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An Approximate PML Applied to Cylindrical and Spherical Coordinate Sectors

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Abstract—This paper proposes an approximate perfectly matched layer (PML) that is applicable to cylindrical and spherical coordinate sectors. The proposed PML is based on complex coordinate stretching, which enables the truncation of FDTD grids, not only at \( \rho \)- and \( r \)-coordinates, but also at \( \phi \)- and \( \theta \)-coordinates. The absorption performance of the PML is demonstrated through numerical simulations.

Index Terms—Perfectly matched layer, complex coordinate stretching.

I. INTRODUCTION

PERFECTLY matched layer (PML) [1] has been widely used as a standard absorbing boundary condition to truncate a computational domain for solving open-region problems by using the finite-difference time-domain (FDTD) method. The PML based on complex coordinate stretching facilitates PML formulation in the framework of Maxwell’s equations [2]. Although the FDTD method has been applied to various coordinate systems, studies on the development of PML for such systems are rather limited. Among the seminal works [2], the time domain equations to be solved are represented as

\[ j \omega \varepsilon E_z = \frac{1}{\rho s_\rho} \frac{\partial}{\partial \rho} (\rho H_\phi) - \frac{1}{\rho s_\phi} \frac{\partial}{\partial \phi} E_\rho, \]

\[ j \omega \mu H_\rho = \frac{1}{s_\rho} \frac{\partial}{\partial \rho} E_\rho, \]

\[ j \omega \mu H_\phi = \frac{1}{s_\phi} \frac{\partial}{\partial \phi} E_\phi, \]

where \( \omega \) is the angular frequency, \( E_z \) and \( H_\rho \) are the tangential electric and magnetic fields, respectively, \( s_\rho \) and \( s_\phi \) are the complex stretching factors, and \( \varepsilon \) and \( \mu \) are the permittivity and permeability, respectively.

II. FORMULATION

A. PML in cylindrical coordinates

In this section we describe the PML applied to computational domain having the form of a cylindrical sector for the two-dimensional TM case. To accomplish this, in addition to the coordinate stretching in the \( \rho \)-direction [3], we introduce stretching in the \( \phi \)-direction normal to the boundary of the sector. The three-dimensional formulation is rather straightforward because the \( z \)-axis is the same as that in the Cartesian system.

The complex stretched variable transformations with respect to \( \alpha \) can be represented as follows.

\[ \alpha \rightarrow \tilde{\alpha} = \int_{\alpha_0}^{\alpha} s_\alpha(\alpha')d\alpha' + \alpha_0 = \alpha - j \frac{\tilde{\alpha}}{\omega}, \]

where \( \alpha = \rho \) and \( \phi \); \( \alpha_0 \) is a constant; \( s_\alpha = 1 - j s_\alpha(\alpha)/\omega \); and \( \tilde{\alpha} = \int_{\alpha_0}^{\alpha} s_\alpha(\alpha')d\alpha' \). It is known that the coordinate \( \phi \) is not a measure of length; therefore, Eq. (1) with \( \alpha = \phi \) can be considered as the transformation of the variable \( \rho \phi \) to \( \rho \tilde{\phi} \).

Maxwell’s equations can then be rewritten as follows:

\[ j \omega \varepsilon_{\tilde{E}_z} = \frac{1}{\rho s_\rho} \frac{\partial}{\partial \rho} (\rho H_\phi) - \frac{1}{\rho s_\phi} \frac{\partial}{\partial \phi} \tilde{E}_\rho, \]

\[ j \omega \mu_{\tilde{H}_\rho} = \frac{1}{s_\rho} \frac{\partial}{\partial \rho} \tilde{E}_\rho, \]

\[ j \omega \mu_{\tilde{H}_\phi} = \frac{1}{s_\phi} \frac{\partial}{\partial \phi} \tilde{E}_\phi, \]

where we adopt the time factor \( e^{j\omega t} \). In a manner similar to [2], the time domain equations to be solved are represented as

\[ \frac{\partial \tilde{E}_\rho}{\partial t} + \sigma_\rho \tilde{E}_\rho = \frac{1}{\varepsilon} \frac{\partial \tilde{H}_\phi}{\partial \rho}, \]

