A degree and forbidden subgraph conditon for a k-contractible edge

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Abstract

An edge in a k-conected graph is said to be k-contractible if the contraction of it results in a k-connected graph. We say that k-connected graph G satisfies " degree-sum conditon" if $\sum_{x \in V(W)} \deg_G(x) \geq 3k+2$ holds for any connected subgraph W of G with $\mid$ W $\mid = 3$. Let k be an integer such that $k \geq 5$. We prove that if a k-connected graph with no $K_1 + C_4$ satisfies degree-sum condition, then it has a k-contractible edge.

1 Introduction

In this paper, we deal with finite undirected graphs with neither self-loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices of G and the set of edges of G, respectibely. Let $V_k(G)$ denote the set of vertices of degree k of G. We denote the degree of $x \in V(G)$ by $d_G(x)$. Let $\Delta(G)$ and $\delta(G)$ denote the maximum degree of G and the minimum degree of G, respectively. For a vertex $x \in V(G)$, we denote by $N_G(x)$ the neighborhood of x in G. Moreover, for a subset $S \subset V(G)$, let $N_G(S) = \bigcup_{x \in S} N_G(x) - S$. For nonintersecting vertex sets $S, T \subseteq V(G)$, we denote the set of edges between S and T by $E_G(S,T)$. We write $E_G(x, S)$ for $E_G(\{x\}, S)$. When there is no ambiguity, we write $V_k, N(x), N(S)$ and $E(S,T)$ for $V_k(G), N_G(x), N_G(S)$ and $E_G(S,T)$, respectively. For a vertex set $S \subseteq V(G)$, we let $G[S]$ denote the subgraph induced by S in G. Let K_n, C_n and P_n denote the complete graph on n vertices, the cycle on n vertices and the path on n vertices, respectively. For a vertex set $S \subseteq V(G)$, we let $G - S$ denote the graph obtained from G by deleting the vertices in S together with the edges incident with them; thus $G - S = G[V(G) - S]$. Let G be a connected graph. A subset $S \subset V(G)$ is said to be a *cutset* of G if $G - S$ is not connected. A cutset S is said to be a k-cutset if $|S| = k$. For a noncomplete connected graph G, the order of a minimum cutset of G is said to be the connectivity of G and the connectivity of G is denoted by $\kappa(G)$. Let k be an integer such that $k \geq 2$ and let G be a graph with $\kappa(G) = k$ and $|V(G)| \geq k+2$. An edge e of G is said to be k-contractible if the contraction of the edge results in a k-connected graph. Note that, in the contraction, we replace each resulting pair of double edges by a simple edge. An edge which is not k-contractible is said to be k-noncontractible. Let $xy \in E(G)$ If $N_G(x) \cap N_G(y) \cap V_k(G) \neq \emptyset$, then xy is said to be trivially k-noncontractible. A k-connected graph is said to be *contraction-critically* if it has no k-contractible edge.

Every edge of a connected graph is 1-contractible. We observe that every 2-connected graph of order at least 4 has a 2-contractible edge. Tutte [8] showed that every 3-connected graph of order at least 5 has a 3-contractible edge. For $k \geq 4$, there are infinitely many contraction-critically k-connected graphs for each k. Hence, if $k \geq 4$, then we cannot expect the existence of a contractible edge in a k-connected graph with no condition. We let a k-sufficient condition stand for a conditon for a k-connected graph to have a k-contractible edge.

There are some k-sufficient conditions involving degree. Edawa [3] proved the following minimum degree k-sufficient condition

Theorem A. *Let $k \geq 2$ be an integer, and let G be a k-connected graph with $\delta(G) \geq \lfloor \frac{5}{4}k \rfloor$. Then G has a k-contractible edge, unless $k \in \{2,3\}$ and G is isomorphic to K_{k+1}.*

Krisell [5] extended Theorem A and proved the following degree sum k-sufficient condition.

Theorem B. *Let G be a k-connected graph for which $d_G(v) + d_G(w) \geq 2\lfloor \frac{5}{4}k \rfloor - 1$. for any pair v, w of G. Then G contains a k-contractible edge.*

Solving a conjecture in [5], Su and Yuan [6] proved the following degree-sum k-sufficient condition which is an extension of Theorem B.

Theorem C. *Let G be a k-connected graph with $k \geq 8$. If $d_G(v) + d_G((w) \geq 2\lfloor \frac{5}{4}k \rfloor - 1$ for any two adjacent vertices v, w, then G has k-contractible edge.*

There are also some k-sufficient conditions involving forbidden subgraphs. We can see that "triangle-free" is a k-sufficient condition; Thomassen [7] pointed out this condition.

Theorem D. *Every k-connected graph with no triangle has k-contractible edge.*

A k-connected graph with no triangle has many k-contractible edges, which indicates the possible existence of a weaker condition involving forbidden subgraphs which guarantees a k-connected graph to have a k-contractible edge. In this direction, Kawarabayashi [4] showed the following, where K_4^- stands for the graph obtained from K_4 by deleting one edge.

Theorem E. *For an odd integer $k \geq 3$, every k-connected graph with no K_4^- has a k-contractible edge.*

Since K_4^- contains a triangle, this is an extension of Theorem D when k is odd. We call the graph $K_1 + 2K_2$ a *bowtie*. Ando, Kaneko, Kawarabayashi and Yoshimoto [1] proved that every k-connected graph with no bowtie has a k-contractible edge, which is also an extension of Theorem D.

