The trade-off relations of a relativistic spin-1/2 particle for two-parameter unitary model with commuting generators

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論文の和文概要

論文題目	可換な生成子を持つ2パラメータユニタリモデルにおける相対論的スピン-1/2粒子のトレードオフ関係
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本論文は、不確定性関係からは導き出せない、可換なオブザーバブルの期待値の推定を量子パラメータ推定問題として扱うことにより、推定誤差間のトレードオフ関係を定量的に導出した。トレードオフ関係を求める新しい方法を提案した。
スピン1/2の相対論的粒子の位置演算子の期待値推定の問題に提案した手法を適用した。
相対論的効果により、観測者の速度に依存してスピンが回転するため、静止系では古典的な（量子力学的でない）モデルのため存在しなかったトレードオフ関係が運動している観測者にとっては、必ず存在するという非自明な結果を得た。

論 文 の 英 文 要 旨

TITLE	The trade-off relations of a relativistic spin-½ particle for two-parameter unitary model with commuting generators
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We treat estimating the expectation values of commuting as a quantum parameter estimation problem. A new method for obtaining the trade-off relation is proposed. As a result, we quantitatively derive a trade-off relation of the commuting observables, which is inconsistent with Heisenberg's uncertainty relation. We applied the proposed approach to the problem of estimating the expectation value of the position operator of a relativistic particle with spin 1/2. We obtained the non-trivial result that the trade-off relation, which does not exist in the rest frame due to the classical (non-quantum mechanical) model, always exists for a moving observer because the relativistic effect causes the spin to rotate depending on the observer's velocity.

Contents

List of Figures

Chapter 1

Introduction

1.1 Background

1.1.1 Uncertainty relation and quantum estimation theory

The uncertainty relation based on the quantum estimation theory was investigated by many authors, see for example [1, 2, 3, 4, 5, 6, 7, 8]. It is known that the one-parameter unitary model with a pure reference state, the Heisenberg-Robertson type uncertainty relation and the uncertainty relation by the parameter estimation has the same form. Further, this approach is more general than the traditional one, since one can derive the uncertainty relation for non-observables. The celebrated energy-time uncertainty relation [9] is a well-defined relation for time and energy when treated by the quantum estimation theory. Many authors in the literature discussed the similarity between the two types of uncertainty relations. In Ref. [6], they showed that the uncertainty relation for a generic full parameter qudit model can be different when derived from the quantum parameter estimation theory. Usually, when the uncertainty relation is discussed, the uncertainty relation of two non-commuting observables is discussed, see for example [10, 11, 12]. Therefore, as far as we know, trade-off relation has not been investigated when the two observables commute.

1.1.2 Quantum information theory with relativity taken into account

Relativistic quantum information theory brings a new direction to physics research. The significance of the effect of relativity on the quantum state is that the state vector for a moving observer changes depending on the observer's motion while the physical state in the rest frame remains the same. As a natural consequence, information that the moving observer obtains changes depending on the motion of the moving observer since the state vector changes. There are studies of quantum information theory with relativity taken into account. The studies in the realm of relativistic quantum information have increased in number in the past. Here, we briefly list some of them. Firstly, the information paradox about black holes is now formulated in the framework of information theory; see recent reviews [15, 16]. Secondly, quantum information in non-inertial frame was investigated [17, 18, 19, 20, 21]. Thirdly, the effect of relativity on Bell's inequality. The degree of Bell's inequality violation was investigated [22, 23, 24, 25, 26]. The entropy changes due to the relativistic effect [27] and its effect on Bell's inequality are also studied, which was initiated in [28].

Among these early studies about the relativity and the quantum information, the papers [22, 23, 29, 30] brought the use of the Wigner rotation [31, 32] into the realm of quantum information. As other examples, the Wigner rotation is used to discuss the limitation given by a quantum entropy in the relativity domain [27]. The entanglement [29, 33, 34, 35, 36, 37, 39, 40] and Bell's inequality [23, 22, 41] are also discussed by using the Wigner rotation. The essence of the Wigner rotation is that it 'rotates' the spin of the relativistic particle by the angle, which is a function of the particle's momentum. Thus, the spin and the momentum couple in a non-trivial way that the Wigner rotation gives.

1.2 Trade-off relation based on relativistic quantum estimation

Based on the previous investigations in relativistic quantum information theory, it is natural to pose a question: what is the effect of the Wigner rotation for parameter estimation of quantum states? To phrase it differently, we ask: how does estimation accuracy change for a moving observer? However, to the best of our knowledge, there has existed no study about the change in estimation accuracy that a moving observer undergoes. We demonstrate how estimation accuracy changes for the moving observer in the framework of the quantum estimation theory [1, 2]. To obtain the limit of estimation accuracy as a function of the moving observer's velocity, we utilize the quantum Fisher information matrix which enables us to quantify the accuracy limit. Among those quantum Fisher information matrices, we consider the symmetric logarithmic derivative (SLD) Fisher information matrix as an indicator of estimation accuracy. As the main result, we obtain the analytical expression of the SLD Fisher information matrix for an arbitrary moving observer as an integral form Eq. (6.52). This then sets the estimation accuracy limits between the observers in the rest frame and in the moving frame. To illustrate our result, we plot the relativistic effect on estimation accuracy in Fig. 6.4. Estimation accuracy obtained by the SLD Fisher information matrix is finite even at the relativistic limit where the velocity vapproaches the speed of light. This suggests that estimation accuracy remains finite at the relativistic limit.

As for the model, we set up a specific pure-state model that describes a single spin-1/2 particle. A parametric model is defined by a two-parameter unitary shift model. We next consider an observer moving at a constant velocity in one direction with respect to the rest frame. The moving observer then makes a measurement to estimate the parameters encoded in the state without accessing the spin degree of freedom. Thus, our parameter model in the moving frame is given by the Wigner rotation followed by the partial trace over the spin. We investigate how estimation accuracy for the moving observer changes as a function of the velocity. In our study, the parameters correspond to the expectation value for the position of the particle. We evaluate the limits for the mean square error (MSE) upon estimating the expectation value of the position operator by the SLD Cramér-Rao (CR) bound and λLD CR bound. As a result, we show analytically that a trade-off relation always exists for the moving observer once we consider relativity.

1.3 Trade-off relation by commuting observables

We also investigate a trade-off relation between two commuting observables based on the two-parameter quantum estimation theory. From what has been said so far, it may seem that such a trade-off relationship is impossible at first glance. However, as we show in this thesis, the quantum estimation theory allows us to derive a non-trivial trade-off relation, or trade-off relation by estimating the expectation values of two commuting observables in the following way.

We first treat the estimation of the expectation value of commuting observables as a quantum parameter estimation problem. Then, we quantitatively derive the trade-off relation between estimation errors, which cannot be derived from the trade-off relation. More specifically, we considered a two-parameter unitary model with the commuting generators. We evaluated the lower bound of the mean square error (MSE) matrix by applying the quantum Cramér-Rao inequality for one-parameter families based on quantum estimation theory. The proposed method in this thesis consists of i) giving a trade-off relation by the components of the MSE matrix and of the quantum Fisher information matrix, and ii) narrowing the lower bound by combining multiple quantum Cramér-Rao inequalities. We confirm the effectiveness of the proposed method.

As for the trade-off relation of the commuting observables, we investigate the cases of the state generated by the unitary transformations with two commuting generators. In other words, we assume X and Y as generators with parameters $\theta = (\theta_1, \theta_2)$ and use the unitary transformations generated by them. The state ρ_θ generated by this transformation from the reference state ρ_0 which is known in advance is defined as follows.

$$\rho_\theta = \mathrm{e}^{-\mathrm{i}\theta_1 X}\mathrm{e}^{-\mathrm{i}\theta_2 Y}\rho_0\mathrm{e}^{\mathrm{i}\theta_1 X}\mathrm{e}^{\mathrm{i}\theta_2 Y}. \tag{1.1}$$

When the reference state ρ_0 is a pure state or a qubit, we show that the trade-off relation does not exist. However, in the three-dimensional case, when ρ_0 is a qutrit, we give conditions for the trade-off relation to be established, construct such an example, and show numerically that the trade-off relation does indeed exist generically.

After we confirmed that the trade-off relation of the commuting generators does exist, we come to a thought that it may be nothing unusual to see the trade-off relation when the generators commute. To see if that is the case, we next investigate the trade-off relation of a model with a infinite degree of freedom.

Our physical model for this purpose is a model of one electron in a uniform magnetic field. We also assume that the electron is in the thermal state, which is a mixed state, because a pure reference state does not provide a trade-off relation when the generators of the transformation commute. Then, we investigate the trade-off relation regarding the position of the electron by the parameter estimation problem of the two-parameter unitary model. In this model, the Heisenberg-Robertson type uncertainty relation [13, 14] of the position operators X, Y of an electron only yields the following trivial inequality.

$$(\Delta X)(\Delta Y) \geq \frac{1}{2}|[X,\ Y]| = 0, \tag{1.2}$$

where [X, Y]:=XY-YX. ΔX denotes the (quantum) standard deviation about X with respect to a state ρ, which is defined by $(\Delta X)^2 = \mathrm{tr}[\rho(X - \langle X\rangle_\rho)^2]$.

To derive the trade-off relation, we are to use a parametric model that describes the position measurement of the particle. Here we utilize the unitary transformation generated by the canonical momenta p_x and p_y with the parameter $\theta = (\theta_1, \theta_2)$. By the same token as in Eq. (1.1), we can write the state ρ_θ generated by the unitary transformation from the reference state ρ_0 which is known in advance as follows.

$$\rho_\theta = \mathrm{e}^{-\mathrm{i}\theta_1 p_x}\mathrm{e}^{-\mathrm{i}\theta_2 p_y}\rho_0\mathrm{e}^{\mathrm{i}\theta_2 p_y}\mathrm{e}^{\mathrm{i}\theta_1 p_x}. \tag{1.3}$$

By using the momenta p_x and p_y as the generators, we obtain the expectation value of the position operators as follows.

$$\langle X\rangle_\theta = \langle X\rangle_0 + \theta_1, \tag{1.4}$$
$$\langle Y\rangle_\theta = \langle Y\rangle_0 + \theta_2, \tag{1.5}$$

where $\langle X\rangle_\theta = \mathrm{tr}[\rho_\theta X]$ and $\langle X\rangle_0 = \mathrm{tr}[\rho_0 X]$. We define $\langle Y\rangle_\theta$ and $\langle Y\rangle_0$ similarly. From the above, it can be seen that estimating the parameters θ_1 and θ_1 is equivalent to measuring the positions X and Y. Note that the generators of this model also commute and $[p_x, p_y] = 0$.

Based on the analysis by the quantum estimation theory, we see that this model is an example of showing the intersections of the SLD and RLD bounds. By using the method explained above, we get non-trivial bounds that give the trade-off relations between the two commuting observables, x and y, unlike the result of Heisenberg-Robertson type uncertainty relation.

Finally, we set up another physical model also with a continuous infinite degree of freedom. In a similar setting for the reference state, we add a moving observer to investigate how relativistic effects affect estimation accuracy. As the reference state, we set up a particle with spin-1/2 in the rest frame, where an observer is not in motion with respect to the physical system. As for the spin state, it is known that it is in the spin down state in the rest frame. As a relativistic effect, it is known that the moving observer observes the change in the wave function as well as the spin rotation once the relativity is taken into account. It is also known that the Wigner rotation describes these changes. This relativistic effect is expected to affect the accuracy of the estimation of the position by the moving observer.

Our method is then applied to this statistical model to show the existence of the trade-off relation between the components of the MSE error matrix. As a statistical model, we place the pure state in the rest frame, obtain the wave function in the moving frame based on relativistic quantum mechanics. Then, the spin degrees of freedom are traced out (averaged).

As a result, we obtain the non-trivial result that a trade-off relation is always established when the observer moves although this model is a classical (non-quantum mechanical) model in the rest frame. Therefore, a trade-off does not exist in the rest frame. In this way, our model is an example that a relativistic effect gives rise to the trade-off relation.

1.4 Derivation of the trade-off relation

Furthermore, in this study, a general method for determining the trade-off relation from quantum estimation theory is also given. This method is used to investigate the uncertainty relation, or trade-off relation. First, we regard the trade-off relation based on the quantum estimation theory as the diagonal components of the MSE matrix [8]. Next, we establish a formulation of the diagonal components of the MSE matrix for the case of two-parameter estimation. We prove that our formulation is the best estimate as long as we are interested in the relation between the diagonal components.

In addition, we propose a use of two different quantum CR bounds, for example, the SLD and the λLD CR bounds so that we can narrow the lower bound region by combining the two. Since this method is not always possible, we give conditions as to when we can use this method as well.

1.5 Outline

The outline of this thesis is as follows. Chapter 2 reviews the definitions of quantum Fisher information and quantum Cramér-Rao inequalities. In chapter 3, we propose the method for deriving the trade-off relation. Our proposal is twofold. First, we give a formula for the trade-off relation given by quantum Cramér-Rao inequalities in the case of the two-parameter model, assuming that we are interested in the relation between the diagonal components of the MSE matrix. This formula gives the trade-off relation in terms of the relation between the diagonal components of the MSE matrix. Next, we apply this formula to the SLD and the λLD Fisher information matrices separately and combine the obtained bounds to narrow down the lower bound. In chapter 4, we pick up the reference state as i) a pure state and ii) a finite-dimensional state and investigate whether the trade-off relation exists when the generators of the unitary transformation commute. We prove that when a pure state and a qubit state are the reference states, there does not exist a trade-off relation if the generators of the unitary transformation commute. However, we find a condition for qutrit states having a trade-off relation. Chapter 5 presents an example in which the trade-off relation exists while the generators of the unitary transformation commute. The physical model is one electron in a magnetic field with a continuous infinite degree of freedom. We assume that the reference state is a mixed (thermal) state. We find that the trade-off relation exists. The strength of the trade-off relation or even the existence of the trade-off relation strongly depends on the expectation value of angular momentum. In chapter 6, we set up a physical model similar to the one we use in chapter 5. Our model here is a massive particle in the x-y plane with its wave function described by a gaussian function and spin-1/2. However, the difference in this model is that we investigate in the framework of relativistic quantum mechanics. For that purpose, we assume that an observer is moving toward the z direction. We evaluate his/her estimation accuracy. This chapter analyzes the problem using the SLD Fisher information matrix only. The RLD Fisher information matrix does not exist. Hence, we do not see a trade-off relation. In chapter 7, we find the formula that gives the λLD and, thus, λLD Fisher information matrix for the non-full-rank model. Then, we apply the formula to the same physical model in chapter 6. By combining the SLD and the λLD CR bounds, we find that a trade-off relation exists with any given spread of the wave function and with any given observer's velocity.

Chapter 2

Quantum Cramér-Rao inequalities and quantum Fisher information matrices

In this chapter, we define a quantum state, a quantum parametric model, quantum CR bounds, and quantum CR type bounds.

2.1 Preliminary

In this section, we explain the notations used in this thesis.

2.1.1 Dirac notation

Ket

In the case of an n-dimensional complex Hibert space $\mathcal{H} = \mathbb{C}^n$, a ket $|\psi\rangle \in \mathbb{C}^n$ is a column vector.

$$|\psi\rangle = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}. \tag{2.1}$$

The dimension of Hilbert space can be extended to infinity, i.e., $\mathcal{H} = \mathbb{C}^\infty$. However, we only consider that the kets of their norm are finite such as

$$\langle\psi|\psi\rangle = \sum_{j=1}^{\infty} |\xi_j|^2 < \infty, \tag{2.2}$$

that is, $|\psi\rangle \in l^2$, where

$$l^2 = \left\{ a = \{a_j\}_{j=1}^{\infty} \in \mathbb{C}^\infty | \sum_{j=1}^{\infty} |a_j|^2 < \infty \right\}. \tag{2.3}$$

Bra

A bra $\langle\psi|$ is defined by the row vector whose entry is the complex conjugation of a ket.

$$\langle\psi| = (\xi_1^*,\ \xi_2^*,\ ,\cdots,\xi_n^*). \tag{2.4}$$

We denote complex conjugate of α by α^*.

Products

The inner product of the two kets $|\phi\rangle$ and ket $|\psi\rangle$ is written as

$$\langle\phi|\psi\rangle = (\eta_1^*,\ \eta_2^*,\ ,\cdots,\eta_n^*)\begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_n\end{pmatrix} = \sum_{i=1}^{n}\eta_i^*\xi_i. \tag{2.5}$$

The $|\psi\rangle\langle\phi|$ is a matrix $[|\psi\rangle\langle\phi|]_{ij}$ and is expressed as

$$|\psi\rangle\langle\phi| = \begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_n\end{pmatrix}(\eta_1,\ \eta_2,\ ,\cdots,\eta_n), \tag{2.6}$$

$$[|\psi\rangle\langle\phi|]_{ij} = \xi_i\eta_j^*. \tag{2.7}$$

2.1.2 Hermite conjugate

The Hermite conjugate of a matrix M is denoted by $M^\dagger$ which is obtained by taking the transpose of M and by taking the complex conjugate of each element.

$$[M^\dagger]_{ij} = ([M]_{ji})^*, \tag{2.8}$$

2.1.3 Positivity

Positive semi-definite

A matrix M is positive semi definite $\Leftrightarrow$ $\langle x|M|x\rangle \geq 0$ for all $|x\rangle \in \mathcal{H}$.
In this case, we denote $M \geq 0$.

Positive definite

A matrix M is positive definite $\Leftrightarrow$ $\forall\,|x\rangle \in \mathcal{H},$ if $|x\rangle \neq 0,\ \langle x|M|x\rangle > 0.$
In this case, we denote $M > 0$.

2.2 Quantum state, measurement, Born rule

2.2.1 Quantum state

A quantum state is represented by a matrix called state, ρ. The state ρ is positive semi definite and its trace is 1. Mathematically, $\rho > 0$ and $\mathrm{tr}\rho = 1$.

2.2.2 Measurement

We define the concept of measurement in quantum theory. In quantum theory, measurement is mathematically described by a set of matrices called positive operator-valued measure (POVM), Π.

$$\Pi = \{\Pi_x\}_{x \in \mathcal{X}}, \tag{2.9}$$

where $\mathcal{X}$ is an index set of measurement outcomes. Π_x are non-negative and satisfy the normalization condition.

$$\sum_{x \in \mathcal{X}} \Pi_x = \mathrm{I}, \tag{2.10}$$

where I is the identity matrix.

2.2.3 Born rule

This rule is one of the axioms of quantum theory and often called Born rule. The probability of getting outcome x, $p(x|\Pi)$ is given by

$$p(x|\Pi) = \mathrm{tr}(\rho \Pi_x). \tag{2.11}$$

From the positivity of state ρ and the positivity of POVM Π_x, $p(x|\Pi)$ is positive for all x. The summation over x gives 1, i.e.,

$$\sum_{x \in \mathcal{X}} p(x|\Pi) = \sum_{x \in \mathcal{X}} \mathrm{tr}(\rho \Pi_x) = \mathrm{tr}(\rho \sum_{x \in \mathcal{X}} \Pi_x) = \mathrm{tr}(\rho \mathrm{I}) = 1. \tag{2.12}$$

2.3 Model, MSE matrix, and quantum information matrices

2.3.1 Model

Let us define a quantum statistical model as a parametric family of quantum states on a Hilbert space $\mathcal{H}$.

$$\mathcal{M} := \{\rho_\theta | \theta \in \Theta \subset \mathbb{R}^n\}, \tag{2.13}$$

where $\theta = (\theta_1, \theta_2, \cdots, \theta_n)$.

MSE matrix

The mean square error (MSE) matrix $\mathsf{V}_\theta[\Pi, \hat{\theta}] = \left[\mathsf{V}_{\theta, ij}[\Pi, \hat{\theta}]\right]$ is defined by

$$\mathsf{V}_{\theta, ij}[\Pi, \hat{\theta}] = \sum_{x \in \mathcal{X}} \mathrm{tr}(\rho_\theta \Pi_x)(\hat{\theta}_i(x) - \theta_i)(\hat{\theta}_j(x) - \theta_j), \tag{2.14}$$

where $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_n)$ is an estimator.

The set of a POVM and an estimator, Π and $\hat{\theta}$ are called locally unbiased at θ if

$$\sum_{x \in \mathcal{X}} \hat{\theta}_i(x) \mathrm{tr}(\rho_\theta \Pi_x) = \theta_i, \tag{2.15}$$

and

$$\sum_{x \in \mathcal{X}} \hat{\theta}_i(x) \frac{\partial}{\partial \theta_j} [\mathrm{tr}(\rho_\theta \Pi_x)] = \delta_{i,j}, \tag{2.16}$$

hold for all i, j.

2.3.2 Qutantum Fisher information

SLD, RLD, and λLD

The symmetric logarithmic derivative (SLD), $L_{\mathrm{S}\,\theta, i}$, the right logarithmic derivative (RLD), $L_{\mathrm{R}\,\theta, i}$, and the λ logarithmic derivative (λLD) $L_{\lambda\,\theta, i}$ [42, 43] are defined by the solutions of the following equations.

$$\partial_i \rho_\theta = \frac{1}{2}(\rho_\theta L_{\mathrm{S}\,\theta, i} + L_{\mathrm{S}\,\theta, i} \rho_\theta), \tag{2.17}$$

$$\partial_i \rho_\theta = \rho_\theta L_{\mathrm{R}\,\theta, i}, \tag{2.18}$$

$$\partial_i \rho_\theta = \frac{1+\lambda}{2} \rho_\theta L_{\lambda\,\theta, i} + \frac{1-\lambda}{2} L_{\lambda\,\theta, i} \rho_\theta, \tag{2.19}$$

where

$$\partial_i \rho_\theta = \frac{\partial_i \rho_\theta}{\partial \theta_i}. \tag{2.20}$$

From Eq. (2.17), we can show the following.