\[ \frac{\partial \tilde{E}_\phi}{\partial t} + \sigma_\phi \tilde{E}_\phi = \frac{1}{\varepsilon} \frac{\partial \tilde{H}_\rho}{\partial \phi}, \]

\[ \tilde{E}_z = \tilde{E}_\rho + \tilde{E}_\phi, \]

\[ \frac{\partial \tilde{E}_\rho}{\partial t} + \sigma_\rho \tilde{E}_\rho = \frac{1}{\mu} \frac{\partial \tilde{H}_\phi}{\partial \rho}, \]

\[ \frac{\partial \tilde{E}_\phi}{\partial t} + \sigma_\phi \tilde{E}_\phi = \frac{1}{\mu} \frac{\partial \tilde{H}_\rho}{\partial \phi}, \]

\[ \frac{\partial \tilde{H}_\rho}{\partial t} + \sigma_\rho \tilde{H}_\rho = \frac{1}{\mu} \frac{\partial \tilde{E}_\phi}{\partial \rho}, \]

\[ \frac{\partial \tilde{H}_\phi}{\partial t} + \sigma_\phi \tilde{H}_\phi = \frac{1}{\mu} \frac{\partial \tilde{E}_\rho}{\partial \phi}, \]
\[ \frac{\partial H_\rho}{\partial t} + \tilde{\sigma}_\rho H_\rho = \frac{\partial \tilde{H}_\rho}{\partial t}, \]  
\[ \frac{\partial H_\phi}{\partial t} = \rho \frac{\partial H_\phi}{\partial t} + \tilde{\sigma}_\rho H_\phi. \]  

Note that the field variables on the left-hand side of the above equations must be updated by finite differentiation. Therefore, for the time-domain solution, it is necessary to introduce the following auxiliary field variables: \( \tilde{E}_{z\rho}, \tilde{E}_{z\phi}, \tilde{E}_z, \tilde{H}_\rho, \) and \( \tilde{H}_\phi. \)

### B. Simplified PML in cylindrical coordinates

The second term on the right-hand side in Eq. (2) and Eq. (3) absorb the wave components propagating along the \( \phi \)-direction. These terms influence the PML normal to \( \phi \) where \( s_\rho \neq 1 \) and \( s_\phi = 1 \). Therefore, the operator \( \frac{1}{s_\rho s_\phi} \) can be approximated by \( \frac{1}{s_\rho} \). This approximation would affect only the four corners of FDTD grids where PMLs normal to the \( \rho \) and \( \phi \)-axes are overlapped. For example, Eq. (2) becomes

\[ j\omega \varepsilon E_z = \frac{1}{s_\rho} \frac{\partial (\rho H_\rho)}{\partial \rho} - \frac{1}{s_\phi} \frac{\partial H_\rho}{\partial \phi}. \]  

In the time domain, Eqs. (6), (7), and (9) are rewritten as

\[ \frac{\partial E_{z\phi}}{\partial t} + \sigma_\phi E_{z\phi} = -\frac{1}{\varepsilon} \frac{\partial H_\rho}{\partial \phi}, \]  
\[ E_z = E_{z\phi} + E_z, \]  
\[ \frac{\partial H_\rho}{\partial t} + \sigma_\rho H_\rho = -\frac{1}{\mu} \frac{\partial E_z}{\partial \phi}. \]

Moreover, \( E_{z\phi} \) must be updated before the operation of Eq. (15) by

\[ \rho \frac{\partial E_{z\phi}}{\partial t} + \tilde{\sigma}_\rho E_{z\phi} = \frac{\partial \tilde{E}_{z\phi}}{\partial t}, \]

and thus, Eq. (8) is rendered redundant. In this formulation, the number of update equations decreases by one whereas that of the auxiliary field variables reduces to the following four, \( \tilde{E}_{z\rho}, E_{z\phi}, E_{z\phi}, \) and \( H_\phi \).