Theorem F. *If a k-connected graph contains no K_1+2K_2, then it has a k-contractible edge.*

Theorems D, E and F deal with forbidden subgraph k-sufficient conditions. On the other hand, Theorem A gives a minimum degree k-sufficient condition and Theorem C gives a degree-sum k-sufficient conditon. However, if we restrict ourselves to a class of graphs that satisfy some forbidden subgraph conditions, then we may relax the minimum degree bound in Theorem A and the degree-sum bound in Theorem C. The following forbidden subgraph condition relaxes the minimum degree bound (see Ando and Kawarabayashi [2]). Let K_5^- be the graph obtained from K_5 by removing one edge.

Theorem G. *Let k be an integer such that $k \geq 5$. Let G be a k-connected graph which contains neither K_5^- nor $5k_1 + P_3$. If $\delta(G) \geq k+1$, then G has a k-contractible edge.*

Note that if $k \geq 5$, then $\lfloor \frac{5}{4}k \rfloor \geq k+1$. There is a k-regular contraction-critically k-connected graph which contains neither K_5^- nor $5K_1 + P_3$. Hence we cannot replace $\delta(G) \geq k+1$ by $\delta(G) \geq k$ in Theorem G. In this sense, the minimum degree bound in Theorem G is sharp.

In the same direction, Yingqiu and Liang recently proved the following [9].

Theorem H. *For $k \geq 5$, let G be a k-connected graph which contains no K_1+C_4. If $d_G(v) + d_G(w) \geq 2k+2$ for any two adjacent vertices v, w, then G has a k-contractible edge.*

In this paper we prove an extension of Theorem H which involves 3-degree-sum condition. For a connected subgraph W of G, we set $d_G(W) = \sum_{x \in V(W)} d_G(x)$.

Theorem 1. *Let k be an integer such that $k \geq 5$, and G be a k-connected graph which contains no $K_1 + C_4$. If $d_G(W) \geq 3k+2$ hold for any connected subgraph W of G with $|W| = 3$, then G has a k-contractible edge.*

To conclude this section, we give two examples of contraction critically k-connected graphs. The first one shows that, for each $k \geq 5$, there is a contraction critically k-connected graph with no $K_1 + C_4$ and $d_G(W) \geq 3k + 1$ hold for any connected subgraph W of G with $|W| = 3$. The second shows that, for each even integer $k \geq 8$, there is a contraction critically k-connected graph such that it contains $K_1 + C_4$ and $d_G(W) \geq 3k+2$ hold for any connected subgraph W of G with $|W| = 3$. There example shows that neither the degree-sum condition nor the forbidden subgraph condition of Theorem 1 can be dropped.

Example 1.

Let G be the graph illustrated in Figure 1. Then we observe that each edge of G is trivially noncontractible and G contains no K_1+C_4. Hence G is a contraction-critically 5-connected graph which satisfies the forbidden subgraph condition of Theorem 1.

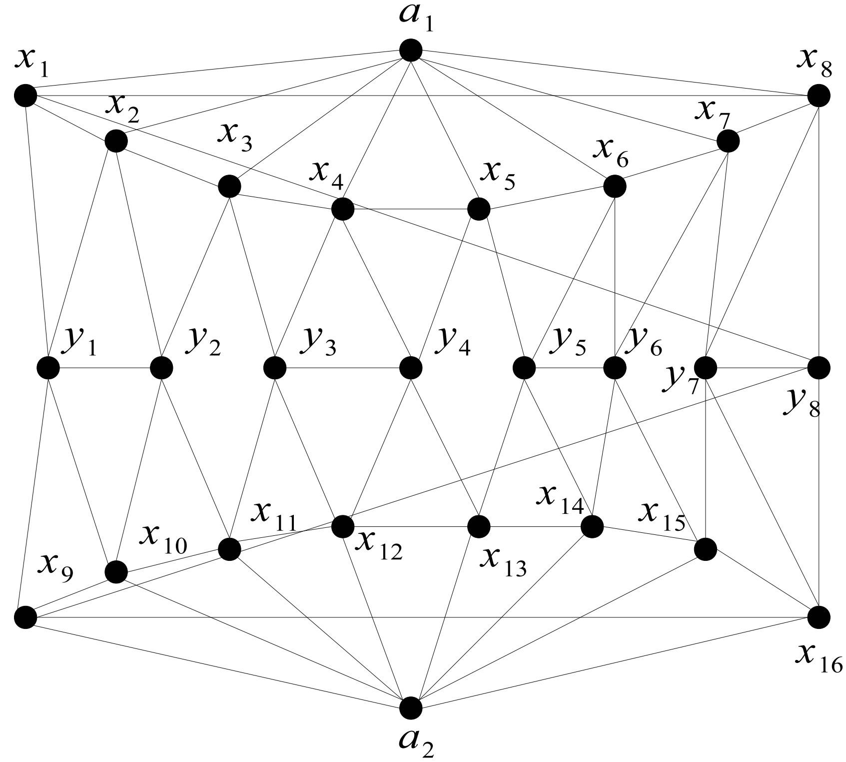

Figure 1: graph G

Example 2.