Theorem 2.3.1 (Theorem SLD: Hermitian)
The SLD is Hermitian when the model is full rank.

$$L^\dagger_{\mathrm{S}\,\theta, i} = L_{\mathrm{S}\,\theta, i}. \tag{2.21}$$

Proof

The SLD is defined by Eq. (2.17), i.e.,

$$\partial_i \rho_\theta = \frac{1}{2}(\rho_\theta L_{\mathrm{S}\,\theta,i} + L_{\mathrm{S}\,\theta,i}\rho_\theta), \tag{2.22}$$

By taking Hermite conjugate on both the left and right-hand sides, we have

$$\partial_i \rho_\theta^\dagger = \frac{1}{2}(\rho_\theta L_{\mathrm{S}\,\theta,i} + L_{\mathrm{S}\,\theta,i}\rho_\theta)^\dagger, \tag{2.23}$$

$$\Longleftrightarrow \partial_i \rho_\theta = \frac{1}{2}(L_{\mathrm{S}\,\theta,i}^\dagger \rho_\theta + \rho_\theta L_{\mathrm{S}\,\theta,i}^\dagger), \tag{2.24}$$

$$\Longleftrightarrow \partial_i \rho_\theta = \frac{1}{2}(\rho_\theta L_{\mathrm{S}\,\theta,i}^\dagger + L_{\mathrm{S}\,\theta,i}^\dagger \rho_\theta). \tag{2.25}$$

We use $\rho_\theta^\dagger = \rho_\theta$. By comparing Eqs. (2.22, 2.25), we have

$$L_{\mathrm{S}\,\theta,i} = L_{\mathrm{S}\,\theta,i}^\dagger. \tag{2.26}$$

This is because the SLD is unique when ρ_θ is a full rank. Therefore, the SLD is Hermitian. □

As for the relation among the SLD, RLD, and λLD, the λLD includes the SLD and the RLD as the special cases of $\lambda = 0$ and 1, respectively.

SLD, RLD, and λLD inner products

Let us define three kinds of inner products, the SLD, RLD, and λLD inner products.

$$\langle X, Y\rangle_{\rho_\theta}^{\mathrm{S}} = \frac{1}{2}\mathrm{tr}[\rho_\theta(YX^\dagger + X^\dagger Y)], \tag{2.27}$$

$$\langle X, Y\rangle_{\rho_\theta}^{\mathrm{R}} = \mathrm{tr}\rho_\theta(YX^\dagger), \tag{2.28}$$

$$\langle X, Y\rangle_{\rho_\theta}^{\lambda} = \frac{1+\lambda}{2}\mathrm{tr}\rho_\theta(YX^\dagger) + \frac{1-\lambda}{2}\mathrm{tr}\rho_\theta(X^\dagger Y), \tag{2.29}$$

where $X,\ Y$ are any linear operators on $\mathcal{H}$.

SLD, RLD, and λLD Fisher information matrices

Let us denote the SLD, RLD, and λLD Fisher information matrices by $J_{\mathrm{S}\,\theta}$, $J_{\mathrm{R}\,\theta,}$, and $J_{\lambda\,\theta,}$, respectively. Then, the SLD, RLD, and λLD Fisher information matrices are defined by

$$[J_{\mathrm{S}\,\theta}]_{ij} = \langle L_{\mathrm{S}\,\theta,i}, L_{\mathrm{S}\,\theta,j}\rangle_{\rho_\theta}^{\mathrm{S}} = \frac{1}{2}\mathrm{tr}[\rho_\theta(L_{\mathrm{S}\,\theta,i}L_{\mathrm{S}\,\theta,j} + L_{\mathrm{S}\,\theta,j}L_{\mathrm{S}\,\theta,i})], \tag{2.30}$$

$$[J_{\mathrm{R}\,\theta}]_{ij} = \langle L_{\mathrm{R}\,\theta,i}, L_{\mathrm{R}\,\theta,j}\rangle_{\rho_\theta}^{\mathrm{R}} = \mathrm{tr}(\rho_\theta L_{\mathrm{R}\,\theta,j}L_{\mathrm{R}\,\theta,i}^\dagger), \tag{2.31}$$

$$[J_{\lambda\,\theta}]_{ij} = \langle L_{\lambda\,\theta,i}, L_{\lambda\,\theta,j}\rangle_{\rho_\theta}^{\lambda} = \frac{1+\lambda}{2}\mathrm{tr}(\rho_\theta L_{\lambda\,\theta,j}L_{\lambda\,\theta,i}^\dagger) + \frac{1-\lambda}{2}\mathrm{tr}(\rho_\theta L_{\lambda\,\theta,i}^\dagger L_{\lambda\,\theta,j}), \tag{2.32}$$

respectively. In Eq. (2.30), we use $L_{\mathrm{S}\,\theta,i} = L_{\mathrm{S}\,\theta,i}^\dagger$.

Theorem 2.3.2 (SLD Fisher information matrix: Real matrix)
The SLD Fisher information matrix is a real matrix.

Proof

The SLD Fisher information matrix J_S is defined by Eq. (2.30), i.e.,

$$[J_{S\,\theta}]_{ij} = \langle L_{S\,\theta,i}, L_{S\,\theta,j}\rangle^S_{\rho_\theta} = \frac{1}{2}\mathrm{tr}[\rho_\theta(L_{S\,\theta,i}L_{S\,\theta,j} + L_{S\,\theta,j}L_{S\,\theta,i})]. \tag{2.33}$$

The first term is a complex conjugate of the second term on the right hand side.

$$\because\ [\mathrm{tr}(\rho_\theta L_{S\,\theta,i}L_{S\,\theta,j})]^* = \mathrm{tr}[(\rho_\theta L_{S\,\theta,i}L_{S\,\theta,j})^\dagger] = \mathrm{tr}(L^\dagger_{S\,\theta,j}L^\dagger_{S\,\theta,i}\,\rho^\dagger_\theta). \tag{2.34}$$

Since the SLD is a Hermitian matrix, we obtain

$$[\mathrm{tr}(\rho_\theta L_{S\,\theta,i}L_{S\,\theta,j})]^* = \mathrm{tr}(\rho_\theta L_{S\,\theta,j}L_{S\,\theta,i}) \tag{2.35}$$

Therefore, $[J_{S\,\theta}]_{ij}$ is real. □

2.3.3 Quantum Cramér-Rao inequality

In the following, we present a theorem about λLD Cramér-Rao inequality.

Theorem 2.3.3 (Theorem: λLD Cramér-Rao inequality)
For any locally unbiased estimator $(\Pi, \hat{\theta})$ at θ, its MSE matrix satisfies

$$\mathsf{V}_\theta[\Pi, \hat{\theta}] \geq J^{-1}_{\lambda\,\theta}, \tag{2.36}$$

where $J_{\lambda\,\theta}$ is the λLD Fisher information matrix.

A proof of this theorem is given in Appendix A.
As noted above, the SLD and RLD Fisher information matrices, $J_{S\,\theta}$ and $J_{R\,\theta}$ corresponds to the cases when $\lambda = 0$ and 1, respectively.

2.3.4 Quantum Cramér-Rao type inequality

In a quantum parameter estimation problem, it is desirable to find the precision bound called the most informative bound defined by

$$C^{\mathrm{MI}}_\theta[W] = \min_{\Pi,\hat{\theta}:\,\mathrm{l.u.\ at\ }\theta} \mathrm{tr}\{W\mathsf{V}[\Pi, \hat{\theta}]\}. \tag{2.37}$$

The abbreviation "l.u." stands for locally unbiased. It is not always possible to find an explicit expression of C^{MI}_θ. In general, we cannot minimize the MSE matrix directly, one possible way is to minimize the weighted trace of the MSE matrix. The inverse of the quantum Fisher information matrix gives a lower bound for the weighted trace of the MSE matrix. The next theorem states this as follows.

Theorem 2.3.4 (Scalar quantum Cramér-Rao bound)
Given a weight matrix $W > 0$, the weighted trace of the MSE matrix for any locally unbiased estimator at θ obeys the scalar inequality

$$\mathrm{tr}(W\mathsf{V}_\theta[\hat{\Pi}, \theta]) \geq C_\theta^\lambda[W], \tag{2.38}$$

where

$$C_\theta^\lambda[W] = \mathrm{tr}\{W\mathrm{Re}(J_{\lambda\,\theta}^{-1})\} + \mathrm{tr}|\sqrt{W}\mathrm{Im}(J_{\lambda\,\theta}^{-1})\sqrt{W}|, \tag{2.39}$$

$$\mathrm{Re}(X) = \frac{1}{2}(X + X^*), \tag{2.40}$$

$$\mathrm{Im}(X) = \frac{1}{2\mathrm{i}}(X - X^*). \tag{2.41}$$

$\mathrm{tr}|X|$ *denotes the trace of absolute values of eigenvalues of the matrix X. X^* has a complex conjugate of all components of X as its components, i.e.,* $[X^*]_{ij} = ([X]_{ij})^*$.

In general a bound for the weighted trace of the MSE matrix is called the CR type bound. We use the formulation in Eq. (2.38) for the discussion in the following chapter. Here again, we remark that the SLD and RLD CR type bounds, $C_\theta^{\mathrm{S}}[W]$ and $C_\theta^{\mathrm{R}}[W]$ corresponds to $C_\theta^\lambda[W]$ when $\lambda = 0$ and 1, respectively.

Chapter 3

Trade-off formulation

In this chapter, we derive the error trade-off relation between the diagonal components of the MSE matrix for the two-parameter model. We use a bound obtained by the weighted trace of the MSE matrix which we refer to Cramér-Rao (CR) type bound. We show that we can use the diagonal weight matrix to calculate the bound for the two-parameter model when we are interested in the trade-off relation between the diagonal components of the MSE matrix. By using the diagonal weight matrix as in [8], we derive a CR inequality for a two-parameter model in a general form than the one we gave in [44].

We also propose using two quantum CR bounds to narrow down the lower bound region when they have intersections. We demonstrate the use of the symmetric logarithmic derivative (SLD) and the right logarithmic derivative (RLD) CR bounds. It is, of course, applicable to the SLD and λ logarithmic derivative (λLD) CR bounds. We give the conditions for the intersections to exist explicitly.

3.1 MSE region and locally unbiasedness

We define $\mathcal{V}_{\text{l.u.}}$, the set of positive matrix of any MSE matrices $\mathsf{V}_\theta[\Pi, \hat{\theta}]$ as

$$\mathcal{V}_{\text{l.u.}} = \{V \in \mathbb{R}^{n \times n} | V = \mathsf{V}_\theta[\Pi, \hat{\theta}];\ \Pi,\ \hat{\theta} \text{ are locally unbiased at } \theta\}. \tag{3.1}$$

$\mathcal{V}_{\text{l.u.}}$ depends on θ in general. However, we only discuss the cases of a fixed θ. Thus, θ-dependence is omitted. We can always find the estimater $\hat{\theta}$ for Π such that and $(\Pi, \hat{\theta})$ is locally unbiased at θ.

Theorem 3.1.1 ($\mathcal{V}_{\text{l.u.}}$ and classical Fisher information matrix)
$\mathcal{V}_{\text{l.u.}}$ is also expressed as

$$\mathcal{V}_{\text{l.u.}} = \{\mathsf{V} \in \mathbb{R}^{n \times n} | \mathsf{V} = (J_\theta[\Pi])^{-1},\ \Pi \text{ is POVM.}\} \tag{3.2}$$

where $J_\theta[\Pi]$ is a (classical) Fisher information matrix. It is explicitly written as

$$J_{\theta, ij}[\Pi] = \sum_{x \in \mathcal{X}} p_\theta(x|\Pi) \partial_i l_\theta(x|\Pi) \partial_j l_\theta(x|\Pi). \tag{3.3}$$

With Eq. (2.11)*, i.e., $p_\theta(x|\Pi) = \mathrm{tr}(\rho_\theta \Pi_x)$, $l_\theta(x|\Pi)$ is defined by $l_\theta(x|\Pi) = \log p_\theta(x|\Pi)$.*

Proof

We define $\hat{\theta}_{\mathrm{l.u.}\,i}(x)$ by

$$\hat{\theta}_{\mathrm{l.u.}\,i}(x) = \theta_i + \sum_{j=1}^{n} J^{-1}_{\theta,\,ji}[\Pi]\partial_j \log p_\theta(x|\Pi). \tag{3.4}$$

Then, we first show the $(\Pi, \hat{\theta}_{\mathrm{l.u.}\,i}(x))$ is locally unbiased at θ. We show $E_\theta[\hat{\theta}_{\mathrm{l.u.}\,i}(x) - \theta_i] = 0$ as follows.

$$E_\theta[\hat{\theta}_{\mathrm{l.u.}\,i}(x) - \theta_i] = \sum_{x\in\mathcal{X}}(\mathrm{tr}(\rho_\theta\Pi)[\theta_i + \sum_{j=1}^{n} J^{-1}_{\theta,\,ji}[\Pi]\partial_j \log p_\theta(x|\Pi) - \theta_i], \tag{3.5}$$

$$\Longleftrightarrow E_\theta[\hat{\theta}_{\mathrm{l.u.}\,i}(x) - \theta_i] = \sum_{x\in\mathcal{X}} p_\theta(x|\Pi)\sum_{j=1}^{n} J^{-1}_{\theta,\,ji}[\Pi]\frac{\partial_j p_\theta(x|\Pi)}{p_\theta(x|\Pi)}, \tag{3.6}$$

$$\Longleftrightarrow E_\theta[\hat{\theta}_{\mathrm{l.u.}\,i}(x) - \theta_i] = \sum_{x\in\mathcal{X}}\sum_{j=1}^{n} J^{-1}_{\theta,\,ji}[\Pi]\partial_j p_\theta(x|\Pi), \tag{3.7}$$

$$\Longleftrightarrow E_\theta[\hat{\theta}_{\mathrm{l.u.}\,i}(x) - \theta_i] = \sum_{j=1}^{n} J^{-1}_{\theta,\,ji}[\Pi]\partial_j \sum_{x\in\mathcal{X}} p_\theta(x|\Pi), \tag{3.8}$$

$$\Longleftrightarrow E_\theta[\hat{\theta}_{\mathrm{l.u.}\,i}(x) - \theta_i] = 0 \tag{3.9}$$

We use $\sum_{x\in\mathcal{X}} p_\theta(x|\Pi) = 1$. From Eq. 3.9, we have

$$E_\theta[\hat{\theta}_{\mathrm{l.u.}\,i}(x)] = \theta_i, \tag{3.10}$$

and

$$\partial_j E_\theta[\hat{\theta}_{\mathrm{l.u.}\,i}(x)] = \delta_{ij}. \tag{3.11}$$

Therefore, $(\Pi, \hat{\theta}_{\mathrm{l.u.}\,i}(x))$ is locally unbiased at θ. The MSE matrix $\mathsf{V}_\theta[\Pi,\hat{\theta}] = \left[\mathsf{V}_{\theta,\,ij}[\Pi,\hat{\theta}]\right]$ is defined by

$$\mathsf{V}_{\theta,\,ij}[\Pi,\hat{\theta}] = \sum_{x\in\mathcal{X}} \mathrm{tr}(\rho_\theta\Pi_x)(\hat{\theta}_i(x) - \theta_i)(\hat{\theta}_j(x) - \theta_j). \tag{3.12}$$

We use $(\Pi, \hat{\theta}_{\mathrm{l.u.}\,i}(x))$ to calculate the MSE matrix $\mathsf{V}_{\theta,\,ij}[\Pi,\hat{\theta}]$.

$$\mathsf{V}_{\theta,\,ij}[\Pi,\hat{\theta}] = \sum_{x\in\mathcal{X}} p_\theta(x|\Pi)\sum_{k=1}^{n} J^{-1}_{\theta,\,ki}[\Pi]\partial_k \log p_\theta(x|\Pi)\sum_{l=1}^{n} J^{-1}_{\theta,\,lj}[\Pi]\partial_l \log p_\theta(x|\Pi), \tag{3.13}$$

$$\Longleftrightarrow \mathsf{V}_{\theta,\,ij}[\Pi,\hat{\theta}] = \sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{x\in\mathcal{X}} p_\theta(x|\Pi)\partial_k \log p_\theta(x|\Pi)\partial_l \log p_\theta(x|\Pi) J^{-1}_{\theta,\,ki}[\Pi]J^{-1}_{\theta,\,lj}[\Pi] \tag{3.14}$$

$$\Longleftrightarrow \mathsf{V}_{\theta,\,ij}[\Pi,\hat{\theta}] = \sum_{k=1}^{n}\sum_{l=1}^{n} J_{\theta,\,kl}[\Pi]J^{-1}_{\theta,\,lj}[\Pi]J^{-1}_{\theta,\,ki}[\Pi] \tag{3.15}$$

$$\Longleftrightarrow \mathsf{V}_{\theta,\,ij}[\Pi,\hat{\theta}] = \sum_{k=1}^{n} \delta_{kj}J^{-1}_{\theta,\,ki}[\Pi] \tag{3.16}$$

$$\Longleftrightarrow \mathsf{V}_{\theta,\,ij}[\Pi,\hat{\theta}] = J^{-1}_{\theta,\,ji}[\Pi] \tag{3.17}$$

$$\tag{3.18}$$

From $\mathsf{V}_{\theta,ij}[\Pi,\hat{\theta}] = \mathsf{V}_{\theta,ji}[\Pi,\hat{\theta}]$, we have

$$\mathsf{V}_{\theta,ij}[\Pi,\hat{\theta}] = J^{-1}_{\theta,ij}[\Pi]. \quad \square \tag{3.19}$$

3.2 Intersection of two bounds

Let us consider two different CR type bounds, $C^1_\theta[W]$ and $C^2_\theta[W]$ which are any two CR type bounds. Let $\mathcal{V}^i$ be

$$\mathcal{V}^i = \{\mathsf{V} \in \mathbb{R}^{n\times n} | \mathsf{V} \geq 0,\ {}^\forall W \in \mathcal{W},\ \mathrm{tr}[W\mathsf{V}] \geq C^i_\theta[W]\} \quad (i = 1,\ 2), \tag{3.20}$$

where

$$\mathcal{W} = \{W \in \mathbb{R}^{n\times n} | W > 0\}. \tag{3.21}$$

W is a weight matrix which is positive definite, $W > 0$. By its definition of $\mathcal{V}_{\text{l.u.}}$, we have a relation,

$$\mathcal{V}_{\text{l.u.}} \subset (\mathcal{V}^1 \cap \mathcal{V}^2). \tag{3.22}$$

Therefore, if $\mathcal{V}^1$ is not a subset of $\mathcal{V}^2$ or if $\mathcal{V}^2$ is not a subset of $\mathcal{V}^1$, i.e., if the intersection of the sets $\mathcal{V}^1$ and $\mathcal{V}^2$ satisfy $\mathcal{V}^1 \cap \mathcal{V}^2 \neq \mathcal{V}^i \quad (i = 1, 2)$, we can obtain a more precise bound than the one given by each bound separately. This situation corresponds to the case when there exist two bounds that have intersections. We revisit this concept in Section 3.6.

Let us consider the case of $C^1_\theta[W]$ and $C^2_\theta[W]$ being the SLD and RLD CR bounds, $C^{\mathrm{S}}_\theta[W]$ and $C^{\mathrm{R}}_\theta[W]$, respectively. From Eq. (3.22),

$$\mathcal{V}_{\text{l.u.}} \subset (\mathcal{V}^{\mathrm{S}} \cap \mathcal{V}^{\mathrm{R}}), \tag{3.23}$$

where

$$\mathcal{V}^{\mathrm{Q}} = \{\mathsf{V} \in \mathbb{R}^{n\times n} | \mathsf{V} \geq 0, {}^\forall W \in \mathcal{W}, \mathrm{tr}[W\mathsf{V}] \geq C^{\mathrm{Q}}_\theta[W]\} \quad (\mathrm{Q} = \mathrm{S},\ \mathrm{R}, \lambda). \tag{3.24}$$

$\mathcal{V}^{\mathrm{S}}$ and $\mathcal{V}^{\mathrm{R}}$ are the sets of MSE matrices defined by the SLD and RLD CR bounds, respectively. From (3.23), by combining two error trade-off relations given by the SLD and RLD CR bounds, we can determine the shape of an error trade-off relation more accurately when the SLD and RLD CR bounds have intersections. They may have intersections, because there is no ordering between J_{R}^{-1} and J_{S}^{-1} in general as the general relationship between the SLD and RLD Fisher information matrices [46]

$$J_{\mathrm{S}}^{-1} \geq \mathrm{Re}\,(J_{\mathrm{R}}^{-1}), \tag{3.25}$$

indicates. The inverse of the RLD Fisher information matrix J_{R}^{-1} may have complex components. Note, however, that this inequality alone does not give a conclusive argument whether an error trade-off relation exists or not. This is because the quantum CR inequality is not tight unless certain special conditions are satisfied. For example, if the model is a gaussian shift model, it is known that the RLD CR bound is achievable and dominant over the SLD CR bound.

3.3 Cramér-Rao (CR) type bound: two-parameter model

In this section, we focus on the two-parameter model. Therefore, $V, W \in \mathbb{R}^{2\times 2}$.

From Eq. (2.39), the quantum CR type bound $C^{\mathrm{Q}}_{\theta}[W]$ is expressed by

$$C^{\mathrm{Q}}_{\theta}[W] = \mathrm{tr}\{W\mathrm{Re}(J_{\mathrm{Q}}{}_{\theta}^{-1})\} + \mathrm{tr}|\sqrt{W}\mathrm{Im}(J_{\mathrm{Q}}{}_{\theta}^{-1})\sqrt{W}|. \tag{3.26}$$

where $\mathrm{tr}|X|$ denotes the trace of absolute values of eigenvalues of the matrix X.

We will consider the following form of a CR type bound. Let us introduce two matrices, A and B here. The matrix A is positive definite, i.e., $A > 0$. The matrix B, is a skew symmetric matrix, i.e., $B^{\mathrm{T}} = -B$. $A,\ B \in \mathbb{R}^{2\times 2}$.

$$C^{\mathrm{Q}}_{\theta}[W] = \mathrm{tr}(WA) + \mathrm{tr}|\sqrt{W}B\sqrt{W}|. \tag{3.27}$$

To see the correspondence to Eq. (3.26) , we set

$$A = \mathrm{Re}(J_{\mathrm{Q}}{}_{\theta}^{-1}), \tag{3.28}$$

$$B = \mathrm{Im}(J_{\mathrm{Q}}{}_{\theta}^{-1}). \tag{3.29}$$

Their examples are shown in sections 3.5.1 and 3.5.2.

As given in Appendix B.1, the eigenvalues of $\sqrt{W}B\sqrt{W}$, $\lambda_{\pm}$ are expressed as

$$\lambda_{\pm} = \pm \mathrm{i}\sqrt{\det W \det B}. \tag{3.30}$$

Thus, we have

$$C^{\mathrm{Q}}_{\theta}[W] = \mathrm{tr}(WA) + 2\sqrt{\det W \det B}. \tag{3.31}$$

Therefore, the quantum CR type inequality

$$\mathrm{tr}(W\mathsf{V}) \geq C^{\mathrm{Q}}_{\theta}[W] \tag{3.32}$$

is expressed as

$$\mathrm{tr}[W(\mathsf{V} - A)] \geq 2\sqrt{\det W \det B}. \tag{3.33}$$

As given in Appendix B.1, we have an inequality that holds for any weight matrices, $W \in \mathcal{W}$,

$$\det(\mathsf{V} - A) \geq \det B, \tag{3.34}$$

and for any diagonal weight matrices, $W \in \tilde{\mathcal{W}}$,

$$(\mathsf{V}_{11} - A_{11})(\mathsf{V}_{22} - A_{22}) \geq \det B. \tag{3.35}$$

A derivation is given in Appendix B.1.

