzy model with the controller, which considers the rational functions. The closed-loop system containing the model and the controller is represented in the descriptor form. To stabilize the system, a homogeneous functions' method has been presented in this thesis. The Lyapunov candidate has been chosen as the homogeneous Lyapunov function in which the matrix of decision variables is a homogeneous matrix. By considering the properties of Euler's homogeneity relation, the differential of the Lyapunov function has been able to be extracted. The rest part of the stabilization has been analyzed by considering the stabilization method of the descriptor design methodology for T-S fuzzy model. However, the bilinear term appears in the stabilization conditions and makes it impossible to solve directly by SOSTOOL. Therefore, the path-following approach has been applied to solve the conditions. An example has been presented to show the comparison between the proposed method and the polynomial fuzzy model without descriptor form. The result has been proven that the proposed method obtains more relaxed results when in the same operation domain.

Because the path-following method may not find the optimal result for solving the stabilization constraints, the thesis has tried other descriptor form design methodology for polynomial FMB control design. The controller for polynomial fuzzy model has been changed to the PDC-based polynomial controller, which shares the membership function with the

polynomial fuzzy model.

In Chapter 4, a polynomial fuzzy model with PDC-based polynomial controller has been presented and has been represented in the descriptor form. The Lyapunov candidate for stabilization has been chosen as a Lyapunov function which is commonly used in the research for polynomial fuzzy model. The stabilization has been extracted by considering the definition of the vector \tilde{x} and the vector of the membership function. The properties of congruence transformation have also been considered for stabilization analysis. Compared with the polynomial fuzzy model without descriptor form (previous study), in the same operation domain, the matrices' dimension in the proposed method is higher than the previous study, though. The number of the SOS constraints from the proposed method is smaller than the previous study. The contrast shows that the proposed method is more suitable with the cases that contain more rules, but the state vector's dimension should be small. Moreover, compared with the method in Chapter 3, the proposed method has not contained the bilinear or non-convex term. The proposed method does not need to use the path-following algorithm to solve the conditions. Two examples have been provided, including a numerical example and an application example. The numerical shows that the feasibility (relaxation) of the proposed approach is similar to the existing polynomial FMB control design approach, though. The smaller number of constraints means that the proposed method still held the advantage when compared with the existing polynomial FMB control design approach. The application example gives a bicycle dynamic system. The proposed method has been successfully made the system stable by setting the operation domain of the bicycle's angle and angle speed.

Chapter 5 has considered the same model, controller, and Lyapunov function as Chapter 4. The improvement is that the matrices which contain fuzzy slack variables have been brought into the stabilization. The fuzzy slack matrices have made the Lyapunov candidate be rewritten into a new form and produce a new stabilization analysis. The SOS conditions born from the stabilization analysis contain double fuzzy summation, which can be taken as co-positivity problem. Thus, the copositive relaxation has been applied and made the double fuzzy summation disappear from the conditions. A numerical example has been presented to compare the existing polynomial FMB control design approach, the descriptor design methodology for T-S fuzzy system, and the proposed approach. The summary has been shown that the proposed method can obtain the more relaxed result than the rest two approaches in some cases.

Furthermore, for the special cases that all membership functions are functions of the states being not related to the inputs, Chapter 5 has proposed the second stabilization design approach which applies the novel fuzzy Lyapunov function. This kind of Lyapunov function contains an inverse of fuzzy summation matrix. Since the differential of the Lyapunov function contains membership function's differential terms in stabilization, the thesis has proposed a method to extract the time derivative of membership function. After extracting the differential of membership function, the sector nonlinearity method has been applied to deal with the rest part of the membership function, which considers its maximum and minimum values in the operation domain. Because the first method of Chapter 5 is seen as a special case of the proposed method, the thesis has proven that the proposed method is always more relaxed than the first method in Chapter 5. Two numerical examples have been presented to show the comparison with the proposed method, the previous method, and previous studies. The first one is a polynomial example, and the comparison is to compare with the proposed method, existing polynomial FMB control design approach, and the first method of Chapter 5. The results have been shown that the proposed method performs best from them. The second one is an LMI example to compare with the proposed method, the existing polynomial FMB control design approach, the first method of Chapter 5, and the descriptor design methodology for T-S fuzzy system. The proposed method has also obtained the best result. Finally, there have been three common examples that compare the four proposed Theorems in the thesis with the previous studies. The results have shown that Theorem 4 has the best performance than other proposed Theorems and previous studies beyond the special case.