Let G be the graph illustrated in Figure 2. Let $S_1 = \{x_1, x_2, x_3, x_4\}$, $S_2 = \{a_1, a_2, a_3, a_4\}$ and $S_3 = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. Then we observe $S = S_1 \cup S_2$ is a 8-cutset of G. We also observe that each vertex of S_1 has degree 8 and end vertex of $V(G) - S_1$ has degree more than 10. Since each vertex of S_1 has degree 8, we see that each edge of $E_G(S_1, S_3) \cup E(G[S_3])$ is trivially noncontractible. Since $E(G) = E_G(S_1, S_3) \cup E(G[S_3]) \cup E(G[S])$ and S is an 8-cutset of G, we see that G has no contractible edge. Let W be a connected subgraph of G such that $|W| = 3$. Since W is connected, we observe that $|V(W) \cap S_1| \leq 2$ and $|V(W) \cap (S_2 \cup S_3)| \geq 1$, which implies $d_G(W) \geq 3 \times 8 + 2 = 24$. Hence G is a contraction-critically 8-connected graph which satisfies the degree-sum condition of Theorem 1.

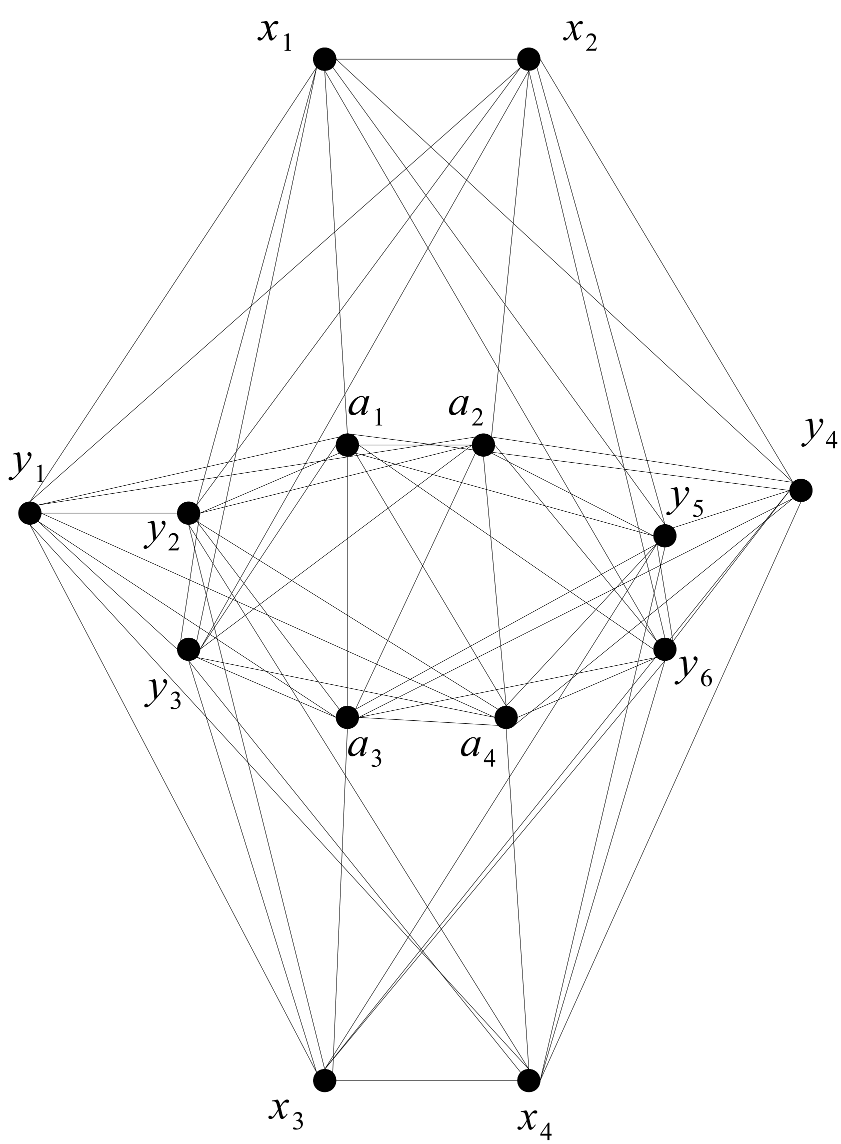

Figure 2: graph G

Similarly we can construct a contraction-critically k-connected graph which satisfied the degree-sum condition of Theorem 1, for any k such that $k \geq 9$

2 Preriminaries

In this section we give some more definitions and preliminary results.

For a graph G, we write $|G|$ for $|V(G)|$. For a subgraph H of graph G, when there is no ambiguity, we sometimes write simply H for $V(H)$. So $N_G(H)$ and $G-H$ mean $N_G(V(H))$ and $G-V(H)$, respectively. For a vertex x of G, we write $E_G(x)$ for $E_G(\{x\}, V(G)-\{x\})$. When there is no ambiguity we write $E(x)$ for $E_G(x)$. Hence $E(x)$ stands for the set of edges incident with x. We denote the set of end vertices of an edge e of G by $V(e)$. For a connected subgraph W of G, we denote the degree sum of W by $d_G(W)$, that is $d_G(W) = \sum_{x\in V(W)} d_G(x)$.