From (3.34) and (3.35), equivalent expressions for $\mathcal{V}^{\mathrm{CR}}$ and $\tilde{\mathcal{V}}^{\mathrm{CR}}$ are written as

$$\mathcal{V}^{\mathrm{CR}} = \{\mathsf{V} \in \mathbb{R}^{2\times 2} | {}^{\forall} i, \mathsf{V}_{ii} - A_{ii} > 0, \det[\mathsf{V} - A] \geq \det B\}, \tag{3.36}$$

$$\tilde{\mathcal{V}}^{\mathrm{CR}} = \{\mathsf{V} \in \mathbb{R}^{2\times 2} | {}^{\forall} i, \mathsf{V}_{ii} - A_{ii} > 0, (\mathsf{V}_{11} - A_{11})(\mathsf{V}_{22} - A_{22}) \geq \det B\}. \tag{3.37}$$

where $A = [A_{ij}]$.

3.4 Trade-off relation between diagonal components of MSE matrices: two-parameter model

In the following, if a trade-off relation between the diagonal components of the MSE matrix exist, we consider the trade-off relation exists. For two-parameter models, as far as the investigation of the relation between the diagonal components of the MSE matrix is concerned, we show that we can use the diagonal weight matrices and arbitrary weight matrices.

3.4.1 Choice of MSE region

For any given weight matrix $W > 0$, a quantum CR type bound $C^{\mathrm{Q}}_{\theta}[W]$ satisfies the relation Eq. (2.38)

$$\mathrm{tr}(W\mathsf{V}_{\theta}[\theta, \hat{\Pi}]) \geq C^{\mathrm{Q}}_{\theta}[W]. \tag{3.38}$$

We define two sets of weight matrices $\mathcal{W}$ and $\tilde{\mathcal{W}}$, the sets of arbitrary weight matrices and those of diagonal weight matrices, respectively. They are defined by

$$\mathcal{W} := \{W \in \mathbb{R}^{n\times n} | W > 0\}, \tag{3.39}$$

$$\tilde{\mathcal{W}} := \{W \in \mathbb{R}^{n\times n} | W = \mathrm{diag}(w_1, w_2, \cdots, w_n) > 0\}. \tag{3.40}$$

$W = \mathrm{diag}(w_1, w_2, \cdots, w_n)$ constitutes a diagonal weight matrix W with its diagonal components $[W]_{ii} = w_i$. We introduce a diagonal weight matrix because

$$\mathrm{tr}(W\mathsf{V}_{\theta}[\Pi, \hat{\theta}]) = \sum_{j=1}^{n} w_j \mathsf{V}_{\theta}[\Pi, \hat{\theta}]_{jj} \tag{3.41}$$

holds if $W \in \tilde{W}$. With this relation, we can determine relation between the diagonal components of the MSE matrix.

Let us assume that a trade-off relation is defined by a trade-off relation between the diagonal components of the MSE matrix. To investigate the trade-off relation between the diagonal components of the MSE matrix, we can use the elements of the $\tilde{\mathcal{W}}$ as weight matrices. With these sets of the weight matrices $\mathcal{W}$ and $\tilde{\mathcal{W}}$, we can also define the set of all possible MSE matrices $\mathcal{V}^{\mathrm{CR}}$ and $\tilde{\mathcal{V}}^{\mathrm{CR}}$.

$$\mathcal{V}^{\mathrm{CR}} = \{\mathsf{V} \in \mathbb{R}^{n\times n} | \mathsf{V} \geq 0, {}^{\forall} W \in \mathcal{W}, \mathrm{tr}(W\mathsf{V}_{\theta}[\Pi, \hat{\theta}]) \geq C^{\mathrm{Q}}_{\theta}[W]\}, \tag{3.42}$$

$$\tilde{\mathcal{V}}^{\mathrm{CR}} = \{\mathsf{V} \in \mathbb{R}^{n\times n} | \mathsf{V} \geq 0, {}^{\forall} W \in \tilde{\mathcal{W}}, \mathrm{tr}(W\mathsf{V}_{\theta}[\Pi, \hat{\theta}]) \geq C^{\mathrm{Q}}_{\theta}[W]\}. \tag{3.43}$$

The sets $\mathcal{V}^{\mathrm{CR}}$ and $\tilde{\mathcal{V}}^{\mathrm{CR}}$ are the sets of all possible MSE matrices derived by using the arbitrary weight matrices and the arbitrary diagonal weight matrices, respectively. Therefore, the relation between is expressed by

$$\mathcal{V}^{\mathrm{CR}} \subset \tilde{\mathcal{V}}^{\mathrm{CR}}. \tag{3.44}$$

3.4.2 Sets of diagonal components of MSE matrices by arbitrary weight matrices and by diagonal matrices

Hereafter, we focus on the 2-parameter model only. We define the two sets of diagonal components of the MSE matrices. One is the diagonal components of the MSE matrix derived by the arbitrary weight matrix, and the other by using the diagonal weight matrix. Then, we show those two sets are identical.

Let us define the following two sets $\mathcal{D}^{\mathrm{CR}}$ and $\tilde{\mathcal{D}}^{\mathrm{CR}}$ as follows.

$$\mathcal{D}^{\mathrm{CR}} = \{\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} | {}^{\exists} v \in \mathbb{R},\ V = \begin{pmatrix} v_1 & v \\ v & v_2 \end{pmatrix} \in \mathcal{V}^{\mathrm{CR}}\}, \tag{3.45}$$

$$\tilde{\mathcal{D}}^{\mathrm{CR}} = \{\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} | {}^{\exists} u \in \mathbb{R},\ U = \begin{pmatrix} u_1 & u \\ u & u_2 \end{pmatrix} \in \tilde{\mathcal{V}}^{\mathrm{CR}}\}, \tag{3.46}$$

By using alternative expressions, Eqs. (3.36, 3.37), $\mathcal{D}^{\mathrm{CR}}$ and $\tilde{\mathcal{D}}^{\mathrm{CR}}$ can also written as

$$\mathcal{D}^{\mathrm{CR}} = \{\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} | {}^{\forall} i, v_i - A_{ii} > 0\ \wedge^{\exists} v,\ (v_1 - A_{11})(v_2 - A_{22}) - (v - A_{12})^2 \geq \det B\}, \tag{3.47}$$

$$\tilde{\mathcal{D}}^{\mathrm{CR}} = \{\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} | {}^{\forall} i, u_i - A_{ii} > 0 \wedge (u_1 - A_{11})(u_2 - A_{22}) \geq \det B\}. \tag{3.48}$$

$\mathcal{D}^{\mathrm{CR}}$ and $\tilde{\mathcal{D}}^{\mathrm{CR}}$ are the sets of diagonal components of the MSE matrices obtained with using arbitrary weight matrices and diagonal weight matrices, respectively.

Theorem 3.4.1 ($\mathcal{D}^{\mathrm{CR}} = \tilde{\mathcal{D}}^{\mathrm{CR}}$)
$\mathcal{D}^{\mathrm{CR}}$ and $\tilde{\mathcal{D}}^{\mathrm{CR}}$ are identical.

$$\tilde{\mathcal{D}}^{\mathrm{CR}} = \mathcal{D}^{\mathrm{CR}}. \tag{3.49}$$

Proof

Using Eq. (3.47), we see a following equivalent expressions of $\mathcal{D}^{\mathrm{CR}}$.

$$\mathcal{D}^{\mathrm{CR}} = \{\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} | {}^{\forall} i, v_i - A_{ii} > 0\ \wedge^{\exists} v,\ (v_1 - A_{11})(v_2 - A_{22}) - (v - A_{12})^2 \geq \det B\} \tag{3.50}$$

$$\Longleftrightarrow \mathcal{D}^{\mathrm{CR}} = \{\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} | {}^{\forall} i, v_i - A_{\,ii} > 0\ \wedge (v_1 - A_{11})(v_2 - A_{22}) \geq \det B\}. \tag{3.51}$$

From Eq. (3.48), we have

$$\tilde{\mathcal{D}}^{\mathrm{CR}} = \{\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} | {}^{\forall} i, u_i - A_{ii} > 0 \wedge (u_1 - A_{11})(u_2 - A_{22}) \geq \det B\}. \tag{3.52}$$

From (3.51) and (3.48), we obtain

$$\tilde{\mathcal{D}}^{\mathrm{CR}} = \mathcal{D}^{\mathrm{CR}}. \tag{3.53}$$

Although there is a relation $\mathcal{V}^{\mathrm{CR}} \subset \tilde{\mathcal{V}^{\mathrm{CR}}}$, $\mathcal{D}^{\mathrm{CR}}$ and $\tilde{\mathcal{D}}^{\mathrm{CR}}$ are turned out to be identical. To obtain the trade-off relation between the diagonal components of MSE matrix, we can use (3.35), i.e.,

$$(\mathsf{V}_{11} - A_{11})(\mathsf{V}_{22} - A_{22}) \geq \det B, \tag{3.54}$$

because $\mathcal{D}^{\mathrm{CR}}$ and $\tilde{\mathcal{D}}^{\mathrm{CR}}$ are identical. □

3.5 Examples: SLD CR bound, RLD CR bound, λLD CR bound, and Nagaoka bound for two-parameter model

Using the result in the previous section, we derive the inequality for the diagonal components which results from SLD CR bound, RLD CR bound, and Nagaoka bound for a qubit system.

3.5.1 SLD CR bound

Since the SLD Fisher infromation matrix $J_{\mathrm{S}\,\theta}^{-1}$ is a real matrix, Eq. (3.26) gives the SLD CR bound $C_\theta^{\mathrm{S}}[W]$ as follows.

$$C_\theta^{\mathrm{S}}[W] = \mathrm{tr}(W J_{\mathrm{S}\,\theta}^{-1}). \tag{3.55}$$

Therefore, $A = J_{\mathrm{S}\,\theta}^{-1}$ and $B = 0$ in (3.26). From (3.35), we obtain

$$(\mathsf{V}_{11} - J_{\mathrm{S}\,11}^{-1})(\mathsf{V}_{22} - J_{\mathrm{S}\,22}^{-1}) \geq 0. \tag{3.56}$$

where $J_{\mathrm{S}}^{-1} = [J_{\mathrm{S}\,ij}^{-1}]$. Then, we obtain

$$\mathsf{V}_{11} - J_{\mathrm{S}\,11}^{-1} \geq 0, \quad \mathsf{V}_{22} - J_{\mathrm{S}\,22}^{-1} \geq 0. \tag{3.57}$$

3.5.2 RLD CR bound

From Eq. (3.26), the RLD CR bound $C_\theta^{\mathrm{R}}[W]$ is written as

$$C_\theta^{\mathrm{R}}[W] = \mathrm{tr}(W \mathrm{Re}[J_{\mathrm{R}\,\theta}^{-1}]) + \mathrm{tr}|\sqrt{W} \mathrm{Im}[J_{\mathrm{R}\,\theta}^{-1}]\sqrt{W}|. \tag{3.58}$$

Unlike the SLD CR type bound, the second term in the right hand side appears because the RLD Fisher information matrix $J_{\mathrm{R}\,\theta}$ is a complex matrix in general. Therefore, $A = \mathrm{Re}[J_{\mathrm{R}}^{-1}]$ and $B = \mathrm{Im}[J_{\mathrm{R}}^{-1}]$ in (3.26). From (3.35),

$$(\mathsf{V}_{11} - J_{\mathrm{R}\,11}^{-1})(\mathsf{V}_{22} - J_{\mathrm{R}\,22}^{-1}) \geq |\mathrm{Im}(J_{\mathrm{R}\,12}^{-1})|^2, \tag{3.59}$$

where $J_{\mathrm{R}\,\theta}^{-1} = [J_{\mathrm{R}\,ij}^{-1}]$. Here, we use $\det(\mathrm{Im}[J_{\mathrm{R}}^{-1}]) = |\mathrm{Im}(J_{\mathrm{R}\,12}^{-1})|^2$.

3.5.3 λLD CR bound

In the same way as we obtain the RLD CR bound, we have

$$(\mathsf{V}_{11} - J_{\lambda}{}^{-1}_{11})(\mathsf{V}_{22} - J_{\lambda}{}^{-1}_{22}) \geq |\mathrm{Im}(J_{\lambda}{}^{-1}_{12})|^2. \tag{3.60}$$

3.5.4 Nagaoka bound: qubit

For a qubit system, Nagaoka bound $C_\theta^{\mathrm{N}}[W]$ [45] is written as

$$C_\theta^{\mathrm{N}}[W] = \mathrm{tr}(WJ_{\mathrm{S}}^{-1}) + 2\sqrt{\det W \det J_{\mathrm{S}}^{-1}}. \tag{3.61}$$

By setting

$$B = \sqrt{\det J_{\mathrm{S}}^{-1}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.62}$$

we have $B^{\mathrm{T}} = -B$ and $\det B = \det J_{\mathrm{S}}^{-1}$. Then, we can write $C_\theta^{\mathrm{N}}[W]$ as

$$C_\theta^{\mathrm{N}}[W] = \mathrm{tr}(WA) + 2\sqrt{\det W \det B}, \tag{3.63}$$

where $A = \mathrm{Re}(J_{\mathrm{S}}^{-1}) = J_{\mathrm{S}}^{-1}$. Therefore, we have

$$\mathrm{tr}[W(\mathsf{V} - A)] \geq 2\sqrt{\det W \det B}. \tag{3.64}$$

Eq. (3.64) has the same form as Eq. (3.33). From Eq. (3.35), we obtain

$$(\mathsf{V}_{11} - J_{\mathrm{S}}{}^{-1}_{11})(\mathsf{V}_{22} - J_{\mathrm{S}}{}^{-1}_{22}) \geq \det J_{\mathrm{S}}^{-1} \tag{3.65}$$

3.6 Error trade off relation by the SLD and λLD CR bounds: two-parameter model

We now move to our central idea of this chapter. From Eq.(3.23), we can obtain a more precise bound by combining the SLD and λLD CR bounds when they have intersections. When we use the SLD and λLD CR bounds as quantum CR bounds, we denote the sets of the diagonal components as $\tilde{\mathcal{D}}^{\mathrm{S}}$ and $\tilde{\mathcal{D}}^{\lambda}$, respectively. From Eq. (3.57), $\tilde{\mathcal{D}}^{\mathrm{S}}$ is expressed as

$$\tilde{\mathcal{D}}^{\mathrm{S}} = \{\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} | {}^\forall i, v_i - J_{\mathrm{S}\theta}{}^{-1}_{ii} > 0\} \tag{3.66}$$

$\tilde{\mathcal{D}}^{\lambda}$ is expressed as

$$\tilde{\mathcal{D}}^{\lambda} = \{\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} | {}^\forall i, v_i - J_{\lambda}{}^{-1}_{11} > 0 \ \wedge (v_1 - J_{\lambda}{}^{-1}_{11})(v_2 - J_{\lambda}{}^{-1}_{22}) \geq |\mathrm{Im}(J_{\lambda}{}^{-1}_{12})|^2\}. \tag{3.67}$$

If $\tilde{\mathcal{D}}^{\mathrm{S}} \cap \tilde{\mathcal{D}}^{\lambda} \neq \tilde{\mathcal{D}}^{\mathrm{S}}$ or $\tilde{\mathcal{D}}^{\mathrm{S}} \cap \tilde{\mathcal{D}}^{\lambda} \neq \tilde{\mathcal{D}}^{\lambda}$ holds, the MSE components $v_1 = V_{11}$ and $v_2 = V_{22}$ fall in the set $\tilde{\mathcal{D}}^{\mathrm{S}} \cap \tilde{\mathcal{D}}^{\lambda}$. We can make the CR bound narrow down by using two sets, $\tilde{\mathcal{D}}^{\mathrm{S}}$ and $\tilde{\mathcal{D}}^{\lambda}$.

Let us consider the case plotting V_{11} in the horizontal axis and V_{22} in the vertical axis. If $\det B = \det(\mathrm{Im} J_\lambda{}^{-1})$ is not zero,

$$(v_1 - J_{\lambda}{}^{-1}_{11})(v_2 - J_{\lambda}{}^{-1}_{22}) \geq |\mathrm{Im}(J_{\lambda}{}^{-1}_{12})|^2 \tag{3.68}$$

gives a hyperbola in the plot. If the hyperbola intersects with the lines $\mathsf{V}_{11} = J_{\mathrm{S}}{}^{-1}_{11}$ and $\mathsf{V}_{22} = J_{\mathrm{S}}{}^{-1}_{22}$, we call the situation as *intersections exists*. See Figure 3.1 for the occurrence of intersections of the two bounds: the SLD CR and λLD1 CR bounds. On the other hand, if $\tilde{\mathcal{D}}^{\mathrm{S}} \subset \tilde{\mathcal{D}}^{\lambda}$ holds, there is no intersections. In this case, we cannot conclude that the trade-off relation exists. See Fig. 3.1 for the two bounds that do not have intersection: the SLD and λLD2 CR bounds.

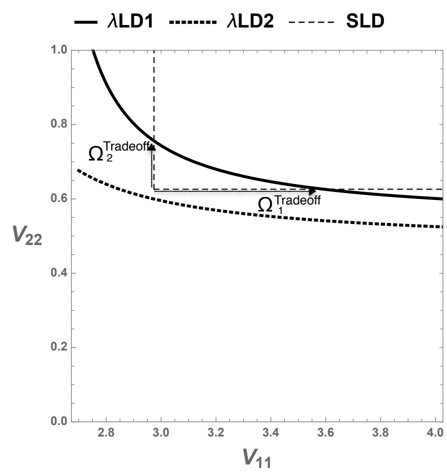

Figure 3.1: Two λLD bounds. The SLD bound is dominant over the λLD2 bound. No trade-off relation exist. The λLD1 bound goes above the SLD bound approximately in the range $3 < \mathsf{V}_{11} < 3.6$. The trade-off relation exists in the range.

3.7 Conditions for the intersection of the SLD and λLD CR bounds to exist

In the following, we discuss the conditions for the intersection of the SLD and λLD CR bounds to exist. We drop θ in the quantum Fisher information since we consider the cases of a fixed θ.

Since the SLD CR bound cannot give a trade-off relation, we need a trade-off relation given by the λLD. From the discussion in the previous subsections, when we are interested in the diagonal components of the MSE matrix only, the λLD CR inequality is expressed as Eq. (3.59),

$$(\mathsf{V}_{11} - J_{\lambda}{}^{-1}_{11})(\mathsf{V}_{22} - J_{\lambda}{}^{-1}_{22}) > \left|\mathrm{Im}\, J_{\lambda}{}^{-1}_{12}\right|^2. \tag{3.69}$$

In this case, the trade-off relation results from the λLD CR inequality exists if and only if

$$\left|\mathrm{Im}\, J_{\lambda}{}^{-1}_{12}\right|^2 \neq 0. \tag{3.70}$$

By defining $\delta := J_{\lambda\,12} - J_{\lambda\,21}$ and with $J_{\lambda\,12}$ being the complex conjugate of $J_{\lambda\,21}$, we have an equivalent condition.

Lemma 3.7.1 (Condition 1: Existence of trade-off relation that results from λLD)

$$\delta = J_{\lambda\,12} - J_{\lambda\,21} \neq 0, \tag{3.71}$$

where $J_\lambda = [J_{\lambda\,ij}]$.

As explained in Section 3.6, if $\tilde{\mathcal{D}}^{\mathrm{S}} \cap \tilde{\mathcal{D}}^{\lambda} \neq \tilde{\mathcal{D}}^{\mathrm{S}}$ or $\tilde{\mathcal{D}}^{\mathrm{S}} \cap \tilde{\mathcal{D}}^{\lambda} \neq \tilde{\mathcal{D}}^{\lambda}$ holds, the MSE components V_{11} and V_{22} fall in the set $\tilde{\mathcal{D}}^{\mathrm{S}} \cap \tilde{\mathcal{D}}^{\lambda}$. In that case, we narrow the region where V_{11} and V_{22} exist by combining the SLD and λLD CR inequality Eqs.(3.57, 3.59). We remark here that we exclude the case where $\tilde{\mathcal{D}}^{\mathrm{S}} \cap \tilde{\mathcal{D}}^{\lambda} \neq \tilde{\mathcal{D}}^{\lambda}$ because we can conclude the trade-off relation can be detected by the λLD CR bound only. Condition 2 below is about the relation explained above.

Lemma 3.7.2 (Condition 2: Existence of intersection between SLD and λLD CR bounds)

The intersection of the sets $\tilde{\mathcal{D}}^{\mathrm{S}}$ and $\tilde{\mathcal{D}}^{\lambda}$ are not equal to $\tilde{\mathcal{D}}^{\mathrm{S}}$ if and only if

$$\Delta := |\mathrm{Im} J^{-1}_{\lambda\,12}|^2 - (J^{-1}_{\mathrm{S}\,11} - J^{-1}_{\lambda\,11})(J^{-1}_{\mathrm{S}\,22} - J^{-1}_{\lambda\,22}) > 0. \tag{3.72}$$

Proof

We derive the condition for the trade-off relation determined by both λLD and the SLD CR bounds to exist. We remark that the λLD includes the λLD as a special case $\lambda = 1$. This proof includes the case of the RLD CR bound.

Let the MSE matrix V be

$$\mathsf{V} = \begin{pmatrix} \mathsf{V}_{11} & \mathsf{V}_{12} \\ \mathsf{V}_{21} & \mathsf{V}_{22} \end{pmatrix}. \tag{3.73}$$

Then, λLD Cramér-Rao (CR) bound is given by

$$(\mathsf{V}_{11} - J^{-1}_{\lambda\,11})(\mathsf{V}_{22} - J^{-1}_{\lambda\,22}) \geq |\mathrm{Im} J^{-1}_{\lambda\,12}|^2, \tag{3.74}$$

where $[J^{-1}_{\lambda}]_{jk} = J^{-1}_{\lambda\,jk}$.