6.2 Future Planes

Because the novel fuzzy Lyapunov function can have the best performance than the proposed three Theorems, we will try the modification on this part. As the stabilization analysis of the novel fuzzy Lyapunov function brings the issue of membership functions' time derivatives, the research will add this term to the current PDC-based control design. In the common cases, this term cannot be implemented to the controller since it requires the future information of the input signals and may need to consider the observer-based controller and feedback. Therefore, bringing the time derivatives of membership functions to the controller makes it

necessary to construct the model under the special case that

$$h_i(\mathbf{z}) = h_i(\tilde{\mathbf{x}}) \quad \forall i$$

where $\tilde{\mathbf{x}}$ is the vector that is the vector that is not related to the input vector, which means that the membership function is the function that has no relationship to the input vector. Beyond the special case, no future information is needed to estimate the derivative of the states in the membership function. The future information can be predicted by the current state. It can be expected that this method can also obtain better results than the current research Theorem 4. The next step may be adding the constrain to membership function, since the research's design methodologies are all constructed locally. In this thesis, only the differential of membership functions is added to the upper and lower bound. This concept will be added into the membership function "without" differentiation. Last but not least, the region-of-attraction (ROA)'s analysis is considered since design methodologies are all locally constructed. Finding the ROA makes graphing the phase plot more easily. Not only tries to find the ROA, but the research also searches for the method to extend the ROA that the system can calculate.

The four design methodologies are proposed for the polynomial FMB control systems, which are in "type-1". As potential improvements, the polynomial FMB control systems which are in "type-2" like [67] presented can also apply to the design methodologies as the extension. Moreover, even the so-called "type-3" polynomial FMB control systems can be applied under the issue of how to apply the interval type-3 membership function [68] into the proposed descriptor form design methodologies is solved. Besides, since the technique of polynomial fuzzy models can be applied to neural network, [69], [70] shows the possibilities for the proposed descriptor form design methodologies' application.

The main purpose of the thesis is to design the stabilization control for polynomial fuzzy model by transforming the closed-loop system into descriptor form. The stabilization results can be extended to achieve other specifications such as H_∞ , H_2 , guarantee cost, et. al. in the future. In addition, different types of controllers can be applied to the model. Currently, only rational and PDC-based controllers are considered in the thesis. For instance, the controller with observer-based or decimal controllers, which are more product class, may be a choice.

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LIST OF PUBLICATIONS

Journal paper:

1. **FAN-NONG Yu**, Ying-Jen Chen, Kazuo Tanaka, Motoyasu Tanaka, and Shun-Hung Tsai, "Descriptor Form Design Methodology for Polynomial Fuzzy-Model-Based Control Systems," *International Journal of Fuzzy Systems*, July, 2021.
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International conference papers:

1. **FAN-NONG Yu**, Ying-Jen Chen, Kazuo Tanaka, Motoyasu Tanaka, "A Polynomial Fuzzy Descriptor System Approach for Rational Fuzzy Control Design," in *2019 International Conference on Fuzzy Theory and Its Applications (iFUZZY)*, Tamsui, Taiwan, Nov, 2019.
2. **FAN-NONG Yu**, Ying-Jen Chen, Kazuo Tanaka, Motoyasu Tanaka, and Shun-Hung Tsai, "A Descriptor System approach for Polynomial Fuzzy-Model-Based Control Design," in *International Conference on System Science and Engineering (ICSSE)*, Taipei, Taiwan, pp. 7-12, Sep. 2020.