Let k be an integer such that $k \geq 2$ and let G be a graph with $\kappa(G) = k$. Recall that an edge of G is said to be k-noncontractible if it is not k-contractible. An edge is k-noncontractible if and only if there is a k-cutset S which contained $V(e) \subseteq S$. An induced subgraph A of G is called a $fragment$ if $|N_G(A)| = k$ and $V(G)-(A\cup N_G(A)) \neq \emptyset$. In other words, a fragment A is nonempty union of components of $G-S$ where S is a k-cutest of G such that $V(G)-(A\cup S) \neq \emptyset$. By the definition, if A is a fragment of G, then $G-(A\cup N_G(A))$ is also a fragment of G. Let $\overline{A}$ stand for $G-(A\cup N_G(A))$. For a k-noncontractible edge e of G, a fragment A of G is said to be a fragment with respect to e if $V(e) \subseteq N_G(A)$. For a set of edges $F \subseteq E(G)$, we say that A is a fragment with respect to F if A is a fragment with respect to some $e \in F$. A fragment A with respect to F is said to be *minimum* if there is no fragment B other than A with respect to F such that $|B| < |A|$. If $|A| = 1$ and $|A| = 2$, then a fragment A is said to be a $trivial\ fragment$ and 2-$fragment$, respectively. Moreover, if $|A| \geq \lceil \frac{k+1}{2} \rceil$, then a fragment A is said to be a large fragment.

For an edge e of G, we write the cardinality of a minimum fragment with respect to e by $\eta(e)$. Namely, $\eta(e) = min\{|A| \mid A \text{ is a fragment with respect to } e\}$. For a vertex x of G, we set $\eta(x) = max\{\eta(e) \mid e \in E(x)\}$. By the definition $\eta(x) = 1$ if and only if each edge of $E(x)$ is trivially noncontractible. We set $E_L(G) = \{e \in E(G) \mid \eta(e) \geq \lfloor \frac{k+1}{2} \rfloor\}$. Hence $E_L(G)$ is the set of edges e for which each fragment with respect to e is large.

In this paper we will frequently mention a specific graph, $K_1 + C_4$. For convenience we write $(x_1, x_2x_3x_4x_5)$, for a graph $H \cong K_1 + c_4$, with $V(H) = \{x_1, x_2, x_3, x_4, x_5\}$,$d_H(x_1) = 4$ and $H[\{x_2, x_3, x_4.x_5\}] \cong C_4$ with $x_2x_3, x_3x_4, x_4x_5, x_5x_2 \in E(H)$.

In the proof Theorem 1 we will use the following Lemmas. The proofs of three Lemmas are not difficult.

Lemma 1. *Let G be a k-connected graph, and let A and B be fragments of G. Let $S = N_G(A)$ and $T = N_G(B)$. Then the following hold. If $A \cap B \neq \emptyset$, then $|S \cap B| \geq |\overline{A} \cap T|$.*

Lemma 2. *Let G be a graph and let W be a subset of $V(G)$. Then $\sum_{x\in V(G)-W} |N_G(x)\cap W| = \sum_{y\in W} d_G(y) - 2|E(W)|$.*

Ando and Kawarabayashi proved the following Lemma 3 in [2], which plays a fundamental roll in the proof of Theorem 1.

Lemma 3. *Let G be a k-connected graph and let A be a a minimum fragment with respect to $E_L(G)$. Suppose that A has a vertex x such that $E(x) \cap E_L(G) \neq \emptyset$ Then each edge in $E(x) \cap E_L(G)$ is k-contractible.*

3 The proof of Theorem 1

In this section we give a proof of Theorem 1. Let k be an integer, such that $k \geq 5$. Assume that G is a contraction critically k-connected graph with no $K_1 + C_4$ such that $d_G(x) + d_G(y) + d_G(z) \geq 3k + 2$ hold for any connected subgraph W of G with $V(W) = \{x, y, z\}$. Since G is contraction critically k-connected, there is a k-cutset S of G such that $e \in E(G[S])$ for every edge e of $E(G)$.

Claim 1. *Let S be a k-cutset of G, and let A be a minimum fragment of $G - S$. If $|A| \notin \{1, 2\}$, then $|A| \geq \lceil \frac{k+1}{2} \rceil$.*

Proof.
Clearly, we have either $|A| \in \{1, 2\}$ or $|A| \geq 3$. We prove that if $|A| \geq 3$, then $|A| \geq \lceil \frac{k+1}{2} \rceil$. Set $S = N_G(A)$, and let $H = G[S \cup V(A)]$. For any $w \in V(A)$, we have $N_G(w) = N_H(w)$. Thus $d_G(w) = d_H(w)$. By the minimality of A, A is connected. Since $|A| \geq 3$, A contains a connected subgraph W with $|W| = 3$. Denote $X = V(W) = \{x_1, x_2, x_3\}$, $Q = V(H) - X$. Since W is connected, without loss of generality we may assume that $x_1x_2, x_2x_3 \in E(W)$. We show $|N_G(x_1) \cap N_G(x_2) \cap N_G(x_3)| \leq 1$. Assume $|N_G(x_1) \cap N_G(x_2) \cap N_G(x_3)| \geq 2$, say $N_G(x_1) \cap N_G(x_2) \cap N_G(x_3) \supset \{u, v\}$. Then we observe that $G[\{u, x_1, x_2, x_3, v\}] \supset K_1 + C_4 = (x_2, x_1ux_3v)$, which contradicts the forbidden subgraph condition. Therefore $|N_G(x_1) \cap N_G(x_2) \cap N_G(x_3)| \leq 1$. So, at most one vertex of $A \cup S - X$ is adjacent to all vertices of X, and every other vertex of $V(H) - X$ is adjacent to at most two vertices of X. By the degree sum condition, we have that $d_G(x_1) + d_G(x_2) + d_G(x_3) \geq 3k + 2$. If $x_1x_3 \notin E(G)$ then by Lemma 2, we have

$$\begin{aligned} 3k + 2 &\leq \sum_{x \in X} d_G(x) = 2|E(W)| + |E(X, Q)| \\ &\leq 2 \times 2 + 3 + 2(|A| + |S| - 3 - 1) \\ &= 2|A| + 2k + 1 \end{aligned}$$