For now, we are interested in the boundary of λLD bound. In the following discussion, we use the equation for the boundary,

$$(\mathsf{V}_{11} - J^{-1}_{\lambda\,11})(\mathsf{V}_{22} - J^{-1}_{\lambda\,22}) = |\mathrm{Im} J^{-1}_{\lambda\,12}|^2. \tag{3.75}$$

Figure 3.1 shows the cases in which the trade-off exists and does not. This is a conceptual diagram. The SLD CR bound is dominant over the λLD2 CR bound. No trade-off relation exists. The λLD1 CR bound goes above the SLD CR bound approximately in the range $3 < \mathsf{V}_{11} < 3.6$. The trade-off relation exists in the range. Therefore, as we can see in the figure, if the value of the V_{22} component of λLD bound is bigger than that of the SLD bound at $\mathsf{V}_{11} = J^{-1}_{\mathrm{S}\,11}$, the trade-off relation exists. By substituting $J^{-1}_{\mathrm{S}\,11}$ in V_{11} in Eq. (3.75), we obtain

$$\mathsf{V}_{22}|_{\mathsf{V}_{11}=J^{-1}_{\mathrm{S}\,11}} = \frac{|\mathrm{Im}J^{-1}_{\lambda\,12}|^2}{J^{-1}_{\mathrm{S}\,11} - J^{-1}_{\lambda\,11}} + J^{-1}_{\lambda\,22}. \tag{3.76}$$

From the condition for the trade-off relation to be established, $\mathsf{V}_{22}|_{\mathsf{V}_{11}=J^{-1}_{\mathrm{S}\,11}} - J^{-1}_{\mathrm{S}\,22} > 0$, we have

$$\mathsf{V}_{22}|_{\mathsf{V}_{11}=J^{-1}_{\mathrm{S}\,11}} - J^{-1}_{\mathrm{S}\,22} = \frac{|\mathrm{Im}J^{-1}_{\lambda\,12}|^2}{J^{-1}_{\mathrm{S}\,11} - J^{-1}_{\lambda\,11}} + J^{-1}_{\lambda\,22} - J^{-1}_{\mathrm{S}\,22} \tag{3.77}$$

$$= \frac{|\mathrm{Im}J^{-1}_{\lambda\,12}|^2 - (J^{-1}_{\mathrm{S}\,11} - J^{-1}_{\lambda\,11})(J^{-1}_{\mathrm{S}\,22} - J^{-1}_{\lambda\,22})}{J^{-1}_{\mathrm{S}\,11} - J^{-1}_{\lambda\,11}} > 0. \tag{3.78}$$

A relation

$$J^{-1}_{\mathrm{S}} \geq \mathrm{Re}J^{-1}_{\lambda} \tag{3.79}$$

holds in general. Therefore,

$$J^{-1}_{\mathrm{S}\,11} - J^{-1}_{\lambda\,11} \geq 0, \tag{3.80}$$

$$J^{-1}_{\mathrm{S}\,22} - J^{-1}_{\lambda\,22} \geq 0, \tag{3.81}$$

holds. Since $\tilde{\mathcal{D}}^{\mathrm{S}} \cap \tilde{\mathcal{D}}^{\lambda} = \tilde{\mathcal{D}}^{\lambda}$ holds when $J^{-1}_{\mathrm{S}} = \mathrm{Re}J^{-1}_{\lambda}$, We exclude the case where $J^{-1}_{\mathrm{S}} = \mathrm{Re}J^{-1}_{\lambda}$. Then, from Eq. (3.78), we have

$$\Delta = |\mathrm{Im}J^{-1}_{\lambda\,12}|^2 - (J^{-1}_{\mathrm{S}\,11} - J^{-1}_{\lambda\,11})(J^{-1}_{\mathrm{S}\,22} - J^{-1}_{\lambda\,22}) > 0. \quad \square \tag{3.82}$$

Here, we remark on the achievability of the SLD and λLD CR bound. We assume that neither of them is achievable. However, if $\Delta > 0$ holds, we regard that we detect the existence of a trade-off relation given by an achievable bound if it exists because the achievable bound is above the CR bound given by the SLD and λLD bounds in the $\mathsf{V}_{11} - \mathsf{V}_{22}$ chart such as Fig. 3.1.

3.8 Strength of trade-off relation $\Omega^{\mathrm{Tradeoff}}$

As indicators of the strength of the trade-off relation, we choose $\Omega^{\mathrm{Tradeoff}}_1$ and $\Omega^{\mathrm{Tradeoff}}_2$ shown in Fig. 3.1. The strength of trade-off relation $\Omega^{\mathrm{Tradeoff}}_1$ and $\Omega^{\mathrm{Tradeoff}}_2$ become larger, and the bound gets narrowed down. We consider that the trade-off relation becomes more significant.

We define $\Omega^{\mathrm{Tradeoff}}_1$ and $\Omega^{\mathrm{Tradeoff}}_2$ by

$$\Omega^{\mathrm{Tradeoff}}_1 = \begin{cases} \mathsf{V}_{11}|_{\mathsf{V}_{22}=J^{-1}_{\mathrm{S}\,22}} - J^{-1}_{\mathrm{S}\,11} & \text{if } \Delta > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.83}$$

$$\Omega^{\mathrm{Tradeoff}}_2 = \begin{cases} \mathsf{V}_{22}|_{\mathsf{V}_{11}=J^{-1}_{\mathrm{S}\,11}} - J^{-1}_{\mathrm{S}\,22} & \text{if } \Delta > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.84}$$

The $\Omega^{\mathrm{Tradeoff}}_1$ and $\Omega^{\mathrm{Tradeoff}}_2$ are the strength of trade-off relation in the direction of θ_1 and θ_2. Figure 3.1 shows $\Omega^{\mathrm{Tradeoff}}_1$ and $\Omega^{\mathrm{Tradeoff}}_2$. When $\Omega^{\mathrm{Tradeoff}}_1 = \Omega^{\mathrm{Tradeoff}}_2$, we just use $\Omega^{\mathrm{Tradeoff}} = \Omega^{\mathrm{Tradeoff}}_1 = \Omega^{\mathrm{Tradeoff}}_2$. $\Omega^{\mathrm{Tradeoff}}_2$ has the distance between the intersection and the boundary of the SLD CR bound. As shown in Fig. 3.1, if the $\Omega_1^{\text{Tradeoff}}$ and $\Omega_2^{\text{Tradeoff}}$ becomes larger, the trade-off relation becomes more significant. We regard these as an indicator of the strength of the tradeoff relation.

Chapter 4

Trade-off relation given by unitary transformation with commuting generators: Finite dimension

In the previous chapter, we establish a way to derive a trade-off relation from a given model and provide conditions for a trade-off relation to exist. In this chapter, by using those methods, we check if a trade-off relation exists for a pure state, a mixed state of qubit state, and a mixed state of qutrit state when the generators of the unitary transformation commute. As a result, we find no trade-off exists for the pure state and the qubit cases. However, a trade-off relation can be established under certain conditions for the qutrit. We construct an example and conduct a numerical analysis. Since we use unitary models only, we drop θ in the quantum Fisher information matrices in this chapter.

4.1 Model and error trade-off relation

Let us consider arbitrary finite dimensional system. We consider the two-parameter unitary transformation with the generators X and Y , i.e.,

$$U(\theta_1,\, \theta_2) = \mathrm{e}^{-\mathrm{i}X\theta_1 - \mathrm{i}Y\theta_2}. \tag{4.1}$$

We denote the two-parameter family of states generated from the state ρ_0 as ρ_θ.

$$\rho_\theta = U(\theta_1,\, \theta_2)\rho_0 U^\dagger(\theta_1,\, \theta_2). \tag{4.2}$$

The state ρ_0 is called as a reference state. In this thesis, we mainly consider the case of the commuting generators, $[X,\, Y] = 0$ unless stated explicitly.

4.2 Reference state: Pure state

We first consider the case that the reference state is a pure state, i.e., $\rho_0 = |\psi_0\rangle\langle\psi_0|$. From Eqs. (4.1, 4.2), ρ_θ is expressed as

$$\rho_\theta = \mathrm{e}^{-\mathrm{i}X\theta_1 - \mathrm{i}Y\theta_2} |\psi_0\rangle\langle\psi_0| \mathrm{e}^{\mathrm{i}X\theta_1 + \mathrm{i}Y\theta_2}. \tag{4.3}$$

Therefore, we have

$$\partial_1 |\psi_\theta\rangle = -\mathrm{i}X |\psi_\theta\rangle, \tag{4.4}$$
$$\partial_2 |\psi_\theta\rangle = -\mathrm{i}Y |\psi_\theta\rangle, \tag{4.5}$$

where

$$\partial_j |\psi_\theta\rangle = \frac{\partial}{\partial\theta_j} |\psi_\theta\rangle, \quad (j = 1, 2), \tag{4.6}$$
$$|\psi_\theta\rangle = \mathrm{e}^{-\mathrm{i}X\theta_1 - \mathrm{i}Y\theta_2} |\psi_0\rangle. \tag{4.7}$$

The SLD $L_{\mathrm{S}\,\theta,i}$ are given by [48]

$$L_{\mathrm{S}\,\theta,1} = 2\partial_1(|\psi_\theta\rangle\langle\psi_\theta|) = -2\mathrm{i}X\rho_\theta + 2\mathrm{i}\rho_\theta X = 2\mathrm{i}[\rho_\theta, X], \tag{4.8}$$
$$L_{\mathrm{S}\,\theta,2} = 2\partial_2(|\psi_\theta\rangle\langle\psi_\theta|) = -2\mathrm{i}Y\rho_\theta + 2\mathrm{i}\rho_\theta Y = 2\mathrm{i}[\rho_\theta, Y]. \tag{4.9}$$

If $[L_{\mathrm{S}\,\theta,1}, L_{\mathrm{S}\,\theta,2}]|\psi_\theta\rangle = 0$ holds, the SLD CR bound is achievable [49], therefore, no trade-off relation exists. $[L_{\mathrm{S}\,\theta,1}, L_{\mathrm{S}\,\theta,2}]|\psi_\theta\rangle$ is obtained as follows.

$$[L_{\mathrm{S}\,\theta,1}, L_{\mathrm{S}\,\theta,2}]|\psi_\theta\rangle = 4\langle\psi_\theta|[X, Y]|\psi_\theta\rangle|\psi_\theta\rangle. \tag{4.10}$$

A derivation of Eq. (4.10) is given in Appendix C.1. Hence, if X and Y commute, i.e., if $[X, Y] = 0$ holds, we have

$$[L_{\mathrm{S}\,\theta,1}, L_{\mathrm{S}\,\theta,2}]|\psi_\theta\rangle = 0. \tag{4.11}$$

4.3 Reference state: Qubit state

In this section, we first discuss the quantum CR bounds of the qubit state for the general case, i.e., $[X, Y] \neq 0$. Then, we discuss the case of $[X, Y] = 0$.

4.3.1 General case $[X, Y] \neq 0$

We consider the case of a single qubit in a mixed state. We first consider the general two-parameter unitary model to get insight into the problem. By using the Bloch vector, we can express the reference state ρ_0 as

$$\rho_0 = \frac{1}{2}(\mathrm{I} + \vec{s}_0 \cdot \vec{\sigma}). \tag{4.12}$$

where $|\vec{s}_0| < 1$. $\vec{\sigma}$ is a vector whose components are the Pauli matrices, i.e., $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) = (\sigma_1, \sigma_2, \sigma_3)$. The Pauli matrices are defined by

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.13}$$

And the matrix I is an identity matrix

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.14}$$

By its definition, we immediately see that the Pauli matrices are traceless, i.e.,

$$\mathrm{tr}(\sigma_j) = 0, \quad (j = 1, 2, 3). \tag{4.15}$$

The state ρ_θ is given by (4.2), i.e.,

$$\rho_\theta = \mathrm{e}^{-\mathrm{i}X\theta_1 - \mathrm{i}Y\theta_2} \rho_0 \mathrm{e}^{\mathrm{i}X\theta_1 + \mathrm{i}Y\theta_2}. \tag{4.16}$$

The generators X, Y can also be expanded with using Pauli matrices.

$$X = x_0\,\mathrm{I} + \vec{x} \cdot \vec{\sigma}, \tag{4.17}$$

$$Y = y_0\,\mathrm{I} + \vec{y} \cdot \vec{\sigma}. \tag{4.18}$$

The inverse of SLD and RLD Fisher information matrices, J_S^{-1} and J_R^{-1} are explicitly written as

$$J_\mathrm{S}^{-1} = \frac{4}{\det J_\mathrm{S}} \begin{pmatrix} |\vec{y} \times \vec{s}_0|^2 & -(\vec{x} \times \vec{s}_0) \cdot (\vec{y} \times \vec{s}_0) \\ -(\vec{x} \times \vec{s}_0) \cdot (\vec{y} \times \vec{s}_0) & |\vec{x} \times \vec{s}_0|^2 \end{pmatrix}, \tag{4.19}$$

$$J_\mathrm{R}^{-1} = J_\mathrm{S}^{-1} + \frac{4}{\det J_\mathrm{S}} \begin{pmatrix} 0 & -\mathrm{i}|\vec{s}_0|^2[\vec{s}_0 \cdot (\vec{x} \times \vec{y})] \\ \mathrm{i}|\vec{s}_0|^2[\vec{s}_0 \cdot (\vec{x} \times \vec{y})] & 0 \end{pmatrix}, \tag{4.20}$$

where $\det J_\mathrm{S}$ is the determinant of J_S, and it is

$$\det J_\mathrm{S} = 16\,|\vec{s}_0|^2 |\vec{s}_0 \cdot (\vec{x} \times \vec{y})|^2. \tag{4.21}$$

Derivations of J_S^{-1} and J_R^{-1} is given in Section C.2. As shown in (4.19) and (4.20), $J_\mathrm{S}^{-1} = \mathrm{Re}\, J_\mathrm{R}^{-1}$ holds. It follows that our qubit model is D-invariant. Let us remark on the D-invariant models. It is known that the RLD CR inequality is saturated when the model is D-invariant [50, 51, 52, 53, 54, 55], which is valid at least in the asymptotic setting. There is no intersection of the RLD and SLD CR bounds in the D-invariant models because the RLD CR bound is dominant over the SLD CR bound. If the model is D-invariant and if the imaginary part of the off-diagonal components of the RLD Fisher information matrix is not zero, there is only a trade-off relation that results from Condition 1. In the following, we mainly investigate the non-asymptotic setting unless stated explicitly. This is in contrast to the previous study [8], where the authors focused on the D-invariant model, i.e., $\mathcal{V}^\mathrm{S} \cap \mathcal{V}^\mathrm{R} = \mathcal{V}^\mathrm{R}$.

Therefore, the RLD CR bound is asymptotically achievable and gives a trade-off relation. As explained earlier, the SLD and RLD CR bounds do not intersect, but the trade-off relation exists in the asymptotic setting.

As for the Nagaoka bound, for a two-parameter qubit model, which is known to be achievable [56, 57] in the non-asymptotic setting, an inequality regarding the diagonal components of the MSE matrix can be derived. The inequality of the Nagaoka bound (3.65) is written as

$$(\mathsf{V}_{11} - J_{\mathrm{S}}{}^{-1}_{11})(\mathsf{V}_{22} - J_{\mathrm{S}}{}^{-1}_{22}) > \frac{1}{\det J_{\mathrm{S}}}. \tag{4.22}$$

From (3.59) and (4.20), we obtain the inequality of the RLD CR bound.

$$(\mathsf{V}_{11} - J_{\mathrm{S}}{}^{-1}_{11})(\mathsf{V}_{22} - J_{\mathrm{S}}{}^{-1}_{22}) > \frac{|\vec{s}_0|^2}{\det J_{\mathrm{S}}}. \tag{4.23}$$

We used $J_{\mathrm{R}}{}^{-1}_{11} = J_{\mathrm{S}}{}^{-1}_{11}$ and $J_{\mathrm{R}}{}^{-1}_{22} = J_{\mathrm{S}}{}^{-1}_{22}$. Since $|\vec{s}_0|^2 < 1$, the Nagaoka bound is tighter than the RLD CR bound.

4.3.2 Commuting generators' case

Next, we derive a relationship between X and Y, or $\vec{x}$ and $\vec{y}$ when X and Y commute. From (4.17) and (4.18), the commuting relation of X and Y is given by

$$[X,\, Y] = [\vec{x}\cdot\vec{\sigma},\, \vec{y}\cdot\vec{\sigma}] = 2\mathrm{i}(\vec{x}\times\vec{y})\cdot\vec{\sigma}. \tag{4.24}$$

It immediately follows that the necessary and sufficient condition for X and Y to commute is $\vec{x}\times\vec{y} = \vec{0}$, i.e., $\vec{x}$ and $\vec{y}$ are parallel. Therefore, there is no trade-off relation because the unitary transformation is no longer a two-parameter model.

4.4 Reference state: Qutrit state

Let us consider a qutrit system, the three-dimensional system. To avoid non-regular models, the models we consider are full-rank. Other regularity conditions are also imposed implicitly. Furthermore, the models are two-parameter unitary models.

First, we check if a trade-off relation given by the RLD CR bound exists. $(i,\ j)$ component of the RLD Fisher information matrix, $J_{\mathrm{R},\,ij}$ is defined by

$$J_{\mathrm{R},\,ij} = \mathrm{tr}\left(\rho_0 L_{\mathrm{R},\,j} L_{\mathrm{R},\,i}^{\dagger}\right), \tag{4.25}$$

where $\partial_i\rho_\theta\,|_{\theta=0} = \rho_0 L_{\mathrm{R},\,i}$. We choose $\theta = 0$ because there is no θ dependence in unitary models. When X_i and X_j commutes, by using $\partial_i\rho_\theta\,|_{\theta=0} = L_{\mathrm{R},\,i}^{\dagger}\rho_0$, (4.1), and (4.2), we obtain

$$J_{\mathrm{R},\,ij} = -\mathrm{tr}\left([X_j,\, \rho_0][X_i,\, \rho_0]\,\rho_0^{-1}\right), \tag{4.26}$$

where $X_1 = X$ and $X_2 = Y$. With this, Condition 1 is expressed as

$$\delta = \mathrm{tr}\left(\left[[X,\, \rho_0]\,,\, [Y,\, \rho_0]\right]\rho_0^{-1}\right), \tag{4.27}$$

and thus, it is relatively easy to check this condition analytically. We stress that having commuting generators, $[X, Y] = 0$ does not immediately imply $\delta = 0$. Since X and Y commute, they are simultaneously diagonalizable. Without the loss of generality, for the calculation of δ, we can use the representation so that both X and Y can be diagonalized.

$$\rho_0 = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix}, \tag{4.28}$$

$$X = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}, \tag{4.29}$$

$$Y = \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{pmatrix}. \tag{4.30}$$

By using (4.26), δ is calculated as follows.

$$\delta = (\det \rho_0)^{-1} \left(\rho_{12}\rho_{23}\rho_{31} - \rho_{21}\rho_{32}\rho_{13}\right) \left[(\vec{y} \times \vec{x}) \cdot \vec{1}\right], \tag{4.31}$$

where $\vec{x} = (x_1, x_2, x_3)$, $\vec{y} = (y_1, y_2, y_3)$, and $\vec{1} = (1, 1, 1)$. The condition of no trade-off relation, $\delta = 0$ holds when

$$\mathrm{Im}\,(\rho_{12}\rho_{23}\rho_{31}) = 0, \tag{4.32}$$

or

$$(\vec{y} \times \vec{x}) \cdot \vec{1} = 0. \tag{4.33}$$

Violation of these conditions and (3.72), i.e.,

$$\Delta = \left|\mathrm{Im}\, J_{\mathrm{R}\,12}^{-1}\right|^2 - (J_{\mathrm{R}\,11}^{-1} - J_{\mathrm{S}\,11}^{-1})(J_{\mathrm{R}\,22}^{-1} - J_{\mathrm{S}\,22}^{-1}) > 0. \tag{4.34}$$

are the necessary and sufficient conditions to have a non-trivial error trade-off relation. In the case of qutrit, we cannot give an explicit expression of Δ in general. But, we can straightforwardly obtain Δ numerically.

The following subsections give examples of reference states that give non-trivial error trade-off relations. One of them gives a relatively high possibility. Our primary interest is investigating the error trade-off relation for a given commuting X and Y.

4.4.1 Example: reference state with multi-parameter

As one of the simplest examples, we pick an example with pure imaginary off-diagonal components as a reference state ρ_0 with five real variables v_1, v_2, v_3, u_1, u_2, and u_3. ($v_1 + v_2 + v_3 = 1$).

$$\rho_0 = \frac{1}{3} \begin{pmatrix} v_1 & -\mathrm{i}\sqrt{u_1} & \mathrm{i}\sqrt{u_2} \\ \mathrm{i}\sqrt{u_1} & v_2 & -\mathrm{i}\sqrt{u_3} \\ -\mathrm{i}\sqrt{u_2} & \mathrm{i}\sqrt{u_3} & v_3 \end{pmatrix}. \tag{4.35}$$

We choose the reference state ρ_0 as above, because imaginary parts of the off-diagonal components of the reference state ρ_0 are important to satisfy Condition 1 as seen in (4.32). We calculate Δ in Condition 2 with using the reference state ρ_0 defined by (4.35) of which refer-

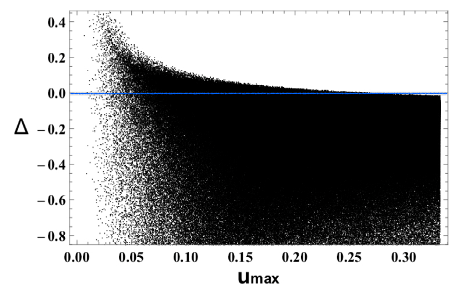

Figure 4.1: Δ as a function of u_{max}, the maximum of u_1, u_2, and u_3 in (4.35). $\vec{x} = (1,\ 2,\ 3)$, $\vec{y} = (1.5,\ 5,\ 1)$.

ence state parameters are generated by random numbers. We pick those which satisfy $\mathrm{tr}\rho_0 = 1$ and $\rho_0 > 0$ and calculate the RLD and SLD Fisher information matrices J_S and J_R. The RLD Fisher information matrix is obtained by using (4.26). The SLD Fisher information calculation is done in the standard method. (See for example, [58, 59].) The number of samples generated is on the order of 10^6. Figure 4.1 shows Δ as a function of u_{max}, the maximum of u_1, u_2, and u_3. There exists a region $\Delta > 0$. The ratio of obtaining $\Delta > 0$ out of all of the samples generated is 3.0%. Figures 4.2 and 4.3 show Δ as a function of λ_{min} and λ_{max}, respectively. λ_{min} and λ_{max} are the minimum and maximum of eigenvalues of ρ_0, respectively. For Δ to be positive, λ_{min} and λ_{max} must be in a certain range. λ_{min} is more than about 0.13 and λ_{max} is less than about 0.58.