### C. Simplified PML in spherical coordinates

In the spherical coordinates, a similar formulation is prescribed, as follows. The coordinate transformation is the same as Eq. (1). As stated earlier, the coordinate transformation can be considered as \( r \theta \rightarrow r \theta \) and \( r \sin \theta \phi \rightarrow r \sin \theta \phi \). As a result, the Maxwell-Ampere’s law can be rewritten as

\[ \frac{1}{r s_\theta} \frac{\partial H_\phi}{\partial \theta} + \frac{\cot \theta}{r} H_\phi - \frac{1}{s_\phi} \frac{\partial H_\theta}{\partial \phi} = j\omega \varepsilon E_r, \]  
\[ \frac{1}{r s_\theta} \frac{\partial H_r}{\partial \phi} + \frac{1}{s_\phi} \frac{\partial (\tilde{\rho} H_\theta)}{\partial \phi} = j\omega \varepsilon E_\theta, \]  
\[ \frac{1}{r s_\theta} \frac{\partial (\tilde{\rho} H_\phi)}{\partial \theta} - \frac{1}{s_\phi} \frac{\partial H_r}{\partial \phi} = j\omega \varepsilon E_\phi. \]

In addition, the transformation of Faraday’s law is similar according to the duality of Maxwell’s equations. In Eq. (18), to improve absorption performance, the equation mapped onto the stretched coordinate space is \( \frac{1}{r s_\theta} \frac{\partial H_\phi}{\partial \theta} + \frac{\cot \theta}{r} H_\phi \) as opposed to \( \frac{1}{r \sin \theta \sin \phi} \). It is noted that the \( r \)-coordinate is not transformed in terms of differentiation with respect to \( \theta \) and \( \phi \), as discussed in the previous section.

### III. Numerical results

#### A. Cylindrical PML

For evaluating the reflection of the developed PML, the waveforms are measured at an observation point. The computational domain is defined by \( \rho_1 \leq \rho \leq \rho_0 (= \rho_1 + R_p) \) and \( 0 \leq \phi \leq \phi_0 \), where \( \rho_1 = 21.2, R_p = 25.6, \) and \( \phi_0 = 5\pi/8 \). The domain is discretized by \( N_\rho \times N_\phi = 640 \times 1000 \) cells. The temporal step is chosen as \( \Delta t = 0.99/|c_0/\sqrt{\Delta \rho} + (\rho \Delta \phi)^{-1}| \), where \( c_0 = 5/\sqrt{\mu} \) is the speed of light in vacuum. The PML with \( L \)-sublayers lies inside the domain. Moreover, the fields are excited by a current source with current density \( J_z \) \([A/m^2]\) at \( (\rho_0, \phi_0) \), spread over an area corresponding to one cell \( \Delta S(= \Delta \rho \times \rho_0 \Delta \phi) \); here, \( \rho_0 = 27 \) and \( \phi_0 = 3\pi/16 \). The waveform of \( J_z \Delta S \), defined as \( f(t) \), is given by

\[ f(t) = -\frac{t - t_0}{\sigma} \exp \left( -\frac{(t - t_0)^2}{2\sigma^2} \right), \]

where \( \sigma = 12\Delta t \) and \( t_0 = 6\sigma \). The observation point is located at \( \rho_1 = 26, \phi_1 = \phi_0, \) and \( L = 16 \). The absorption parameters \( \sigma_\rho \) and \( \sigma_\phi \) are chosen as \( \sigma(\alpha) = -\alpha/(\Delta \alpha)^3(M + \)
1, \alpha = \ln R_0/(2L\Delta \alpha), \) where \( \alpha \) is the distance from the interface between PML and the vacuum, and \( \Delta \alpha \) is the cell size.

Figure 1 shows the numerical results of \( |E_z| \) obtained using the proposed PML. The left panel illustrates the difference between the results obtained using the simplified equations discussed in Sec. II-B (solid line) and the equations formulated in Sec. II-A (the broken line). The direct wave is observed at approximately 10 ns, and the reflected waves are observed at approximately 40 and 100 ns; the former is the wave reflected from the PML near \( \rho = \rho_1 \), while the latter is the wave reflected from the PML near \( \phi = 0 \). Both the results coincide with each other, indicating that the simplified formulation is effective. It should be noted that the use of simplified equations decreases the computational time required for updating \( H \) by a factor of 1.5.