Therefore $2|A| \geq k + 3$. Since $|A|$ is an integer, we have $|A| \geq \lceil \frac{k+3}{2} \rceil > \lceil \frac{k+1}{2} \rceil$. If $x_1x_3 \in E(G)$ then by the same arguments, we have

$$\begin{aligned} 3k + 2 &\leq \sum_{x \in X} d_G(x) = 2|E(W)| + |E(X, Q)| \\ &\leq 2 \times 3 + 3 + 2(|A| + |S| - 3 - 1) \\ &= 2|A| + 2k + 1 \end{aligned}$$

Therefore $2|A| \geq k + 1$. Since $|A|$ is an integer, we have $|A| \geq \lceil \frac{k+1}{2} \rceil$. The proof of Claim 1 is completed. $\square$

Claim 2. *Let x be a vertex of G. Let A be a connected 2-fragment with respect to $E(x)$. If $E(x) \cap E_L(G) = \emptyset$, then each fragment with respect to $E_G(x, A)$ is trivial.*

Proof.
Let $A = \{a, b\}$ and let $S = N_G(A)$. Let B be a fragment with respect to $E_G(x, A)$ and let $T = N_G(B)$. Since $E(x) \cap E_L(G) = \emptyset$, Claim 1 assure us that $|B| \in \{1, 2\}$. Assume that $|B| = 2$, say $B = \{u, v\}$. We show $b \in A \cap T$. Assume that $b \in A \cap B$. Then $A \cap \overline{B} = \emptyset$ since $A = \{a, b\}$. If $S \cap \overline{B} = \emptyset$, then $|S \cap \overline{B}| < |A \cap T|$. Hence Lemma 1 assure us that $\overline{A} \cap \overline{B} = \emptyset$, which implies that $\overline{B} = \emptyset$. This contradicts that the choice of B. Hence $S \cap \overline{B} \neq \emptyset$, and we observe $d_G(b) = k$, which implies $A \cap B$ is a fragment with respect to $E_G(x, A)$. This contradicts the fact that B is minimum fragment. Hence $b \notin A \cap B$.
By the similar argument we can show that $b \notin A \cap \overline{B}$. Now it is shown that $b \in A \cap T$. We show $|S \cap B| = 2$, assume $|S \cap B| \leq 1$. Then $|S \cap B| < |A \cap T|$ and Lemma 1 assure us that $\overline{A} \cap B = \emptyset$, which implies that $B = S \cap B$ and $|B| = |S \cap B| \leq 1$. This contradicts the fact that $|B| = 2$. Hence we have $|S \cap B| = 2$ and $S \cap B = \{u, v\}$. Since A is a connected fragment, we see that $ab \in E(G)$. Since B is a minimum fragment with respect to $E_G(x, A)$, we observe that $uv \in E(G)$. Since $uv, ab \in E(G)$, we see that $G[\{a, b, u, v\}]$ has a 4-cycle, if $|\{a, b, u, v\}| \cap V_k(G) \geq 2$, we can find a connected subgraph W of $G[\{a, b, u, v\}]$ such that $|W| = 3$ and $d_G(W) \leq 3k+1$, which contradicts the degree-sum condition. Hence $|\{a, b, u, v\} \cap V_k(G)| \leq 1$. Without loss of generality, we may assume that $\{a, b, u\} \subseteq V_{k+1}(G)$. Then we see that $G[\{z, a, b, u, v\}] \supset K_1 + C_4 = (a, xbvu)$, which contradicts the forbidden subgraph condition. Now Claim 2 is proved. □

Claim 3. *If $x \in V_k(G) \cup V_{k+1}(G)$. Then $E(x) \cap E_L(G) \neq \emptyset$.*