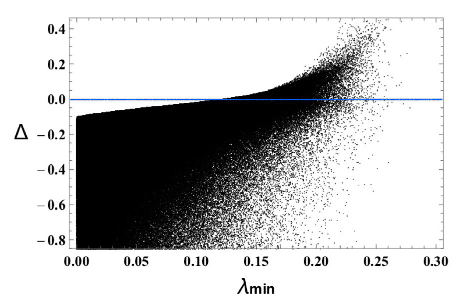

Figure 4.2: Δ as a function of λ_{min}, the minimum of the eigenvalues of ρ_0

4.4.2 Example: one-parameter family of reference states

Next, we set the reference state parameters in (4.35) as $v_1 = v_2 = v_3 = 1$ and $u_1 = u_2 = u_3 = u$ in order to investigate the model more in detail analytically. We pick the reference state parameters as above, because the result of Section 4.4.1 indicates that the eigenvalues of the reference state (4.35) be roughly in the range $1/3 \pm 0.2$ to exhibit the non-trivial trade-off relation. The reference

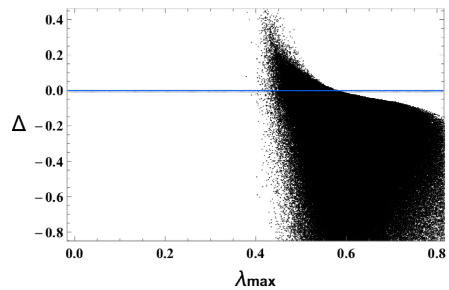

Figure 4.3: Δ as a function of λ_{max}, the maximum of the eigenvalues of ρ_0

state ρ_0 is, then explicitly written as

$$\rho_0 = \frac{1}{3}\mathrm{I} + \frac{1}{3}\sqrt{u}\begin{pmatrix} 0 & -\mathrm{i} & \mathrm{i} \\ \mathrm{i} & 0 & -\mathrm{i} \\ -\mathrm{i} & \mathrm{i} & 0 \end{pmatrix}, \tag{4.36}$$

where I denotes 3×3 identity matrix. The reference state ρ_0 is a sum of the completely mixed state of the qutrit system and a perturbation with one parameter u. The parameter u must be in the range, $0 < u < 1/3$ for the reference state ρ_0 to be positive. We exclude $u = 0$, because $\rho_0 = \mathrm{I}/3$ at $u = 0$.

In the following, we show that the reference state ρ_0 (4.35) always gives a non-trivial trade-off relation with a certain choice of the reference state parameter u and that the possibility of seeing the non-trivial trade-off relation is not small.

Intersections of RLD and SLD CR bounds

From Condition 2 for $\lambda = 1$, the condition for the trade-off relation to exist is as follows. (3.72)

$$\text{Condition 2}: \quad \Delta = \left|\mathrm{Im}\, J_{\mathrm{R}\,12}^{-1}\right|^2 - (J_{\mathrm{R}\,11}^{-1} - J_{\mathrm{S}\,11}^{-1})(J_{\mathrm{R}\,22}^{-1} - J_{\mathrm{S}\,22}^{-1}) > 0.. \tag{4.37}$$

$\Delta > 0$ needs to be satisfied in order to have a non-trivial error trade-off relation. We define a geometrical parameter, ζ as follows.

$$\zeta = \frac{[\vec{1}\cdot(\vec{x}\times\vec{y})]^2}{(\vec{1}\times\vec{x})^2(\vec{1}\times\vec{y})^2}. \tag{4.38}$$

Let $\vec{\xi} = \vec{1}\times\vec{x}$ and $\vec{\eta} = \vec{1}\times\vec{y}$. A vector analysis formula gives an expression,

$$\zeta = \frac{1}{3}\sin^2\theta \le \frac{1}{3}, \tag{4.39}$$

where $\sin\theta = |\vec{\xi}\times\vec{\eta}|/(|\vec{\xi}||\vec{\eta}|)$. $\zeta = 1/3$ when $\theta = \pm\pi/2$. $\zeta = 0$ is excluded, because $\vec{\xi}\times\vec{\eta} = \vec{0}$ gives $\delta = 0$ from (4.33). Therefore, the possible range for the parameter ζ is $0 < \zeta \le 1/3$.

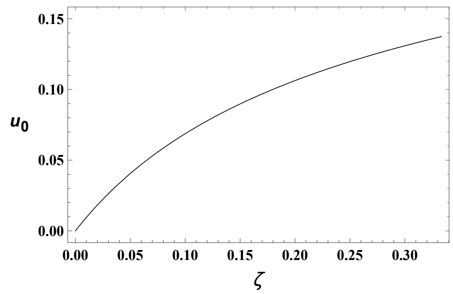

Figure 4.4: Solutoin u_0 that satisfies $F_\zeta(u_0) = 0$ in the range $0 < \zeta \leq 1/3$

We introduce a function of u at a given ζ, $F_\zeta(u)$ as

$$F_\zeta(u) = 16\zeta(3u^2 - 7u + 2)^2 - u(3u^2 - 9u + 8)^2. \tag{4.40}$$

By using $F_\zeta(u)$, Δ is expressed as

$$\Delta = \frac{9}{16\zeta^2|\vec{\xi}|^2|\vec{\eta}|^2(2-u)u(u^2-7u+4)^2} F_\zeta(u). \tag{4.41}$$

The coefficient of $F_\zeta(u)$ in (4.41) is positive finite when $0 < u < 1/3$. In order to investigate the range of u that gives $\Delta > 0$, we can check the condition for $F_\zeta(u) > 0$ instead.

We can analytically show that $F_\zeta(u)$ is a monotonically decreasing function of u and that there is always a unique solution u_0 that satisfies $F_\zeta(u_0) = 0$ when $0 < \zeta \leq 1/3$ and when $\rho_0 > 0$, i.e., $0 < u < 1/3$. A detailed explanation is given in Appendix C.3. Figure 4.4 shows the solution u_0 that satisfies $F_\zeta(u) = 0$. In the region where $u < u_0$ at a given ζ, the non-trivial trade-off relation exists. We can regard u_0 as the upper limit of u that gives a non-trade off relation. It is worth noting that the upper limit of u is almost a half of the maximum of u, $1/3$ at $\zeta = 1/3$. This means that the possibility of realizing non-trivial trade-off relation is not small.

Figure 4.5 shows an example in which the SLD and RLD CR bounds have two intersections. The parameters used are $u = 1/12$, $\vec{x} = (1,\ 2,\ 3)$, and $\vec{y} = (1.5,\ 5,\ 1)$.

Indicators of the strength of the trade-off relation $\Omega_1^{\text{Tradeoff}}$ and $\Omega_2^{\text{Tradeoff}}$ defined in Section 3.8 are calculated as

$$\Omega_1^{\text{Tradeoff}} = \frac{\Delta}{J_{\text{S}}^{22} - J_{\text{R}}^{22}} = \frac{3}{4\,\zeta|\vec{\eta}|^2 u(u^2-7u+4)(3u^2-9u+8)} F_\zeta(u), \tag{4.42}$$

$$\Omega_2^{\text{Tradeoff}} = \frac{\Delta}{J_{\text{S}}^{11} - J_{\text{R}}^{11}} = \frac{3}{4\,\zeta|\vec{\xi}|^2 u(u^2-7u+4)(3u^2-9u+8)} F_\zeta(u). \tag{4.43}$$

The strengths of trade-off relation is proportional to Δ. Figure 4.6 shows $\Omega_1^{\text{Tradeoff}}$ and $\Omega_2^{\text{Tradeoff}}$ as a function of the parameter u. In the range where $\Omega_1^{\text{Tradeoff}} > 0$ or $\Omega_2^{\text{Tradeoff}} > 0$, the non-trivial trade-off relation exists. The strength of trade-off relation becomes stronger as u approaches 0.

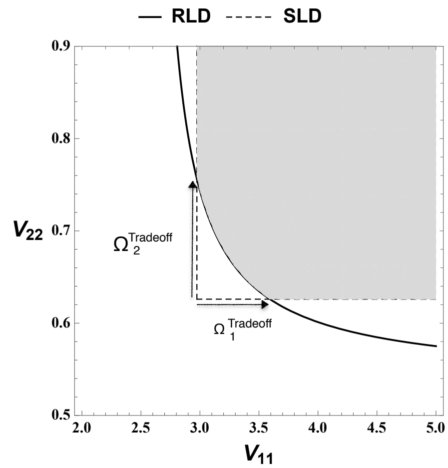

Figure 4.5: Example of RLD and SLD CR bounds with the intersections: the reference state ρ_0 defined by (4.36) with $u = 1/12$, $\vec{x} = (1,\ 2,\ 3)$, $\vec{y} = (1.5,\ 5,\ 1)$. The gray region is the bound determined by both the RLD and the SLD CR bounds .

4.4.3 Discussion

Unlike a qubit reference state or a pure state reference state, there exists a non-trivial trade-off relation for some qutrit reference states even when the generators commute. We show analytically that a non-trivial trade-off relation always exists in a certain range of the reference state parameter u when the reference state ρ_0 is defined by (4.36) that is a sum of the completely mixed state and a perturbation.

Furthermore, the strengths of trade-off relation $\Omega_1^{\text{Tradeoff}}$ and $\Omega_2^{\text{Tradeoff}}$ increase as u approaches 0. This looks counterintuitive, because we can regard u as a small perturbation from 3x3 identity matrix when $u \ll 1$ by the definition of ρ_0, (4.36). This reflects the fact that $\partial_i\rho_\theta$ is not necessarily small when the perturbation itself is small. Since the (i, j) component of the RLD Fisher information matrix is $J_{\mathrm{R},ij} = \mathrm{tr}[\partial_j\rho_\theta L_{R,i}^\dagger]$, the component $J_{\mathrm{R},ij}$ may not be small if $\partial_i\rho_\theta$ is not small.

In a more general case when ρ_0 is expressed by (4.35), we conducted numerical analysis. In this case also, there exists a non-trivial trade-off relation. Furthermore, in the case of four dimensional system with pure imaginary off-diagonal components, we also see a non-trivial trade-off relation by the same numerical analysis as well. With these, we conclude that the error trade-off relation is a generic phenomenon in the sense that it occurs with a finite volume in the spate space.

4.5 Conclusion

We have investigated whether the error trade-off relation exists in the generic two-parameter unitary models for finite dimensional systems with the commuting generators. By analyzing the necessary and sufficient conditions for the SLD and RLD CR bounds to intersect each other, we obtain the necessary and sufficient conditions for the existence of a non-trivial trade-off relation

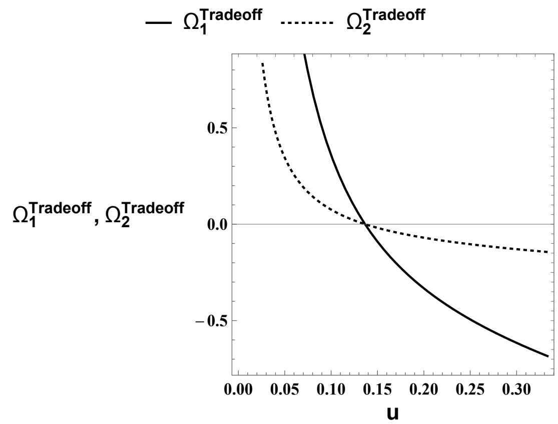

Figure 4.6: $\Omega_1^{\mathrm{Tradeoff}}$ (solid line) and $\Omega_2^{\mathrm{Tradeoff}}$ (dotted line) as a function of the parameter u. $\vec{x}$ and $\vec{y}$ are the same as those used for Fig. 4.5. In the range where $\Omega_1 > 0$, therefore $\Omega_2 > 0$, the non-trivial trade-off relation exists.

based on the SLD and RLD CR bounds for arbitrary finite dimensional system.

By using the conditions, we show two examples of the qutrit system with the non-trivial trade-off relation. The result of the reference state with multi-parameter indicates that the eigenvalues of the reference state be in a certain range. In the other model reference state with one-parameter, we show analytically that a non-trivial trade-off relation always exists in a certain range of the reference state parameter and that the region with the trade-off relation is up to about a half of the allowed region.

In our previous study about the trade-off relation of an infinite dimensional system [44], the bound is also given by both of the SLD and RLD CR bounds when the generators of the unitary transformation with the commuting generators. As shown in Figs. 4.5 and 4.6, we confirmed that what we saw in our previous study is not special, but generic. When the reference state is a pure state or a general qubit state, we disprove the existence of a non-trivial trade-off relation.

Chapter 5

Example of error trade off relation by the SLD and RLD CR bounds: one electron in a magnetic field

In this chapter, we present an example of a physical model that exhibits the trade-off relation in the way that we investigated in the previous chapter. The model we use here is an electron in a uniform magnetic field. We assume the state of the electron is described by the non-relativistic quantum mechanics.

Hence, we investigate the uncertainty relation between two *communing observables* based on the multi-parameter quantum estimation theory [1, 2, 60, 61, 62, 63]. At first sight, one might expect that there cannot be such a trade-off relation. However, as demonstrated in this thesis, the quantum estimation theory enables us to derive a non-trivial trade-off relation for estimating the expectation values of two commuting observables. The result of this chapter is based on the part of [44]. We use the natural unit. Planck constant, $\hbar$, speed of light, c, and the Boltzmann constant, k_{B} are 1, i.e., $\hbar = c = k_{\mathrm{B}} = 1$. Since we use unitary model only, we drop θ in this chapter.

5.1 Parametic model

5.1.1 Hamiltonian

The Hamiltonian H for an electron motion in a uniform magnetic field is

$$H = \frac{1}{2m}(\vec{p} + e\vec{A})^2. \tag{5.1}$$

where $-e$ and m are the charge of an electron ($e > 0$), and the mass of the electron, respectively. $\vec{A}$ is a vector potential. In the following discussion, we use the coordinate representation of operators. The canonical observables describing this systems are p_x, x, p_y, and y. We will investigate the uncertainty relation of an electron motion in a uniform magnetic field $\vec{B} = (0, 0, B)$, $B > 0$. We use the symmetric gauge. Hence the vector potential is written as $\vec{A} = B(-y/2, x/2, 0)$. We can show that the choice of the gauge gives no change in the quantum Fisher information when the magnetic field is uniform.

We will consider the motion in x-y plane only, because z component solution is a plane wave. With a new vector operator, $\vec{\pi} = \vec{p} + e\vec{A}$, our Hamiltonian becomes [64]

$$H = \frac{1}{2m}(\pi_x^2 + \pi_y^2). \tag{5.2}$$

Here we remark that these mechanical (kinetic) momenta satisfy the canonical commutation relation up to a constant factor: $[\pi_x, \pi_y] = -ieB$ [64]. They together with the guiding center operators are the fundamental observables in the study of electrons in strong magnetic fields, see for example [65].

It is known that the operators x, y and p_x, p_y are equally described by the two sets of the creation and annihilation operators, acting on the different Fock spaces, $a, a^\dagger$ and $b, b^\dagger$ such that $[a,\, a^\dagger] = [b,\, b^\dagger] = 1$ with all other commutation relations vanishing [66].

The canonical momenta p_x, p_y and the position x, y in Eq. (5.1) are expressed as

$$p_x = \frac{\mathrm{i}}{2\kappa}\left[(a^\dagger - a) + (b^\dagger - b)\right], \quad p_y = \frac{1}{2\kappa}\left[(a^\dagger + a) - (b^\dagger + b)\right], \tag{5.3}$$

$$x = \frac{\kappa}{2}\left[(a^\dagger + a) + (b^\dagger + b)\right], \quad y = -\frac{\mathrm{i}\kappa}{2}\left[(a^\dagger - a) - (b^\dagger - b)\right]. \tag{5.4}$$

where $\kappa = \sqrt{2(eB)^{-1}}$ has the dimension of length.

The mechanical momenta π_x, π_y in Eq. (5.2) are expressed as

$$\pi_x = \frac{\mathrm{i}}{\kappa}(a^\dagger - a), \quad \pi_y = \frac{1}{\kappa}(a^\dagger + a). \tag{5.5}$$

$$\tag{5.6}$$

As shown in Eq. (5.11) below, κ corresponds to the spread of the probability density of the electron in the lowest Landau level, or LLL.

The Hamiltonian H and z component of the angular momentum L are expressed in terms of the two harmonic oscillators as

$$H = \omega(a^\dagger a + \frac{1}{2}), \tag{5.7}$$

$$L = xp_y - yp_x = a^\dagger a - b^\dagger b, \tag{5.8}$$

where $\omega = eB/m$ is the cyclotron frequency.

5.1.2 States

As the states on which the operators a, $a^\dagger$ and b, $b^\dagger$ act, the number states $|n\rangle_a$ and $|n\rangle_b$ that satisfy

$$a^\dagger a\,|n\rangle_a = n\,|n\rangle_a\,, \quad b^\dagger b\,|n\rangle_b = n\,|n\rangle_b\,, \tag{5.9}$$

are often used. The number states $|0\rangle_a$ and $|0\rangle_b$ are the vacuum states of the harmonic oscillators.

Since the Hamiltonian H does not include b, $b^\dagger$, its energy eigenstate consists of infinite number of the angular momentum eigenstates Eq. (5.8), i.e, the energy eigenstate is degenerated. This state is written as $|0,\, 0\rangle := |0\rangle_a\,|0\rangle_b$ from Eqs. (5.7, 5.8). The wave function of this state is known as the lowest Landau level, or LLL, $\psi_{00}(x, y)$, which is expressed by

$$\psi_{00}(x, y) = \langle x,\, y\,|\,0,\, 0\rangle = C\mathrm{e}^{-\frac{x^2+y^2}{2\kappa^2}}, \tag{5.10}$$

where C is the normalization factor. Then, the position probability density $|\psi_{00}(x,y)|^2$ is

$$|\psi_{00}(x,y)|^2 \propto \mathrm{e}^{-\frac{x^2+y^2}{\kappa^2}}. \tag{5.11}$$

This is a Gaussian distribution with its spread κ and with its peak at $(x,\ y) = (0,\ 0)$.

5.2 Estimation of the position

We use a parametric model that describes the position measurement of the particle. We utilize the unitary transformation generated by the canonical momenta p_x and p_y with the parameter $\theta = (\theta_1, \theta_2)$. We can write the reference state ρ_θ generated by the unitary transformation from the reference state ρ_0 which is known in advance as follows.

$$\rho_\theta = \mathrm{e}^{-\mathrm{i}\theta_1 p_x}\mathrm{e}^{-\mathrm{i}\theta_2 p_y}\rho_0\mathrm{e}^{\mathrm{i}\theta_1 p_x}\mathrm{e}^{\mathrm{i}\theta_2 p_y}. \tag{5.12}$$

Hence, with the reference state ρ_θ, our model $\mathcal{M}$ is written as

$$\mathcal{M} = \{\rho_\theta \,\big|\, \theta = (\theta_1, \theta_2) \in \mathbb{R}^2\}, \tag{5.13}$$

By using the momenta p_x and p_y as the generators, we obtain the expectation value of the position operators as follows.

$$\langle X\rangle_\theta = \langle X\rangle_0 + \theta_1, \tag{5.14}$$

$$\langle Y\rangle_\theta = \langle Y\rangle_0 + \theta_2, \tag{5.15}$$

where $\langle X\rangle_\theta = \mathrm{tr}[\rho_\theta X]$ and $\langle X\rangle_0 = \mathrm{tr}[\rho_0 X]$. The unitary transformations of model makes a shift in the position probability density of the electron by $\theta = (\theta_1,\ \theta_2)$. From Eqs. (5.14, 5.15), we have $\theta_1 = \langle x\rangle_\theta - \langle x\rangle_0$ and $\theta_2 = \langle y\rangle_\theta - \langle y\rangle_0$. Then, the shifted state from the reference state has a sharp peak at $(x,\ y) = (\theta_1,\ \theta_2)$. Therefore, estimating $\langle x\rangle_\theta$ and $\langle y\rangle_\theta$ is equivalent to infer the shift parameters $\theta = (\theta_1,\ \theta_2)$. (Under the assumption that we know in advance the expectation value of the position operators with respect to the reference state ρ_0.) We estimate the unknown parameters θ_1 and θ_2 by making arbitrary measurement, which is unbiased. We then infer the two parameters from the measurement result. We shall use the MSE matrix to measure the estimation accuracy of the position of the electron.

5.3 Trade-off relation of thermal state

As shown in Appendix C.1 and in Section 4.2, the SLD CR bound becomes acheivable for any pure state reference state if the generators of the unitary transformation commute. Therefore, their exists no trade-off relation.

We set up a mixed state as the reference state to see how the noise affects the measurement accuracy of the electron position. For this purpose, as the mixed state, we choose the thermal state. However, in the current system we are considering, there is no unique thermal state, because the energy eigenstate is degenerated. Then, the thermal state of this system is not uniquely specified by the temperature only. To resolve this degeneracy problem, we impose a condition that the expectation value of the angular momentum $\langle L\rangle_0$ is fixed. This is done by introducing a chemical potential.

After constructing the reference state, we use the method explained in Section 3.6 to evaluate the strength of the trade-off relation $\Omega_1^{\mathrm{Tradeoff}}$ and $\Omega_2^{\mathrm{Tradeoff}}$.

5.3.1 Reference state

Given $\langle L\rangle_0$ is fixed at a constant, the reference state ρ_0 is denoted by

$$\rho_0 = Z_{\beta,\mu}^{-1} \mathrm{e}^{-\beta H+\mu L}, \tag{5.16}$$

where $\beta = T^{-1}$ is the inverse temperature and $Z_{\beta,\mu} = \mathrm{tr}\,[\mathrm{e}^{(-\beta H+\mu L)}]$ is the partition function. The parameter μ is the chemical potential, which will be determined later. The role of the chemical potential μ is to keep $\langle L\rangle_0$ constant to avoid complications by the degeneracy of angular momentum. The use of the chemical potential here is the same idea as seen in the grand canonical ensemble of statistical physics where the chemical potential is used to keep the expectation value of the number of particles constant.