The right panel illustrates the waveforms of \( |E_z| \) computed using the simplified formulas in Sec. II-B for the number of sublayers \( L = 8, 16, \) and 32. The location of the observation point is the same as in the previous case. The curve labeled as “Ref” is obtained by setting a large computational domain such that the reflected waves from the grid boundaries are not observed. We see that the reflected wave at \( t = 100 \) ns, which corresponds to the reflection from the PML near \( \phi = 0 \), is sufficiently mitigated for \( L = 16 \). In contrast, the reflection observed at \( t = 40 \) ns, corresponding to the reflection from the PML near \( \rho = \rho_1 \), is not mitigated effectively \((-47.7 \) dB) despite the increase in \( L \).

Figure 2 shows the reflection from the simplified PML for \( L = 8 \) and 16, as a function of the incident angle. Here, \( \rho_1 \) and \( \rho_0 \) denote the convex and concave PMLs, respectively, and \( \phi \) denotes the PML normal to \( \phi \)-direction. The Fourier transform of FDTD results are taken and then normalized by the amplitude of waves propagating with the same distance as the reflected waves, at the frequency corresponding to the sampling number \( N_s = 15 \), i.e., the wavelength equal to 15 cells. Although the reflection from the convex PML is relatively higher, we can conclude that the developed PML is effective.

B. Spherical PML

The computational domain for measuring spurious reflections from the proposed PML is defined as \( r_i \leq r \leq r_o, \theta_1 \leq \theta \leq \theta_2, \) and \( 0 \leq \phi \leq \phi_2 \), where \( r_i = 3.2, r_o = 7.4, \theta_1 = (\frac{1}{2} - \frac{9}{32})\pi, \theta_2 = (\frac{1}{2} + \frac{9}{32})\pi, \) and \( \phi_2 = 3\pi/4 \). This domain is discretized into \( N_r \times N_\theta \times N_\phi = 210 \times 225 \times 300 \) cells, i.e., \( \Delta \tau = (r_o - r_i)/N_r = 0.02 \), and \( \Delta \theta = \Delta \phi = \pi/400; \) the temporal step is \( \Delta t = 0.999/\sqrt{c_0^2 \sqrt{(\Delta \tau)^{-2} + (r_1 \Delta \theta)^{-2} + (r_1 \sin \theta_1 \Delta \phi)^{-2}}} \). A current with dipole moment \( J_0 \Delta V \) [A-m] = \( f(t) \) in Eq. (21) is located at \( (r_x, \theta_x, \phi_x) = (5, \pi/2, 3\pi/8) \) for excitation.

Figure 3 indicates the waveform of \( |E_0| \) observed at \( (r', \theta', \phi') = (4.5, \pi/2, 3\pi/8) \) with \( L = 4, 8, \) and 16 sublayers, and \( M = 2.5, 3.6, \) and 3.7, respectively. The results were computed using the simplified formulas presented in Sec. II-C. The direct wave from the source is observed at approximately 4 ns, and the reflected waves are observed at approximately 13, 20, 27, and 34 ns. Through numerical simulations, we can confirm that the wave at approximately 13 ns is reflected from the PML near \( r = r_i \), and similarly, the waves at approximately 20, 27, and 34 ns correspond to the PML near \( r = r_o, \theta = \theta_{1,2}, \) and \( \phi = \phi_{1,2} \), respectively. We can see that an increase in the sublayers mitigates the reflections down to the given reflection coefficient \( R_0 = 10^{-6} \).

In Fig. 4, the dependence of the reflection of the spherical PML on the incident angle is illustrated for \( L = 8 \) and 16. From the results, we see that the spherical PML developed in this work demonstrates excellent absorption performance.

IV. Conclusion

In this work, we proposed a novel PML formulation that can be applied to cylindrical and spherical sectors as FDTD grids. The numerical results confirm the proposed PML has adequate absorption performance. In addition, we also developed a simplified formulation of the PML, which was demonstrated to exhibit identical performance.

REFERENCES