Proof.
At first, we show $N_G(x) \cap V_k(G) \neq \emptyset$, Assume that $N_G(x) \cap V_k(G) = \emptyset$. Let A be a minimum fragment with respect to $E(x)$. Since $N_G(x) \cap V_k(G) = \emptyset$, we observe that $|A| \geq 2$. Since $E(x) \cap E_L(G) = \emptyset$, Claim 1 assure us $|A| \leq 2$. Hence we see $|A| = 2$. Let B be a minimum fragment with respect to $E_G(x, A)$. Then by Claim 2 we have $|B| = 1$, which contradicts the assumption that $N_G(x) \cap V_k(G) = \emptyset$. Now it is showed that $N_G(x) \cap V_k(G) \neq \emptyset$.
Let $y \in N_G(x) \cap V_k(G)$. Let A be a minimum fragment with respect to xy. We show $|A| \geq 2$. Assume $|A| = 1$, say $A = \{a\}$. Then since $x \in V_k(G) \cup V_{k+1}(G)$ and $y, a \in V_k(G)$, we have $d_G(x) + d_G(y) + d_G(a) \leq 3k + 1$, which contradicts the degree-sum condition. Now it is show that $|A| \geq 2$. Hence $|A| = 2$, say $A = \{a, b\}$. If $\{a, b\} \cap V_k(G) \neq \emptyset$, we see that $d_G(y) + d_G(a) + d_G(b) \leq 3k + 1$, which contradicts the degree-sum condition. Hence we observe that $\{a, b\} \subset V_{k+1}(G)$. Let B be a minimum fragment with respect to ya. Then Claim 2 assure us that $|B| = 1$, say $B = \{u\}$. Since $u \in V_k(G)$ and $x \in V_{k+1}$, We observe that $u \neq x$. That $d(G[\{y, u, a\}]) \leq 3k + 1$, which contradicts the degree-sum condition. The proof of Claim 3 is completed. □

Now we proceed the proof of Theorem 1. Let A be a minimum fragment with respect to $E_L(G)$ and let $S = N_G(A)$. Since G is contraction-critically, Lemma 3 assure us $E(x) \cap E_L(G) = \emptyset$ for any $x \in A$. Hence by Claim 3 we observe that $A \cap (V_k(G) \cup V_{k+1}(G)) = \emptyset$.

Claim 4. *Let $x \in A$ and let $y \in N_G(x) \cap S$. Let B be a minimum fragment with respect to xy. Then $|B| = 1$.*

Proof.
Let $T = N_G(B)$. Since $x \in A$ and G is contraction critically, Lemma 3 assure us that $E(x) \cap E_L(G) = \emptyset$. Hence Claim 1 assure us $|B| \in \{1, 2\}$. Assume $|B| = 2$, say $B = \{u, v\}$. Since $u, v \in V_k(G) \cup V_{k+1}(G)$, Claim 3 assure us that $A \cap B = \emptyset$. If $S \cap B = \emptyset$, then $|S \cap B| < |A \cap T|$, which implies $\overline{A} \cap B = \emptyset$ and $B = \emptyset$. This contradicts the choice of B. Hence $S \cap B \neq \emptyset$. We show $\overline{A} \cap B = \emptyset$, assume $\overline{A} \cap B \neq \emptyset$, say $v \in \overline{A} \cap B$ and $u \in S \cap B$. Since $|S \cap B| = |A \cap T|$, we observe that $|(S \cap B) \cup (S \cap T) \cup (\overline{A} \cap T)| = k$, which implies $v \in V_k(G)$. Let C be a minimum fragment with respect to $E_G(x, B)$. Then, by Claim 2, we know that $|C| = 1$, say $C = \{w\}$. Since $v \in \overline{A}$ and $x \in A$, we see that $xv \notin E(G)$, which implies $w \neq v$, since $xw \in E(G)$. Let $W = G[\{u, v, w\}]$. Since B is minimum, we know that $uv \in E(G)$, which implies that W is connected. Now we observe that $d_G(W) = d_G(w) + d_G(u) + d_G(v) \leq 3k + 1$, which contradicts the degree-sum condition. It is shown that $\overline{A} \cap B = \emptyset$. and $B = S \cap B = \{u, v\}$
Since $N_G(x) \cap B \neq \emptyset$, say $u = N_G(x)$. Let C is a minimum fragment with respect to xu. Then by Claim 2, we have $|C| = 1$, say $C = \{w\}$. We show that $w \neq v$. Assume $w = v$. Then $xv \in E(G)$ and $v \in V_k(G)$. If $vy \in E(G)$, then $v \in N_G(x) \cap N_G(y) \cap V_k(G)$, which contradicts that $\eta(xy) = 1$. Hence $yv \notin E(G)$, which implies $yu \in E(G)$. Since $N_G(y) \cap B \neq \emptyset$. Let C' be a minimum fragment with respect to xv. By Claim 2, we know that $|C'| = 1$, say $C' = \{w'\}$. If $w' = u$, then since $yu \in E(G)$, we see that $u \in N_G(x) \cap N_G(y) \cap V_k(G)$, which contradicts the assumption that $\eta(xy) = 2$. Hence $w' \neq u$. Then $d_G(u) + d_G(v) + d_G(w') \leq 3k + 1$, which contradicts the degree-sum condition. This contradiction proved that $w \neq v$.
If $\{u, v\} \cap V_k(G) \neq \emptyset$, then $W = G[\{u, v, w\}]$ is a connected subgraph of G such that $d_G(u) + d_G(v) + d_G(w) \leq 3k + 1$, which contradicts the degree-sum condition. Hence $\{u, v\} \subset V_{k+1}(G)$.
We show $|A \cap T| \geq 2$. Assume $|A \cap T| = 1$. Then since $|S \cap B| > |A \cap T| = 1$, Lemma 1 assure us $A \cap \overline{B} = \emptyset$, which contradicts the fact that A is a large fragment with $|A| \geq \lceil \frac{k+1}{2} \rceil \geq 3$. It is shown that $|A \cap T| \geq 2$, say $x' \in A \cap T - \{x\}$. Let C' be a minimum fragment with respect to $E_G(x', B)$. Then, Claim 2 assures us that $|C'| = 1$, say $C' = \{w'\}$. Note that $w, w' \in N_G(u)$. Since $u \in V_{k+1}(G)$, if $w \neq w'$, then $W = G[\{w, w', u\}]$ is a connected subgraph of G such that $d_G(u) + d_G(w) + d_G(w') \leq 3k+1$, which contradicts the degree-sum condition. Hence $w = w'$ Then we observe that $xw, x'w \in E(G)$ and $\{x, w, x'\} \subset N_G(u) \cap N_G(v)$, and we find a $K_1 + C_4 = (w, xux'v)$ in G which contradicts the forbidden subgraph condition. The proof of Claim 4 is completed. □