From Eqs. (5.7, 5.8),

$$\rho_0 = Z_{\beta,\mu}^{-1} \mathrm{e}^{-\frac{1}{2}\beta\omega} \mathrm{e}^{-(\beta\omega-\mu)a^\dagger a-\mu b^\dagger b}. \tag{5.17}$$

because H and L commute, i.e., $[H, L] = 0$. By using the Gaussian states which are defined by

$$a\,|z\rangle_a = z\,|z\rangle_a\,,\;\; b\,|z\rangle_b = z\,|z\rangle_b\,, \tag{5.18}$$

the reference state ρ_0 is expressed as

$$\rho_0 = \rho_{0,a} \otimes \rho_{0,b}, \tag{5.19}$$

where $\rho_{0,a}$ and $\rho_{0,b}$ are the thermal states with different temperatures. Explicitly, they are

$$\rho_{0,a} = \frac{1}{2\pi\kappa_\mathrm{a}^2} \int \mathrm{e}^{-\frac{|z|^2}{2\kappa_\mathrm{a}^2}} \,|z\rangle_{a\,a}\langle z\,|\, d^2z, \tag{5.20}$$

$$\rho_{0,b} = \frac{1}{2\pi\kappa_\mathrm{b}^2} \int \mathrm{e}^{-\frac{|z|^2}{2\kappa_\mathrm{b}^2}} \,|z\rangle_{b\,b}\langle z\,|\, d^2z, \tag{5.21}$$

with

$$2\kappa_\mathrm{a}^2 = (\mathrm{e}^{\beta\omega-\mu} - 1)^{-1}, \quad 2\kappa_\mathrm{b}^2 = (\mathrm{e}^{\mu} - 1)^{-1}. \tag{5.22}$$

The derivation of Eqs. (5.19, 5.22) is given in Appendix D.1. It is straightforward to calculate the expectation value $\langle L\rangle_0$ as

$$\langle L\rangle_0 = \mathrm{tr}\,[L\rho_0] = 2\kappa_\mathrm{a}^2 - 2\kappa_\mathrm{b}^2. \tag{5.23}$$

From Eqs. (5.22, 5.23), we obtain

$$(\langle L\rangle_0 + 1)\mathrm{e}^{2\mu} - \langle L\rangle_0(\mathrm{e}^{\beta\omega} + 1)\mathrm{e}^{\mu} + (\langle L\rangle_0 - 1)\mathrm{e}^{\beta\omega} = 0. \tag{5.24}$$

When $\beta\omega$ and $\langle L\rangle_0$ are given, μ is the variable of Eq. (5.24). If $\langle L\rangle_0 = -1$ holds, there exists a unique solution. Whereas there are two solutions for $\langle L\rangle_0 \neq -1$. However, one of them is

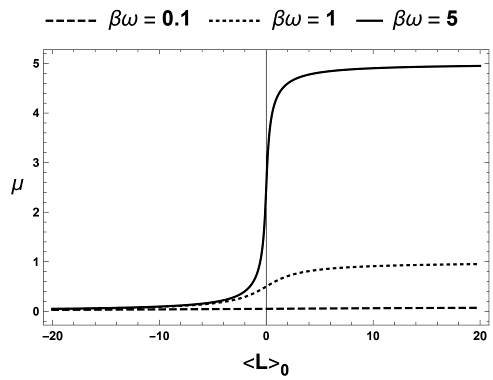

Figure 5.1: The chemical potential μ as a function of the expectation value of the angular momentum $\langle L\rangle_0$ at three different temperature parameters $\beta\omega = 0.1,\ 1,$ and 5. At lower $\beta\omega$ i.e., higher temperature, μ becomes closer to zero, no preference for the angular momentum.

shown to be unphysical giving a negative temperature state. Then, the chemical potential μ as a function of $\langle L\rangle_0$ and $\beta\omega$ is found to be

$$
\mathrm{e}^{\mu} = \begin{cases} \dfrac{2\mathrm{e}^{\beta\omega}}{\mathrm{e}^{\beta\omega}+1} & (\langle L\rangle_0 = -1) \\ \dfrac{1}{2(\langle L\rangle_0+1)}\Big[\langle L\rangle_0(\mathrm{e}^{\beta\omega}+1) + \sqrt{\langle L\rangle_0^2(\mathrm{e}^{\beta\omega}-1)^2+4\mathrm{e}^{\beta\omega}}\Big] & \\ & (\langle L\rangle_0 \neq -1) \end{cases} . \tag{5.25}
$$

Although the solution of Eq. (5.24) has a singular point at $\langle L\rangle_0 = -1$ at a first glance, we can show that the solution the solution for $\langle L\rangle_0 \neq -1$ is continuously connected to the solution for $\langle L\rangle_0 = -1$. We can also show that the first derivative is continuous at $\langle L\rangle_0 = -1$.

Figure 6.1 shows μ as a function of $\langle L\rangle_0$ at $\beta\omega = 0.1$, 1, and 5 from top to bottom. The chemical potential μ as a function of $\langle L\rangle_0$ diverges for $\langle L\rangle_0 \geq 0$ as $\beta\omega$ goes to infinity, i.e., the zero temperature limit. At a special case, $\langle L\rangle_0 = 0$, we see $\mu = \beta\omega/2$ from Eq. (5.25). Explicitly, the zero temperature limit is

$$
\lim_{\beta\to\infty}\mu = \begin{cases} \infty & (\langle L\rangle_0 \geq 0) \\ \log\left[\frac{\langle L\rangle_0 - 1}{\langle L\rangle_0}\right] & (\langle L\rangle_0 < 0) \end{cases} . \tag{5.26}
$$

5.3.2 Trade-off relation

We derive the SLD and the RLD CR bounds. They then provide the trade-off relation for the MSE matrix. The calculations of SLDs and RLDs and their quantum Fisher information matrices are given in Appendix D.2.

Condition for trade-off relation to exist

Let J_{R} and J_{S} be the RLD and the SLD Fisher information matrices with respect to $\rho_\theta = U_\theta \rho_0 U_\theta^\dagger$, respectively. The inverse of J_{R} is calculated as

$$(J_{\mathrm{R}})^{-1} = \frac{\kappa^2}{1+2\kappa_{\mathrm{a}}^2+2\kappa_{\mathrm{b}}^2}\begin{pmatrix} 2\kappa_{\mathrm{a}}^2+2\kappa_{\mathrm{b}}^2+8\kappa_{\mathrm{a}}^2\kappa_{\mathrm{b}}^2 & \mathrm{i}\,(2\kappa_{\mathrm{b}}^2-2\kappa_{\mathrm{a}}^2) \\ -\mathrm{i}\,(2\kappa_{\mathrm{b}}^2-2\kappa_{\mathrm{a}}^2) & 2\kappa_{\mathrm{a}}^2+2\kappa_{\mathrm{b}}^2+8\kappa_{\mathrm{a}}^2\kappa_{\mathrm{b}}^2 \end{pmatrix}$$
$$= \frac{\kappa^2}{1+2\kappa_{\mathrm{a}}^2+2\kappa_{\mathrm{b}}^2}\begin{pmatrix} 2\kappa_{\mathrm{a}}^2+2\kappa_{\mathrm{b}}^2+8\kappa_{\mathrm{a}}^2\kappa_{\mathrm{b}}^2 & \mathrm{i}\,\langle L\rangle_0 \\ -\mathrm{i}\,\langle L\rangle_0 & 2\kappa_{\mathrm{a}}^2+2\kappa_{\mathrm{b}}^2+8\kappa_{\mathrm{a}}^2\kappa_{\mathrm{b}}^2 \end{pmatrix}. \tag{5.27}$$

We use $\langle L\rangle_0 = 2\kappa_{\mathrm{b}}^2 - 2\kappa_{\mathrm{a}}^2$. The explicit expression of κ is $\kappa = \sqrt{2(eB)^{-1}}$. This κ is the spread of the square of the absolute value of the wave function of the ground state.

Hence, we have the RLD CR bound.

$$(\mathsf{V}_{11} - J_{\mathrm{R}}{}^{-1}_{11})(\mathsf{V}_{22} - J_{\mathrm{R}}{}^{-1}_{11}) \geq \kappa^4 \left(\frac{\langle L\rangle_0}{1+2\kappa_{\mathrm{a}}^2+2\kappa_{\mathrm{b}}^2}\right)^2. \tag{5.28}$$

Next, the calculation of the inverse of J_{S} reveals that $(J_{\mathrm{S}})^{-1}$ is a diagonal matrix and that $J_{\mathrm{S}}{}^{-1}_{11}$ is equal to $J_{\mathrm{R}}{}^{-1}_{22}$. $(J_{\mathrm{S}})^{-1}$ is written as

$$(J_{\mathrm{S}})^{-1} = \begin{pmatrix} J_{\mathrm{S}}{}^{-1}_{11} & 0 \\ 0 & J_{\mathrm{S}}{}^{-1}_{22} \end{pmatrix}, \tag{5.29}$$

where

$$J_{\mathrm{S}}{}^{-1}_{11} = J_{\mathrm{S}}{}^{-1}_{22} = \kappa^2 \frac{\frac{1}{2}+2\kappa_{\mathrm{a}}^2+2\kappa_{\mathrm{b}}^2+8\kappa_{\mathrm{a}}^2\kappa_{\mathrm{b}}^2}{1+2\kappa_{\mathrm{a}}^2+2\kappa_{\mathrm{b}}^2}. \tag{5.30}$$

Hence, we have

$$\mathsf{V}_{11} \geq J_{\mathrm{S}}{}^{-1}_{11}, \quad \mathsf{V}_{22} \geq J_{\mathrm{S}}{}^{-1}_{11}. \tag{5.31}$$

There are two cases regarding the ordering between the inverse of RLD and SLD Fisher matrices in terms of the matrix inequality.

We next investigate the condition for trade-off relation to exists, Condition 2 we show in Section 3.7.

$$\Delta = |\mathrm{Im} J_{\mathrm{R}}{}^{-1}_{12}|^2 - (J_{\mathrm{S}}{}^{-1}_{11} - J_{\mathrm{R}}{}^{-1}_{11})(J_{\mathrm{S}}{}^{-1}_{22} - J_{\mathrm{R}}{}^{-1}_{22}) > 0. \tag{5.32}$$

Then, Condition 2, the condition for trade-off relation to exist is obtained as

$$\Delta = \frac{\kappa^4}{4(1+2\kappa_{\mathrm{a}}^2+2\kappa_{\mathrm{b}}^2)^2}(4\langle L\rangle_0^2 - 1) > 0. \tag{5.33}$$

Depending on $\langle L\rangle_0$, we have the following two cases.
Case i). When $|\langle L\rangle_0| \leq 1/2$, the SLD CR bound defines a tighter lower bound. This is because the matrix inequality

$$\Delta J^{-1} := (J_{\mathrm{S}})^{-1} - (J_{\mathrm{R}})^{-1} = \Delta g \begin{pmatrix} 1 & -2\mathrm{i}\,\langle L\rangle_0 \\ 2\mathrm{i}\,\langle L\rangle_0 & 1 \end{pmatrix} \geq 0, \tag{5.34}$$

holds if and only if $|\langle L\rangle_0| \leq 1/2$ is satisfied. Here, Δg is defined by

$$\Delta g := \frac{\kappa^2}{2} \frac{1}{1 + 2\kappa_{\mathrm{a}}^2 + 2\kappa_{\mathrm{b}}^2} > 0. \tag{5.35}$$

Case ii). In the other case, $|\langle L\rangle_0| > 1/2$, however, there is no matrix ordering between the RLD and the SLD Fisher information matrices. This means that both of the RLD CR inequality Eq. (5.28) and the SLD CR inequality (5.31) contribute to the trade-off relation.

Figures 5.2 show examples of the bounds with different $|\langle L\rangle_0|$=1, 1/2, and 0.2. Figure 5.2a shows an example of the bound given by the current analysis with $|\langle L\rangle_0| = 1/2$. The parameters used are $\kappa = 1$, $2\kappa_{\mathrm{a}}^2 = 2$, $2\kappa_{\mathrm{b}}^2 = 1$, and thus $|\langle L\rangle_0| = 1 > 1/2$ holds. Figure 5.2b shows an example of the bound given by the current analysis with $|\langle L\rangle_0| = 1/2$. The parameters used are $\kappa = 1$, $2\kappa_{\mathrm{a}}^2 = 1.5$, $2\kappa_{\mathrm{b}}^2 = 1$, and thus $|\langle L\rangle_0| = 1/2$ holds. The RLD and the SLD CR bounds have only one intersection point. The SLD CR bound is dominant. No trade-off relation exists. Figure 5.2c shows an example of the bound given by the current analysis with $|\langle L\rangle_0| = 1/2$. The parameters used are $\kappa = 1$, $2\kappa_{\mathrm{a}}^2 = 1.2$, $2\kappa_{\mathrm{b}}^2 = 1$, and thus $|\langle L\rangle_0| = 1 < 1/2$ holds. The RLD and the SLD CR bounds have no intersection point. The SLD CR bound is dominant. No trade-off relation exists. The strength of the trade-off relation $\Omega^{\mathrm{Tradeoff}}$ is given by

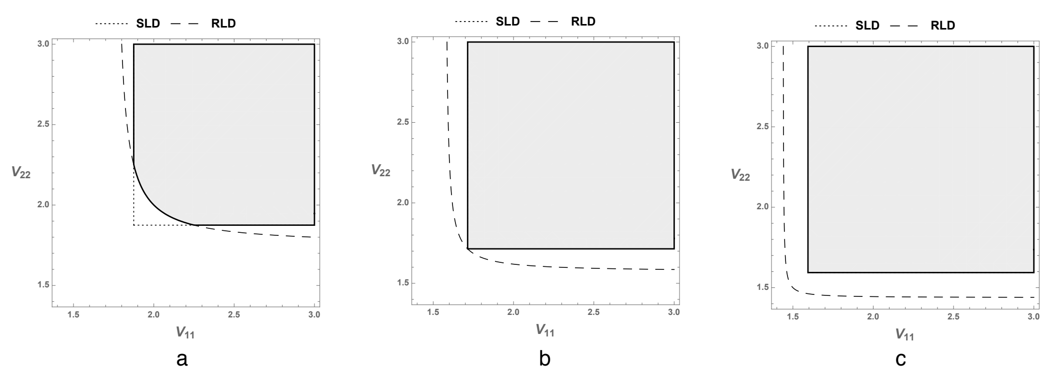

Figure 5.2: Bounds given by quantum Cramér-Rao inequalites. The temperature parameters used are $2\kappa_{\mathrm{a}}^2 = 2$, $2\kappa_{\mathrm{b}}^2 = 1$, and thus $|\langle L\rangle_0| = 1 > 1/2$ for a, $2\kappa_{\mathrm{a}}^2 = 1.5$, $2\kappa_{\mathrm{b}}^2 = 1$, and thus $|\langle L\rangle_0| = 1/2$ for b, $2\kappa_{\mathrm{a}}^2 = 1.2$, $2\kappa_{\mathrm{b}}^2 = 1$, and thus $|\langle L\rangle_0| = 0.2 < 1/2$ for c. The κ is set as 1 for all three cases. The allowed region of the MSE matrix components (V_{11}, V_{22}) is the gray region. The allowed region of model is given by the region covered by both the SLD Cramér-Rao bound (dotted lines) and the RLD CR bound (dashed lines).

$$\Omega^{\mathrm{Tradeoff}} = \frac{\Delta}{J_{\mathrm{S}\,11}^{-1} - J_{\mathrm{R}\,11}^{-1}} = \frac{\kappa^2}{2(1+2\kappa_{\mathrm{a}}^2+2\kappa_{\mathrm{b}}^2)}(4\langle L\rangle_0^2 - 1) = \Delta g(4\langle L\rangle_0^2 - 1), \tag{5.36}$$

if $\Delta > 0$ holds. Otherwise, $\Omega^{\mathrm{Tradeoff}} = 0$. Figure 5.3 shows $\Delta g(4\langle L\rangle_0^2 - 1)$ as a function of $\langle L\rangle_0$ at three different $\beta\omega$'s which are the same as Fig. 5.1.

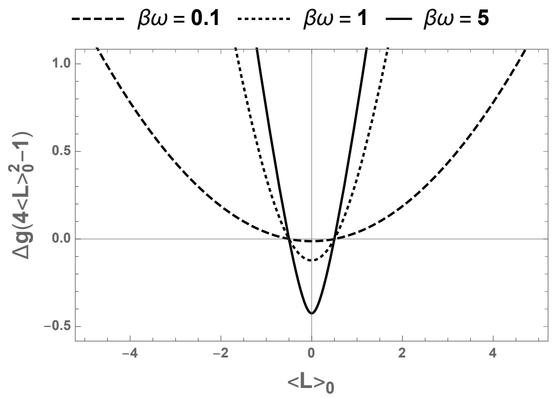

Figure 5.3: $\Delta g(4\langle L\rangle_0^2 - 1)$ as a function of $\langle L\rangle_0$. $\Omega^{\mathrm{Tradeoff}} = \Delta g(4\langle L\rangle_0^2 - 1)$ for $|\langle L\rangle_0| > 1/2$ and $\Omega^{\mathrm{Tradeoff}} = 0$ for $|\langle L\rangle_0| \leq 1/2$.

When $|\langle L\rangle_0| \leq 1/2$, Δ is negative as shown in Eq 5.33, the RLD CR bound stays always below the SLD CR bound. This is consistent with $(J_{\mathrm{S}})^{-1} \geq (J_{\mathrm{R}})^{-1}$ when $|\langle L\rangle_0| \leq 1/2$. At larger $\beta\omega$ (lower temperature), the possible ranges of V_{11} and V_{22} given by the RLD CR bound becomes larger at the same $\langle L\rangle_0$. Finally, we briefly discuss achievability of the trade-off relation above. It is known that the RLD CR bound is (asymptotically) achievable, if and only when the model is D-invariant [54]. This condition is checked by comparing two matrices, the inverse of the RLD Fisher information matrix and the Z matrix ($Z = [Z_{jk}]$) which is defined by [68]

$$Z_{jk} = \mathrm{tr}(\rho_\theta L_{\mathrm{S},\theta}^{k} L_{\mathrm{S},\theta}^{j}), \tag{5.37}$$

where

$$L_{\mathrm{S},\theta}^{j} = \sum_{k=1}^{n} J_{\mathrm{S},\theta\,kj}^{-1} L_{\mathrm{S},\theta,\mathrm{k}}. \tag{5.38}$$

As given in Appendix D.2, $(J_{\mathrm{R}})^{-1}$ and the Z matrix Z are different. Hence, the RLD CR bound is not tight. We next examine if the SLD CR bound is achievable or not. In Refs. [67, 68], the necessary and sufficient conditions are derived for asymptotically achievability of the SLD CR bound. The simplest condition is that the imaginary part of the Z matrix is zero. In our model, this is equivalent to $\langle L\rangle_0 = 0$ which is also equivalent to $\kappa_{\mathrm{a}} = \kappa_{\mathrm{b}}$ [Eq. (5.23)]. When $\langle L\rangle_0 \neq 0$, neither the RLD CR bound nor SLD CR bound is even asymptotically achievable. Therefore, the trade-off relation is not tight, except for the special choice of the parameter, $\langle L\rangle_0 = 0$.

5.4 Conclusion

We have investigated the trade-off relation for estimating x and y components of the position of one electron in a uniform magnetic field. In the present study, the trade-off relation upon estimating the expectation values of the two commuting observables, $(x,\ y)$ was derived in the framework of the quantum estimation theory. As the generators of the unitary transformation, we use a set of canonical momenta, p_x and p_y. Based on the analysis by the quantum estimation theory, we get non-trivial bounds that give the trade-off relations between the two commuting observables, x and y, unlike the result of Heisenberg-Robertson type uncertainty relation.

Before closing this chapter, we make two remarks. First, the CR bound of Model with respect to the thermal state reference state is not achievable except for $\langle L \rangle_0 = 0$. A possible extension might be an analysis by minimizing a weighted trace of the mean square error matrix [8]. However, this method gives asymptotically achievable bound only. Second, for the thermal state with the $\langle L \rangle_0$ constraint, we see the change in the bound shape depending on $\langle L \rangle_0$. We have no clue as to why the bound shape changes at $\langle L \rangle_0 = 1/2$ in a simple physical picture so far. It should be worthwhile seeing why the bound shape changes there.

Chapter 6

Spin-1/2 relativistic particle: SLD CR bound for moving observer [69]

With a similar model in chapter 5, we treat the physical model by taking the relativity [70, 31, 32] into account. Our purpose is to investigate if estimation accuracy is affected and, if so, how much estimation accuracy of a moving observer is affected.

We obtain the accuracy limit for estimating the expectation value of the position of a relativistic particle for an observer moving along one direction at a constant velocity. We set a specific model of a relativistic spin-1/2 particle described by a gaussian wave function with a spin down in the rest frame. To derive the state vector of the particle for the moving observer, we use the Wigner rotation [31, 32] that entangles the spin and the momentum of the particle. Based on this wave function for the moving frame, we obtain the symmetric logarithmic derivative (SLD) Cramér-Rao bound that sets the estimation accuracy limit for an arbitrary observer in motion. It is shown that estimation accuracy decreases monotonically in the observer's velocity when the moving observer does not measure the spin degree of freedom. This implies that the estimation accuracy limit worsens with increasing the observer's velocity, but it is finite even in the relativistic limit. We derive the amount of this information loss by the exact calculation of the SLD Fisher information matrix in an arbitrary moving frame.

6.1 Model

We assume that an observer moves along the z axis with a constant velocity v. We choose the z direction as the moving direction, because we expect that this direction gives the most significant change in the rotation of spin as a massive relativistic spin-1/2 particle on the x-y plane [23]. We use the natural unit, i.e., $\hbar = 1$ and $c = 1$ unless otherwise stated. The mass of the particle is m. As a metric tensor $g_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$, we choose $g_{\mu\nu} = (+1, -1, -1, -1)$.

6.1.1 State in the rest frame

The wave function of the particle is set as a gaussian function of x and y with a plane wave in the z coordinate. For simplicity, we set the wave number, or the momentum along the z direction as zero. To apply the Wigner rotation as described in [31, 32], we mainly use the momentum representation in the following discussion.

The state of the particle is in a known pure state called a reference state. The reference state ρ_0 in the rest frame is

$$\rho_0 = |\Psi_\downarrow\rangle \langle\Psi_\downarrow| , \tag{6.1}$$

$$|\Psi_\downarrow\rangle = \int d^3p\, \varphi_0(p^1)\varphi_0(p^2)\delta(p^3)\, |\vec{p}, \downarrow\rangle , \tag{6.2}$$

where $\delta(p^3)$ denotes the Dirac delta function to represent the plane wave in the z direction. The momentum vector $\vec{p}$ is a spatial part of the four-momentum p^μ, i.e., $\vec{p} = (p^1, p^2, p^3)$. The state vectors $|\vec{p}, \downarrow\rangle$ and $|\vec{p}, \uparrow\rangle$ are the momentum eigenstates with down and up spins, respectively. The $\varphi_0(p)$ is defined by the gaussian function as

$$\varphi_0(p) = \frac{\kappa^{1/2}}{\pi^{1/4}}\, e^{-\frac{1}{2}\kappa^2 p^2}. \tag{6.3}$$

The κ determines the spread of the wave function in the coordinate representation, i.e., the spread of the wave function in the coordinate representation becomes broader as κ increases. The spread κ is a quantity that an experimenter chooses at his/her will.

A quantum parametric model is defined by a two-parameter unitary model as

$$\mathcal{M}_{\mathrm{rest}} = \{\rho_\theta \,\big|\, \theta = (\theta_1, \theta_2) \subset \mathbb{R}^2\}, \tag{6.4}$$

where ρ_θ is generated by the momentum operators in the x and y direction, $\hat{p}^1$ and $\hat{p}^2$, respectively,

$$\rho_\theta = U(\theta)\rho_0 U^\dagger(\theta) = U(\theta)\, |\Psi_\downarrow\rangle \langle\Psi_\downarrow|\, U^\dagger(\theta), \tag{6.5}$$

with

$$U(\theta) = e^{-i\hat{p}^1\theta_1 - i\hat{p}^2\theta_2}. \tag{6.6}$$

The operator $\hat{p}^i$ ($i = 1, 2$) are the momentum operator of ith component, i.e., $\hat{p}^i\, |\vec{p}, \sigma\rangle = p^i\, |\vec{p}, \sigma\rangle$, ($\sigma =\downarrow, \uparrow$). Let us define a state vector $|\Psi_\downarrow(\theta)\rangle$ by

$$|\Psi_\downarrow(\theta)\rangle = U(\theta)\, |\Psi_\downarrow\rangle \tag{6.7}$$

$$= \int d^3p\, \varphi_0(p^1)\varphi_0(p^2)\delta(p^3) e^{-ip^1\theta_1 - ip^2\theta_2}\, |\vec{p}, \downarrow\rangle . \tag{6.8}$$

Then, Eq. (6.5) is expressed as

$$\rho_\theta = |\Psi_\downarrow(\theta)\rangle \langle\Psi_\downarrow(\theta)| . \tag{6.9}$$

The physical implication of the parameter θ is that it is the peak position of the wave function in the coordinate representation. Alternatively, we consider position operators $\hat{x}^j$, which are canonical conjugate of the momentum operators $\hat{p}^j$. ($j = 1, 2$) [71]. From Eq. (6.6), we have

$$U^\dagger(\theta)\hat{x}^j U(\theta) = \hat{x}^j + \theta_j \quad (j = 1, 2). \tag{6.10}$$

The unitary transformation $U(\theta)$ gives a shift by θ_j to a position operator $\hat{x}^j$. By assumption, we know the reference state ρ_0. However, we do not know θ_1 or θ_2. We estimate the parameters θ_1 and θ_2 encoded in $\rho_\theta = U(\theta)\, |\Psi_\downarrow\rangle \langle\Psi_\downarrow|\, U^\dagger(\theta)$. By doing so, we have an estimate for the expectation value of the position operators $\hat{x}^1$ and $\hat{x}^2$ as seen in Eq. (6.10).

The parametric model (6.4) in the rest frame is a classical model in the following sense.

Firstly, two parameters are totally uncorrelated since the state vector (6.8) is also expressed as the tensor product form,

$$|\Psi_{\downarrow}(\theta)\rangle = |\psi_1(\theta_1)\rangle\,|\psi_2(\theta_2)\rangle\,|p^3 = 0\rangle\,|\downarrow\rangle\,, \tag{6.11}$$

with

$$|\psi_j(\theta_j)\rangle = \int dp^j\,\varphi_0(p^j)\mathrm{e}^{-\mathrm{i}p^j\theta_j}\,|p^j\rangle\,, \quad (j = 1, 2). \tag{6.12}$$

Secondly, an optimal measurement to estimate θ_j is the position operator $\hat{x}^j$. Optimal measurements for θ_1 and θ_2 commute and hence we can simultaneously perform the optimal measurement. Thirdly, upon measuring the position operators, the measurement outcomes obey the independent classical gaussian distributions with the mean (θ_1, θ_2) and their variances $(\kappa^2/2, \kappa^2/2)$. Thus, the optimal unbiased estimator is given by the sample mean.

6.1.2 Quantum Fisher information in the rest frame

The symmetric logarithmic derivative (SLD) $L_j(\theta)$ of the pure state model Eq. (6.4) is calculated is given by [48]

$$L_j(\theta) = 2\partial_j[|\Psi_{\downarrow}(\theta)\rangle\,\langle\Psi_{\downarrow}(\theta)|], \tag{6.13}$$

where $\partial_j = \partial/\partial\theta_j$. Throughout this chapter, we consider the SLD only. Hence we omit subscript "s" in this chapter. By a direct calculation, we obtain the commutator of the SLDs as

$$[L_1(\theta),\, L_2(\theta)] = 4(|\partial_1\Psi_{\downarrow}(\theta)\rangle\,\langle\partial_2\Psi_{\downarrow}(\theta)| - |\partial_2\Psi_{\downarrow}(\theta)\rangle\,\langle\partial_1\Psi_{\downarrow}(\theta)|), \tag{6.14}$$

where $|\partial_j\Psi_{\downarrow}(\theta)\rangle = \partial_j\,|\Psi_{\downarrow}(\theta)\rangle$, $(j = 1, 2)$. Similar notations will be used throughout the thesis. We remark that the SLDs do not commute in this particular choice of SLDs.

At first sight, this non-commutativity seems to contradict the fact that the parametric model in the rest frame is a classical one. A resolution is that the choice of the SLDs above is not unique [48]. As an example, we have another choice of the SLDs, $\tilde{L}_j(\theta)$ $(j = 1, 2)$ as follows.

$$\tilde{L}_1(\theta) = 2\partial_1(|\psi_1(\theta)\rangle\,\langle\psi_1(\theta)|) \otimes \mathrm{I}_2 \otimes \mathrm{I}_3 \otimes |\downarrow\rangle\,\langle\downarrow|\,, \tag{6.15}$$

$$\tilde{L}_2(\theta) = \mathrm{I}_1 \otimes 2\partial_2(|\psi_2(\theta)\rangle\,\langle\psi_2(\theta)|) \otimes \mathrm{I}_3 \otimes |\downarrow\rangle\,\langle\downarrow|\,, \tag{6.16}$$

where

$$I_k = \int dp^k\;|p^k\rangle\,\langle p^k|\,, \quad (k = 1,\, 2,\, 3). \tag{6.17}$$

These SLDs $\tilde{L}_j(\theta)$ satisfy the definition of SLD and indeed they do commute each other.

The SLD Fisher information matrix $J(\theta) = [J_{jk}(\theta)]$ is obtained by the formula in [48] as

$$J_{jk} = 4(\langle\partial_j\Psi_{\downarrow}(\theta)|\partial_k\Psi_{\downarrow}(\theta)\rangle + \langle\Psi_{\downarrow}(\theta)|\partial_j\Psi_{\downarrow}(\theta)\rangle\,\langle\Psi_{\downarrow}(\theta)|\partial_k\Psi_{\downarrow}(\theta)\rangle). \tag{6.18}$$

In the following discussion, we drop θ in the SLD Fisher information matrix, because J is independent of θ due to the unitarity of the model. By a straightforward calculation involving the standard gaussian integrals, we have

$$J_{jk} = \frac{2}{\kappa^2}\delta_{jk}, \quad (j, k = 1, 2). \tag{6.19}$$

The alternative SLDs $\tilde{L}_j(\theta)$ Eqs. (6.15) and (6.16) give the same SLD Fisher information matrix Eq. (6.19). The inverse of the SLD Fisher information matrix $J^{-1} = [J^{-1}_{jk}]$ is also diagonal as follows.

$$J^{-1}_{jk} = \frac{\kappa^2}{2}\delta_{jk}. \tag{6.20}$$

The SLD CR inequality is expressed as

$$\mathsf{V} \geq J^{-1}, \tag{6.21}$$

where $\mathsf{V} = [\mathsf{V}_{jk}]$ is the mean square error (MSE) matrix. With Eq. (6.20), we have

$$\mathsf{V}_{11} \geq \frac{\kappa^2}{2}, \quad \mathsf{V}_{22} \geq \frac{\kappa^2}{2}. \tag{6.22}$$

The estimation accuracy limit regarding the expectation value of the position operator is proportional to κ^2 which determines the spread of the wave function in the coordinate representation. It is easy to see J^{-1} approaches the zero matrix as $\kappa \to 0$. At the limit of $\kappa \to 0$, the wave function in the coordinate representation becomes the Dirac delta function. This allows us to estimate the parameter θ without any error.

6.1.3 State in a moving frame

We next consider an observer moving along the z axis with respect to the rest frame. A Lorentz transformation κ from the rest frame to this moving frame is

$$\Lambda = \begin{pmatrix} \cosh\chi & 0 & 0 & -\sinh\chi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\chi & 0 & 0 & \cosh\chi \end{pmatrix}, \tag{6.23}$$

$$\cosh\chi = \frac{1}{\sqrt{1 - v^2}}, \quad \sinh\chi = \frac{v}{\sqrt{1 - v^2}}. \tag{6.24}$$

v is a velocity of the observer moving along the z axis. By this Lorentz transformation, the momentum of the particle is transformed as in classical physics. We define the spatial part of the four-momentum, $\overrightarrow{\Lambda p}$ by

$$\Lambda p = ((\Lambda p)_0, (\Lambda p)_1, (\Lambda p)_2, (\Lambda p)_3) = ((\Lambda p)_0, \overrightarrow{\Lambda p}). \tag{6.25}$$

Then, $\overrightarrow{\Lambda p}$ is given by

$$\overrightarrow{\Lambda p} = \left(\sum_{\mu=0}^{3} \Lambda^1{}_\mu p^\mu,\ \sum_{\mu=0}^{3} \Lambda^2{}_\mu p^\mu,\ \sum_{\mu=0}^{3} \Lambda^3{}_\mu p^\mu \right) \tag{6.26}$$

$$= (p^1, p^2, -p^0 \sinh\chi), \tag{6.27}$$

where $p^0 = \sqrt{m^2 + |\vec{p}|^2}$. The m is the mass of the particle in the rest frame. See for example [31].

For a relativistic spin-1/2 particle, the Lorentz transformation Λ also gives rise to a unitary transformation $U(\Lambda)$ acting on the state vector. This is described by the Wigner rotation [31, 32] (See a short summary in Appendix E.1.). In our model, the state vector in the rest frame is in a spin down state, $|\Psi_\downarrow(\theta)\rangle$. The state vector $|\Psi_\downarrow(\theta)\rangle$ is transformed to $|\Psi^\Lambda(\theta)\rangle$ as

$$|\Psi^\Lambda(\theta)\rangle = U(\Lambda)\,|\Psi_\downarrow(\theta)\rangle = \sum_{\sigma=\downarrow,\uparrow} |\Psi^\Lambda_\sigma(\theta)\rangle\,. \tag{6.28}$$

We remark here that $|\Psi^\Lambda_\sigma(\theta)\rangle$, $(\sigma =\downarrow,\uparrow)$ are not normalized. It is convenient to express the state vectors $|\Psi^\Lambda_\sigma(\theta)\rangle$, $(\sigma =\downarrow,\uparrow)$ as

$$|\Psi^\Lambda_\sigma(\theta)\rangle = |\psi^\Lambda_\sigma(\theta)\rangle\,|\sigma\rangle\,. \tag{6.29}$$

The explicit form of $|\psi^\Lambda_\sigma(\theta)\rangle$ is given by

$$|\psi^\Lambda_\sigma(\theta)\rangle = \int d^3p \sqrt{\frac{(\Lambda p)^0}{p^0}} F_{\theta,\sigma}(p^1, p^2)\delta(p^3)\,|\overrightarrow{\Lambda p}\rangle\,, \tag{6.30}$$

$$F_{\theta,\downarrow}(p^1, p^2) = \varphi_0(p^1)\varphi_0(p^2)\mathrm{e}^{-\mathrm{i}p^1\theta_1 - \mathrm{i}p^2\theta_2} \cos\frac{\alpha(|\vec{p}\,|)}{2}, \tag{6.31}$$

$$F_{\theta,\uparrow}(p^1, p^2) = -\varphi_0(p^1)\varphi_0(p^2)\mathrm{e}^{-\mathrm{i}p^1\theta_1 - \mathrm{i}p^2\theta_2}\mathrm{e}^{\mathrm{i}\phi(p^1,\,p^2)} \sin\frac{\alpha(|\vec{p}\,|)}{2}, \tag{6.32}$$

$$|\vec{p}\,| = \sqrt{(p^1)^2 + (p^2)^2}, \tag{6.33}$$

$$\mathrm{e}^{\mathrm{i}\phi(p^1,\,p^2)} = \frac{p^1}{|\vec{p}\,|} + \mathrm{i}\frac{p^2}{|\vec{p}\,|}, \tag{6.34}$$

$$\cos\alpha(|\vec{p}\,|) = \frac{\sqrt{m^2 + |\vec{p}\,|^2} + m\cosh\chi}{\sqrt{m^2 + |\vec{p}\,|^2}\cosh\chi + m}, \tag{6.35}$$

$$\sin\alpha(|\vec{p}\,|) = -\frac{|\vec{p}\,|\sinh\chi}{\sqrt{m^2 + |\vec{p}\,|^2}\cosh\chi + m}. \tag{6.36}$$

In the expressions above, m denotes the rest mass of the spin-1/2 particle.

The Lorentz boost gives a non-zero probability density of spin up state as shown in Eq. (6.32). This makes the particle spin 'rotate', and hence is called the Wigner rotation. Detailed derivations of Eqs. (6.28), (6.30), (6.31), and (6.32) are given in Appendix E.1.

We remark that the states $|\Psi^\Lambda(\theta)\rangle$ are entangled with respect to the momentum and the spin degrees of freedoms. For the observer moving along the z axis, the spin has a component of spin up which is none at the rest frame, i.e., the spin rotates as the observer moves.

6.2 Parameter estimation: moving frame

We are now in position to discuss parameter estimation in the moving frame. Suppose that a moving observer wishes to estimate the parameter θ encoded in the state Eq. (6.28). The system under discussion has two different degrees of freedoms. One is a continuous part describing the wave function, and the other is the spin. It is natural to measure the continuous degree of freedom to estimate the parameter as the observer does not know whether s/he is in a moving frame or not. In this setting, the moving observer does not have access to the spin degree of freedom. Then, our parametric model is given by tracing out the spin of from the pure state Eq. (6.28).

As comparison, we also give a short account on other possible cases. The first is when the moving observer measures the both degrees of freedoms. This will be discussed in Sec. 6.2.1. The other case is when the spin of the particle is measured only, which will be given in Sec. 6.3.1.

6.2.1 Invariance of quantum Fisher information after the Lorentz boost

We first consider the situation where the moving observer measures the whole state Eq. (6.28). The parametric model for this case is defined as follows.

$$\mathcal{M}_{\text{boost}} = \{\, |\Psi^{\Lambda}(\theta)\rangle\langle\Psi^{\Lambda}(\theta)| \,\big|\, \theta = (\theta_1, \theta_2) \in \mathbb{R}^2\}. \tag{6.37}$$

It is clear that this model is unitary equivalent to the model in the rest frame, since the difference is only given by the unitary transformation $U(\Lambda)$. To phrase it differently, we can regard the model after the Lorentz boost in the different representation. Therefore, the SLD Fisher information matrix is exactly same as in the rest frame, Eq. (6.20). While this is true mathematically, the physical meanings of these two models are different.

Let us further elaborate on physics of the two models; the one in the rest frame Eq. (6.4) and the other Eq. (6.37) in the moving frame. The unitary transformation $U(\Lambda)$ which defines the Wigner rotation is parameter independent, and hence, two parametric models are equivalent. Yet, the significance of the Lorentz transformation is that $U(\Lambda)$ depends on the velocity v of the moving observer with respect to the rest frame. The resulting state-vector after the Lorentz boost Eq. (6.28) indeed depends on v in a non-trivial manner. Furthermore, the wave function in the moving frame is no longer described by the simple gaussian wave function as given in Eqs. (6.31) and (6.32). In particular, the two parameters θ_1 and θ_2 are not described by a tensor product of two independent parametric models as in the rest frame. Nevertheless, we can formally express an optimal measurement for the model after the Lorentz boost by the pair of observables

$$U(\Lambda)\hat{x}^j U^{\dagger}(\Lambda), \quad (j = 1, 2),$$

which obviously commute each other. We will not give further analysis on these observables, but it is obvious that experimental implementation of this optimal measurement is much more complex. It may not be feasible as it will depend on the velocity v.

6.2.2 Parametric model in the moving frame

We now analyze the parametric model when the moving observer does not measure the spin of the particle. By taking the partial trace over the spin σ, we have

$$\rho^\Lambda(\theta) = \mathrm{tr}_\sigma \, |\Psi^\Lambda(\theta)\rangle \langle \Psi^\Lambda(\theta)| \tag{6.38}$$

$$= \sum_{\sigma=\downarrow,\uparrow} \langle \sigma|\Psi^\Lambda(\theta)\rangle \langle \Psi^\Lambda(\theta)|\sigma\rangle \tag{6.39}$$

$$= \sum_{\sigma=\downarrow,\uparrow} |\psi^\Lambda_\sigma(\theta)\rangle \langle \psi^\Lambda_\sigma(\theta)| \,. \tag{6.40}$$

With this $\rho^\Lambda(\theta)$, we define the parametric model of interest as

$$\mathcal{M}^\Lambda = \{\rho^\Lambda(\theta) \,\big|\, \theta = (\theta_1, \theta_2) \in \mathbb{R}^2\}. \tag{6.41}$$

As noted before, the state vectors $|\psi^\Lambda_\sigma(\theta)\rangle$ are unnormalized. Let us evaluate the inner products $\langle \psi^\Lambda_\sigma(\theta)|\psi^\Lambda_\sigma(\theta)\rangle$ to analyze the amplitudes of the each spin state. From Eqs. (6.30), (6.31), and (6.32), by a straightforward calculation shown in Appendix E.2.1 , we obtain the inner products as follows.

$$\langle \psi^\Lambda_\downarrow(\theta)|\psi^\Lambda_\downarrow(\theta)\rangle = \frac{1}{2}(1+\xi), \tag{6.42}$$

$$\langle \psi^\Lambda_\uparrow(\theta)|\psi^\Lambda_\uparrow(\theta)\rangle = \frac{1}{2}(1-\xi), \tag{6.43}$$

$$\langle \psi^\Lambda_\uparrow(\theta)|\psi^\Lambda_\downarrow(\theta)\rangle = 0, \tag{6.44}$$

where

$$\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\varphi_0(p^1)\varphi_0(p^2)|^2 \cos\alpha(|\vec{p}\,|)dp^1dp^2 \tag{6.45}$$

$$= 2\kappa^2 \int_0^\infty dt\, t\, \mathrm{e}^{-\kappa^2 t^2} \frac{\sqrt{m^2+t^2}\sqrt{1-v^2}+m}{\sqrt{m^2+t^2}+m\sqrt{1-v^2}}. \tag{6.46}$$

The ξ is an indicator of the spin rotation by the Lorentz boost as seen in Eqs. (6.42) and (6.43). As the result, it depends only on the observer's velocity v. The smaller ξ becomes, the larger the amplitude of the spin up state. Therefore, the spin rotates. We remark that the state vectors $|\psi^\Lambda_\uparrow(\theta)\rangle$ and $|\psi^\Lambda_\downarrow(\theta)\rangle$ are orthogonal.

By using the normalized state vector $|\bar{\psi}^\Lambda_\sigma(\theta)\rangle$ defined by

$$|\bar{\psi}^\Lambda_\sigma(\theta)\rangle = \frac{|\psi^\Lambda_\sigma(\theta)\rangle}{\sqrt{\langle \psi^\Lambda_\sigma(\theta)|\psi^\Lambda_\sigma(\theta)\rangle}}, \tag{6.47}$$

we write $\rho^\Lambda(\theta)$ as a convex combination of two pure states $|\bar{\psi}^\Lambda_\downarrow(\theta)\rangle\langle\bar{\psi}^\Lambda_\downarrow(\theta)|$ and $|\bar{\psi}^\Lambda_\uparrow(\theta)\rangle\langle\bar{\psi}^\Lambda_\uparrow(\theta)|$, i.e.,

$$\rho^\Lambda(\theta) = \frac{1}{2}(1+\xi)\,|\bar{\psi}^\Lambda_\downarrow(\theta)\rangle\langle\bar{\psi}^\Lambda_\downarrow(\theta)| + \frac{1}{2}(1-\xi)\,|\bar{\psi}^\Lambda_\uparrow(\theta)\rangle\langle\bar{\psi}^\Lambda_\uparrow(\theta)| \,. \tag{6.48}$$

At any given κ, the ξ takes its maximum value 1 at $v = 0$ which means no spin rotation. It takes its minimum value ξ_{rel} in the relativistic limit of $v \to 1$ which corresponds to $v \to c$ in the

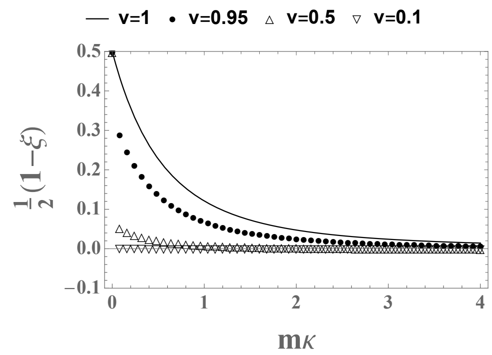

Figure 6.1: Numerically calculated $(1-\xi)/2$ as a function of $m\kappa$ at $v = 1, 0.95, 0.5$, and 0.1. The set of the velocity v is chosen differently to make the distance between the plots more even.

standard unit. An explicit expression of $\xi_{\rm rel}$ is

$$\xi_{\rm rel} = \sqrt{\pi} m\kappa\, \mathrm{e}^{m^2\kappa^2} \mathrm{erfc}(m\kappa), \tag{6.49}$$

where erfc(x) is the complementary error function defined by

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty dt\, \mathrm{e}^{-t^2}. \tag{6.50}$$

The derivations of Eqs. (6.42), (6.43), (6.46), and (6.49) are given in Appendix E.2.1. The probability for the spin up state reaches its maximum $1/2$ at the limit of $\kappa \to 0$ and at the relativistic limit. Figure 6.1 shows the probability of the spin up state $\langle \psi^\Lambda_\uparrow(\theta)|\psi^\Lambda_\uparrow(\theta)\rangle = (1-\xi)/2$ as a function of $m\kappa$ at $v = 0.95, 0.5$, and 0.1. The set of the velocities v is chosen differently to make the distance between the plots more even. Figure 6.1 shows its maximum $(1-\xi_{\rm rel})/2$ as well.