Claim 5. *Let $xy \in E(A)$. Let B be a minimum fragment with respect to xy. Then $|B| = 1$.*

Proof.
Since $x \in A$ and G is contraction critical, Lemma 3 assures us that $E(x) \cap E_L(G) = \emptyset$. Hence Claim 1 assure us $|B| \in \{1, 2\}$. Assume $|B| = 2$, say $B = \{u, v\}$. We show that $u, v \in S \cap B$. Since $u, v \in V_k(G) \cup V_{k+1}(G)$, Claim 3 assure us that $A \cap B = \emptyset$. If $\overline{A} \cap B \neq \emptyset$, then by Lemma 1, we have $|S \cap B| < |A \cap T|$, which implies $\overline{A} \cap B = \emptyset$ and $B = \emptyset$. This contradicts the choice of B. Hence $\overline{A} \cap B = \emptyset$. It is shown that $u, v \in S \cap B$. We show $\{u, v\} \subseteq V_{k+1}(G)$. Assume, $\{u, v\} \cap V_k(G) \neq \emptyset$. Suppose $u \in V_k(G)$. Since $xy, uv \in E(G)$ and note of $N_G(x) \cap B$, $N_G(y) \cap B$, $N_G(u) \cap \{x, y\}$ and $N_G(v) \cap \{x, y\}$ is empty, we observe that $G[\{x, y\} \cup B]$ has a C_4. Without loss of generality we may assume

$C_4 = xuvy$. Let C be a minimum fragment with respect to $E_G(x, B)$. Then Claim 2 assures us that $|C| = 1$, say $C = \{w\}$. If $v = w$, then $v \in N_G(x) \cap N_G(y) \cap V_k((G)$, which contradicts the assumption that $\eta(xy) = 2$. Hence $v \neq w$. Then $W = G[\{u, v, w\}]$ is a connected subgraph of G such that $d_G(u) + d_G(v) + d_G(w) \leq 3k + 1$, which contradicts degree-sum condition. Now it is shown that $\{u, v\} \subseteq V_{k+1}(G)$.
Let D be a minimum fragment with respect to xu. Since Claim 4, $|D| = 1$, say $D = \{w'\}$. Since $u, v \in V_{k+1}(G)$ and $w' \in V_k(G)$, $w' \neq u, v$. Then we observe that $\{x, y, w'\} \subset N_G(u) \cap N_G(v)$, and we find a $K_1 + C_4 = (x, yuw'v)$, which contradicts the forbidden subgraph condition. This contradiction proves Claim 5 □

Now we denote $\mathcal{E} = \{f \in E(G) | V(f) \subset S \cap V_k(G)\}$. Let $f_1, f_2 \in \mathcal{E}$. Since the degree-sum condition, if $V(f_1) \cap V(f_2) \neq \emptyset$, then $f_1 = f_2$.

Claim 6. *For each edge $e \in E[A]$, there exists an edge $f \in \mathcal{E}$ such that $G[V(e) \cup V(f)] \cong K_4$.*

Proof.
Let B be a minimum fragment with respect to xy. By Claim 5, $|B| = 1$, say $B = \{u\}$, then $u \in V_k(G) \cap S$ and $xu, yu \in E(G)$. Let C be a minimum fragment with respect to xu. By Claim 4, $|C| = 1$, say $C = \{v\}$. which implies $v \in V_k(G) \cap S$ and $xv, yv \in E(G)$. Let D be a minimum fragment with respect to uy. By Claim 4, we have $|D| = 1$, say $D = \{w\}$, then $w \in V_k(G) \cap S$ and $xw, yw \in E(G)$. If $w \neq v$ then $d(G[\{u, v, w\}]) = 3k$, which contradicts the degree-sum condition. Hence $w = v$, which implies $uv \in \mathcal{E}$. We observe that $G[\{x, y, u, v\}] \cong K_4$. The proof of Claim 6 is completed. □.

Claim 6 assures us that for each edge $e \in E(A)$, there is an edge $f \in \mathcal{E}$ such that $G[V(e) \cup V(f)] \cong K_4$. For each $e \in E(A)$ choosing a such edge f and setting $\varphi(e) = f$, we define a mapping φ from $E(A)$ to $\mathcal{E}$.

We denote $dist_A(e_1, e_2)$ be the distance between e_1 and e_2 in A. If $dist_A(e_1, e_2) = 1$ then e_1 and e_2 have a common end vertex. If $dist_A(e_1, e_2) = 2$, then there is an edge e_3 between a vertex in $V(e_1)$ and a vertex in $V(e_2)$.

Claim 7. *Let $e_1, e_2 \in E(A)$. If $e_1 \neq e_2$ and $dist_A(e_1, e_2) \leq 2$. then $\varphi(e_1) \neq \varphi(e_2)$.*

Proof.
Assume that $\varphi(e_1) = \varphi(e_2) = f$. Let $V(e_1) = \{x_1, x_2\}$, $V(e_2) = \{x_3, x_4\}$, and $V(f) = \{u, v\}$. At first we consider the case that $dist_A(e_1, e_2) = 1$. Then $V(e_1) \cap V(e_2) \neq \emptyset$, without loss of generality we may assume that $x_2 = x_3$. Then we observe that $x_1u, x_1v, x_2u, x_4u, x_4v \in E(G)$ and $G[\{x_1, x_2, x_4, u, v\}] \supset K_1 + C_4 = (u, x_1vx_4x_2)$, which contradicts the forbidden subgraph condition. Next we consider the case that $dist_A(e_1, e_2) = 2$. In this case all for vertices x_1, x_2, x_3, x_4 any distinct and we may assume that $x_2, x_3 \in E(G)$. Then we observe that $x_1u, x_1v, x_2u, x_3u, x_3v \in E(G)$ and $G[\{x_1, x_2, x_3, u, v\}] \supset K_1 + C_4 = (u, x_1vx_3x_2)$ which contradicts the forbidden subgraph condition. Now Claim 7 is proved. □

Claim 8. *There is an edge $xy \in E(A)$ such that $d_A(x) + d_A(y) \geq \frac{2}{3}k + 3$.*

Proof.
Let W be a subgraph of A which is isomorphic to a path of length 2. Let $W = x_1x_2x_3$.

Subclaim 8.1. $d_A(W) = k + 5$

Proof.
Assume $|N_G(x_1) \cap N_G(x_2) \cap N_G(x_3)| \geq 2$, say $u, v \in N_G(x_1) \cap N_G(x_2) \cap N_G(x_3)$. Then we observe that there is a $K_1 + C_4 = (x_2, x_1ux_3v)$, which contradicts the forbidden subgraph condition. Hence $|N_G(x_1) \cap N_G(x_2) \cap N_G(x_3)| \leq 1$. Hence we have

$$\begin{aligned} |E_G(V(W), S)| &\leq 3 + 2(|S| - 1) \\ &= 2k + 1 \end{aligned}$$

Since $\delta(A) \geq 2$, we have $d_G(W) \geq 3k + 6$. Hence

$$\begin{aligned} d_A(W) &= d_G(W) - |E_G(V(W), S)| \\ &\geq (3k + 6) - (2k + 1) \\ &= k + 5 \end{aligned}$$

Now Subclaim 8.1 is proved.□

Subclaim 8.2. *A contains a subgraph which is isomorphic to P_4.*

Proof.
Assume A has no P_4. Then we see that $dist(e, e') \leq 2$ for any two edge $e, e' \in E(A)$, and A has $K_1, |A| - 1$. Since A has $K_1, |A| - 1$, we see that $\Delta(A) = |A| - 1$. Since $\delta(A) \geq 2$ and $\Delta(A) = |A| - 1$, we have $2|E(A)| \leq (|A| - 1) + 2(|A| - 1) = 3|A| - 3$. Since $dist(e, e') \leq 2$ for any $e, e' \in E(A)$, Claim 7 assures us that $|\varphi(E(A))| = |E(A)|$. Hence $|\mathcal{E}| \geq |\varphi(E(A))| = |E(A)|$, which implies $|S| \geq 2|\mathcal{E}| = 2|E(A)| \geq 3|A| - 3$. Since $|A| \geq \lceil \frac{k+1}{2} \rceil$, we have

$$\begin{aligned} |S| &\geq 3\lceil \frac{k+1}{2} \rceil - 3 \\ &\geq k + \frac{k+3}{2} - 3 \\ &\geq k + 1 \end{aligned}$$

which contradicts the fact $k = |S|$. Now Subclaim 8.2 is proved.□

Subclaim 8.2 assure us that A contains $P_4 = x_1x_2x_3x_4$. Assume $d_A(x_2) + d_A(x_3)\frac{2}{3}k + 3$, and $d_A(x_3) + d_A(x_4) < \frac{2}{3}k + 3$. Since $d_A(x_2) + d_A(x_3) + d_A(x_4) \geq k + 5$, we have $d_A(x_4) \geq \frac{k}{3} + 2$. Then $d_A(x_3) < (\frac{2}{3}k + 3) - d_A(x_4) = \frac{k}{3} + 1$. Hence $d_A(x_1) + d_A(x_2) \geq (k + 5) - d_A(x_3) \geq \frac{2}{3}k + 3$. Now Claim 8 is proved. □

Let A be a minimum fragment with respect to $E_L(G)$. Let $N_G(A) = S$ and let $x, y \in A$. Since Claim 8, We observe that, $d_A(x) + d_A(y) \geq \frac{2}{3}k + 3$. By Claim 7, we have

$$
\begin{aligned}
|S| \geq |\bigcup_{i=1}^{\frac{2}{3}k+2} V(\varphi(i))| &= \sum_{i=1}^{\frac{2}{3}k+2} |V(\varphi(e_i))| \\
&= 2(\frac{2}{3}k+2) \\
&> k
\end{aligned}
$$

This is contradicts that $|S| = k$. This is the final contradiction and proof of Theorem 1 is completed.

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