Let us analyze these state vectors in the coordinate representation. We define the wave function of a particle with spin up in the coordinate representation $\psi^\Lambda_\uparrow(x)$ by

$$\psi^\Lambda_\uparrow(x) = \langle x|\bar{\psi}^\Lambda_\uparrow(\theta)\rangle\,\Big|_{\theta=0}. \tag{6.51}$$

A derivation of its explicit expression is given in Appendix E.3. Figure 6.2 shows numerically calculated densities $|\psi^\Lambda_\uparrow(x)|^2$ for $\kappa = 0.1$ as a function of the position x^1 for $v = 0.99,\ 0.98,\ 0.7$, and 0.1. For simplicity, we set $(\theta_1, \theta_2) = (0, 0)$ and $x^2, x^3 = 0$. To convert the wave function in the moment representation to that in the coordinate representation, we make a Fourier transform of the the wave function in the moment representation. Therefore, $|\psi^\Lambda_\uparrow(x) = \psi^{\Lambda^*}_\uparrow(-x)|$ holds. This is confirmed in Appendix E.3. Then, for the probability density $|\psi^\Lambda_\uparrow(x)|^2 = |\psi^\Lambda_\uparrow(-x)|^2$. Because of this, we plot $\psi^\Lambda_\uparrow(x)$ for $x^1 \geq 0$ in Fig. 6.2.

It is worth noting that the peak of the spin up wave function $\psi^\Lambda_\uparrow(x)$ is no longer at $x^1 = \theta_1 = 0$. To see the dependence of the observer's velocity v on the peak position, we numerically calculate the derivative of $|\psi^\Lambda_\uparrow(x)|^2$. Figure 6.3 shows the derivative of $|\psi^\Lambda_\uparrow(x)|^2$ as a function of position. In this figure, we set $(\theta_1, \theta_2) = (0, 0)$ as well for simplicity. We observe that the faster the observer moves, the further the peak position moves away from $x^1 = \theta_1$. These numerically

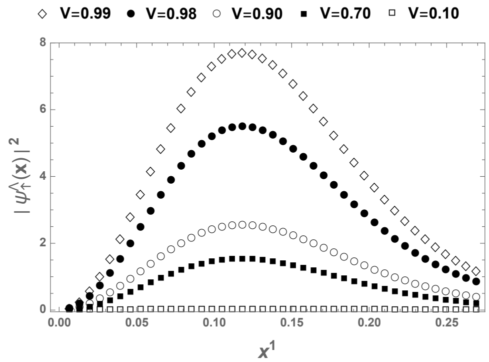

Figure 6.2: Numerically calculated probability density $|\psi^{\Lambda}_{\uparrow}(x)|^2$ for $\kappa = 0.1$ as a function of x^1 at $v = 0.99, 0.98, 0.9, 0.7$ and 0.1.

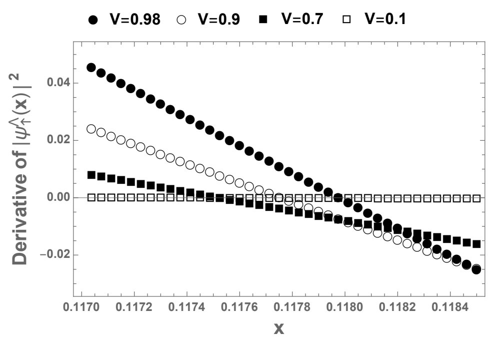

Figure 6.3: Numerically calculated derivative of the probability density $|\psi^{\Lambda}_{\uparrow}(x)|^2$ as a function of x^1 at $v = 0.98, 0.9, 0.7$ and 0.1.

verified facts indicate that the parametric model Eq. (6.41) is a convex mixture of two pure state models. One is centered at θ, and the other is centered at θ+some amount. If one performs the position measurement, the resulting probability distribution is thus given by a convex mixture of two distributions with different locations of the peak. This finding naturally invites us to say that estimation accuracy gets worse for the moving observer.

6.2.3 Quantum Fisher information matrices in the moving frame

SLD Fisher information matrix

The SLD Fisher information matrix $J^{\Lambda} = [J^{\Lambda}{}_{jk}]$ for the model (6.41) is calculated as follows.

$$J^{\Lambda}{}_{jk} = \frac{2}{\kappa^2}(1 - 2\kappa^2\eta^2)\delta_{jk}. \tag{6.52}$$

where

$$\eta = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp^1 dp^2 \frac{(p^1)^2}{|\vec{p}|}[\varphi_0(p^1)\varphi_0(p^2)]^2 \sin\alpha(|\vec{p}|). \tag{6.53}$$

A detailed explanation is given in Appendix E.4. By executing the integration over the angle of the two-dimensional polar coordinate, $\kappa\eta$ is expressed as

$$\kappa\eta = v\int_0^\infty dt \frac{\kappa'^3 t^3 \mathrm{e}^{-\kappa'^2 t^2}}{\sqrt{1+t^2}+\sqrt{1-v^2}}, \tag{6.54}$$

where $\kappa' = m\kappa$. From Eq. (6.52), we have the SLD CR inequality as follows.

$$\mathsf{V}_{11} \geq \frac{\kappa^2}{2}\frac{1}{1-2\kappa^2\eta^2}, \quad \mathsf{V}_{22} \geq \frac{\kappa^2}{2}\frac{1}{1-2\kappa^2\eta^2}. \tag{6.55}$$

As given in Appendix E.5, the denominator in Eq. (6.55), $1-2\kappa^2\eta^2$ is positive, and hence the accuracy limits for V_{11} and V_{22} are always finite.

By comparing the SLD CR inequalities for the rest frame Eq. (6.22) and Eq. (6.55), we see how much estimation accuracy is affected by the Lorentz boost. As an indicator, we take up the ratio of the $(1,1)$ components of $(J^\Lambda)^{-1}$ and $(J)^{-1}$. We define the ratio $\Delta(v)$ by

$$\Delta(v) = \frac{[(J^\Lambda)^{-1}]_{11}}{[(J)^{-1}]_{11}} = \frac{1}{1-2\kappa^2\eta^2}. \tag{6.56}$$

By definition, $\Delta(0) = 1$ for the rest frame. The ratio $\Delta(v)$ quantifies the amount of information loss due to the Lorentz boost. If it is larger, the moving observer can only estimate the parameter less accurately when compared to the rest frame. Figure 6.4 shows the ratio $\Delta(v)$ as a function the moving observer's velocity v at the different spreads of the wave function $\kappa = 0.1$, 0.5, 1.0, and 3.0. The set of the spreads κ is chosen to make the distance between the plots more even. The m is the rest mass of the particle. As shown in Appendix E.5, $\kappa\eta$ is a monotonically decreasing function of $\kappa' = m\kappa$, for any given velocity v. This then implies that $\kappa\eta$ reaches its maximum, $\sqrt{\pi}v/4$ at the limit of $\kappa \to 0$. Therefore, for $\Delta(v)$, we obtain the following inequality.

$$\Delta(v) \leq \lim_{\kappa\to 0}\Delta(v) = \frac{1}{1-\frac{\pi}{8}v^2}. \tag{6.57}$$

6.2.4 Quantum Fisher information matrices at the relativistic limit

SLD Fisher information matrix

We shall analyze the relativistic limit of our result in detail. Firstly, from Eq.(6.57), an upper bound for the relativistic limit of the ratio $\Delta(v)$ is given by

$$\Delta(1) \leq \frac{1}{1-\frac{\pi}{8}} \simeq 1.647. \tag{6.58}$$

This shows that the ratio is always finite.

Next, we calculate an explicit expression for the relativistic limit of the SLD Fisher information matrix $J^{\mathrm{rel}} = \lim_{v\to 1} J^\Lambda$. This is given by

$$J^{\mathrm{rel}}{}_{jk} = \frac{2}{\kappa^2}\left\{1-2\left[\frac{m\kappa}{2}+\frac{\sqrt{\pi}}{4}\mathrm{e}^{m^2\kappa^2}(1-2m^2\kappa^2)\,\mathrm{erfc}(m\kappa)\right]^2\right\}\delta_{jk}. \tag{6.59}$$

Derivation of $J^{\text{rel}} = \lim_{v\to 1} J^{\Lambda}$ Eq. (6.59)

We evaluate the $[J^{\Lambda}]_{jk}$ in the limit of $v \to 1$, i.e.,

$$\lim_{v\to 1}[J^{\Lambda}]_{jk} = \lim_{v\to 1} \frac{2}{\kappa^2}(1 - 2\kappa^2\eta^2)\delta_{jk}. \tag{6.60}$$

Hence, we evaluate the $\kappa\eta$ in the limit of $v \to 1$. Using Eq. (6.54), we have

$$\begin{aligned}\lim_{v\to 1}\kappa\eta &= \lim_{v\to 1} V \int_0^{\infty} dt \frac{\kappa'^3 t^3 \mathrm{e}^{-\kappa'^2 t^2}}{\sqrt{1+t^2} + \sqrt{1-v^2}} \\ &= \lim_{v\to 1} \int_0^{\infty} dt \frac{\kappa'^3 t^3 \mathrm{e}^{-\kappa'^2 t^2}}{\sqrt{1+t^2}} \\ &= \frac{m\kappa}{2} + \frac{\sqrt{\pi}}{4}\mathrm{e}^{m^2\kappa^2}(1 - 2m^2\kappa^2)\,\mathrm{erfc}(m\kappa).\end{aligned} \tag{6.61}$$

We use $\kappa' = m\kappa$. By substituting this result in Eq. (6.60), we obtain Eq. (6.59). □

It is worth noting that the $(J^{\text{rel}})^{-1}$ is finite even at the relativistic limit of $v \to 1$ which corresponds to that of $v \to c$ in the standard unit. To get a further insight into the property of J^{rel}, we consider two different limits in the spread κ of the wave function. We will analyze small and large κ limit of $(J^{\text{rel}})^{-1}$, as the estimation accuracy limit is quantified by the inverse of J^{rel}.

When the spread is extremely broader, $\kappa \gg 1$, with the help of the the asymptotic expansion of the complimentary error function $\mathrm{erfc}(x)$ (see Appendix E.5), an approximate expression of $[(J^{\text{rel}})^{-1}]_{11}$ is written as

$$[(J^{\text{rel}})^{-1}]_{11} \simeq [(J)^{-1}]_{11} + \frac{1}{4m^2}. \tag{6.62}$$

The difference between $[(J^{\text{rel}})^{-1}]_{11}$ and $[(J)^{-1}]_{11}$ is only a constant given by the particle mass.

When the spread is extremely narrower, $\kappa \ll 1$, on the other hand, by using the Taylor expansion (Appendix E.5), we have

$$[(J^{\text{rel}})^{-1}]_{11} \simeq \frac{[(J)^{-1}]_{11}}{1 - \frac{\pi}{8}} \simeq 1.647\,[(J)^{-1}]_{11}. \tag{6.63}$$

As also seen by Eq. (6.57), the relativistic effect for the SLD Fisher information matrix is more prominent when the spread is narrower.

6.3 Discussion

6.3.1 No information left in spin

We show that if the moving observer does not measure the continuous degree of freedom, the observer cannot estimate the parameter shift in the position by the following reasoning. In other words, there is no information left in the spin of the particle. Putting it differently, the Wigner rotation does not transfer the information about the parameter to the spin degree of freedom.

Suppose that the moving observer only measures the spin of the particle. We take the partial trace over the momentum $\vec{p}$ to obtain the reduced state $\rho^{\Lambda}_{\text{spin}}(\theta) = \mathrm{tr}_{\mathrm{p}}\,|\Psi^{\Lambda}(\theta)\rangle\langle\Psi^{\Lambda}(\theta)|$ for this

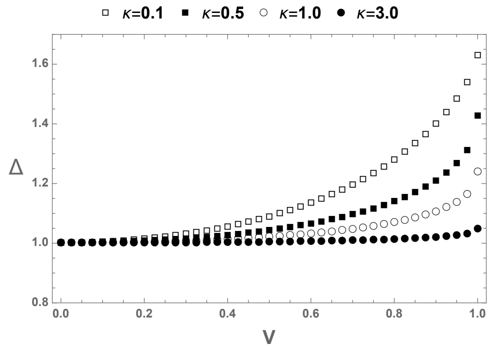

Figure 6.4: Numerically calculated ratio $\Delta(v)$ as a function of v, the velocity of the moving observer at $\kappa = 0.1, 0.5, 1.0$ and 3.0.

case. The parametric model is then given by

$$\mathcal{M}_{\rm spin} = \{\rho^{\Lambda}_{\rm spin}(\theta)\,|\,\theta = (\theta_1, \theta_2) \in \mathbb{R}^2\}. \tag{6.64}$$

A direct calculation with using Eq. (6.28) gives $\rho^{\Lambda}_{\rm spin}(\theta)$ as follows.

$$\rho^{\Lambda}_{\rm spin}(\theta) = \int d^3p\, \langle p|\Psi^{\Lambda}(\theta)\rangle \langle \Psi^{\Lambda}(\theta)|p\rangle \tag{6.65}$$

$$= \int dp^1 dp^2 \sum_{\sigma=\downarrow,\uparrow} \sum_{\sigma'=\downarrow,\uparrow} F_{\theta,\sigma}(p^1, p^2) F^*_{\theta,\sigma'}(p^1, p^2)\, |\sigma\rangle \langle \sigma'|\,. \tag{6.66}$$

From Eqs. (6.31) and (6.32), we see that the integrand $F_{\theta,\sigma}(p^1, p^2)F^*_{\theta,\sigma'}(p^1, p^2)$, $(\sigma, \sigma' =\downarrow, \uparrow)$ does not depend on the parameter θ, since the phases cancel each other. Thus, in this situation, we cannot estimate the parameter of the model (6.64) at all, since the reduced state $\rho^{\Lambda}_{\theta,\rm spin}$ does not depend on the parameter.

6.3.2 Effects of the Wigner rotation

We now discuss the effect of the Wigner rotation on estimation accuracy in our model. As we have seen in Sec. 6.2.2, the Wigner rotation gives the amplitudes of both the spin up and spin down states When a moving observer measures the momentum only, the observer ends up seeing the effect of the Wigner rotation as the mixture of two different pure states Eq. (6.30). This then gives rise to the information loss, as the measurement of the momentum only is not complete. This information loss for the moving observer is, of course, expected. This is because the effect of the Wigner rotation followed by the partial trace is a completely-positive and trace-preserving map. Therefore, the SLD Fisher information should decrease by the monotonicity of the quantum Fisher information. One of the non-trivial findings of our thesis is that the explicit formula for this information loss as a function of the velocity of the observer.

We further elaborate on the parametric model for a moving observer. The wave function of the spin up state $\psi^{\Lambda}_{\uparrow}(x)$, which does not exist in the rest frame, appears due to the Wigner rotation. The peak of the probability density $|\psi^{\Lambda}_{\uparrow}(x)|^2$ no longer exists at $(x^1, x^2) = (\theta_1, \theta_2)$. Our numerical calculation indicates that it moves further away from the point (θ_1, θ_2) as the velocity of the observer v increases (Fig. 6.2 and Fig. 6.3). Because of this extra peak, estimation of the expectation values of the position operators is disturbed; therefore, the SLD CR bound increases. The ratio of the upper bound of the moving frame to the rest frame is given by $(1 - 2\kappa^2\eta^2)^{-1}$, where η is explicitly expressed as the integral form.

Next, we comment on the role of the spread of the wave function. In the rest frame, κ should be as small as possible to have better estimation accuracy. In Fig. 6.4, the ratio of estimation accuracies $\Delta(v)$ is shown to be monotonically decreasing in κ for a fixed velocity v. This means that the information loss for a moving observer is reduced by choosing relatively large κ. However, this results in losing estimation accuracy as the wider spread in general enables us less accurate estimation. Therefore, we expect the existence of a tradeoff relationship for a moving observer to design the best spread to gain the best information available. The investigation of this tradeoff will be subject to the future work.

The relativistic limit of the SLD Fisher information is also a rather unexpected result. In Fig. 6.2, we numerically evaluated the relativistic behaviors of the density for the spin up state. As the velocity approaches to the speed of light, we observe that the hight of the peak increases rapidly. This implies that the peak diverges in the relativistic limit. This is partially because the Lorentz transformation (6.24) does not have a well-defined limit. However, the SLD Fisher information matrix remains finite even in this limit, which is calculated by the derivatives of the state. Thus, the SLD CR bound does not diverge even at the relativistic limit.

Finally, we briefly discuss achievability of the SLD CR bounds. We show that the SLD CR bound in the rest frame is achievable. When an observer is moving and does not measure the spin, the derived SLD CR bound (6.55) is not achievable. This is shown by checking the weak commutativity condition [67, 68]. In Appendix E.4, we calculate this condition and find that $\mathrm{tr}(\rho^{\Lambda}(\theta)[L^{\Lambda}{}_1(\theta), L^{\Lambda}{}_2(\theta)]) = 8\mathrm{i}\xi\eta^2 \neq 0$. Therefore, the SLD CR bound in the moving frame is not achievable even asymptotically. A further investigation of asymptotically and non-asymptotically achievable bounds shall be presented in due course.

6.4 Conclusion

We obtain the accuracy limit for estimating the expectation value of the position of a relativistic particle for an observer moving along one direction at a constant velocity. We evaluate estimation accuracy of the position by the SLD CR bound. Estimation accuracy is degraded by increasing the observer's velocity. We see that this is because the spin up state appears in the moving frame while the spin down state only in the rest frame. Furthermore, it stays finite even at the relativistic limit. Since we show that the SLD CR bound is not achievable, it is not guaranteed that a tight CR bound gives a finite bound at the relativistic limit. However, since the Wigner rotation can be expressed as a rotation matrix that acts on a state vector, we expect that any divergent behavior will not arise from the result of applying the Wigner rotation to a state vector with a finite spread. To confirm the finiteness of estimation accuracy at the relativistic limit, it is crucial to obtain an achievable bound as future work.

Chapter 7

Trade-off relation of spin-1/2 relativistic particle given by λLD Fisher information matrix

In chapter 6, we only investigate the SLD CR bound. Hence, we cannot find a trade-off relation. In this chapter, we evaluate another type of quantum Fisher information matrix called λLD Fisher information matrix [42, 43] so that we can see a trade-off relation. By combining the SLD CR bound and the λLD CR bound, we successfully demonstrate a trade-off relation always exists for a moving observer. We owe the λLD to see that the trade-off relation exists since the RLD does not exist in this model because this model is not full rank .

7.1 Trade-off relation by λLD Fisher information matrix

7.1.1 Parametric model

We consider the same parametric model as in chapter 6, which is defined in Eq. (6.41),

As a reference state, we use the state generated by taking a partial trace over the spin, $\rho^{\Lambda}(\theta)$, i.e.,

$$\mathcal{M}^{\Lambda} = \{\rho^{\Lambda}(\theta) \,\big|\, \theta = (\theta_1, \theta_2) \in \mathbb{R}^2\}, \tag{7.1}$$

where

$$\rho^{\Lambda}(\theta) = \frac{1}{2}(1+\xi)\,|\bar{\psi}^{\Lambda}_{\downarrow}(\theta)\rangle\,\langle\bar{\psi}^{\Lambda}_{\downarrow}(\theta)| + \frac{1}{2}(1-\xi)\,|\bar{\psi}^{\Lambda}_{\uparrow}(\theta)\rangle\,\langle\bar{\psi}^{\Lambda}_{\uparrow}(\theta)|\,. \tag{7.2}$$

7.1.2 λLD Fisher information matrix

Let us first remind the λLD and λLD Fisher information. The λLD Eq. (2.19) is defined by a solution of the following equation.

$$\partial_i\rho_\theta = \frac{1+\lambda}{2}\rho_\theta L_{\lambda\,\theta,i} + \frac{1-\lambda}{2}L_{\lambda\,\theta,i}\rho_\theta. \tag{7.3}$$

By using λLD, $J_{\mathrm{R}\,\theta,}$ Eq. (2.32) is defined by

$$[J_{\lambda\,\theta}]_{i\,j} = \langle L_{\lambda\,\theta,\,i}, L_{\lambda\,\theta,\,j}\rangle^{\lambda}_{\rho_\theta} = \frac{1+\lambda}{2}\mathrm{tr}\rho_\theta(L_{\lambda\,\theta,\,j}L^{\dagger}_{\lambda\,\theta,\,i}) + \frac{1-\lambda}{2}\mathrm{tr}\rho_\theta(L^{\dagger}_{\lambda\,\theta,\,i}L_{\lambda\,\theta,\,j}), \tag{7.4}$$

As shown in Appendix E.2.2, $|\psi^{\Lambda}_{\uparrow}(\theta)\rangle$ and $|\psi^{\Lambda}_{\downarrow}(\theta)\rangle$ are orthogonal, i.e., $\langle\psi^{\Lambda}_{\uparrow}(\theta)|\psi^{\Lambda}_{\downarrow}(\theta)\rangle = 0$. This model consists of two states only, and they are orthogonal. $\mathcal{M}^{\Lambda}$ is not a full rank model; we need a formula for this case. A derivation of the λLD and λLD Fisher information matrix for a non-full rank deficient model is given in Appendix F.1.

With using the formula Eq. (F.17) given in Appendix F.1, we obtain the λLD Fisher information matrix as follows. A detailed derivation is given in Appendix F.2.

λLD **Fisher information matrix** J_λ

The λLD Fisher information matrix J_λ is expressed as

$$J_\lambda = \frac{2}{\kappa^2(1-\lambda^2)(1-\lambda^2\xi^2)}\begin{pmatrix} 1-2\kappa^2\eta^2-\lambda^2\xi^2 & -2\mathrm{i}\kappa^2\eta^2\lambda\xi \\ 2\mathrm{i}\kappa^2\eta^2\lambda\xi & 1-2\kappa^2\eta^2-\lambda^2\xi^2 \end{pmatrix}. \tag{7.5}$$

Determinant of J_λ**,** $\det(J_\lambda)$

The determinant of J_λ, $\det(J_\lambda)$ is expressed as

$$\det(J_\lambda) = \frac{4}{\kappa^2}\frac{(1-2\kappa^2\eta^2)^2-\lambda^2\xi^2}{(1-\lambda^2)(1-\lambda^2\xi^2)}. \tag{7.6}$$

Since $\det(J_\lambda)$ diverges at $\lambda = 1$ and -1. The range of λ is set as $-1 < \lambda < 1$. To show $\det(J_\lambda) > 0$, we utilize the following inequality. We show next that $\det(J_\lambda) > 0$. We use the relation between $\kappa\eta$ and ξ.

Lemma 7.1.1 (Relation between $\kappa\eta$ and ξ I)

$$\xi + \frac{2\eta}{mv} = 1 + \frac{\sqrt{\pi}}{2\kappa'}\mathrm{e}^{\kappa'^2}\mathrm{erfc}(\kappa'), \tag{7.7}$$

where $\kappa' = m\kappa$*.*

Proof

ξ and $\kappa\eta$ are defined as follows.

$$\kappa\eta = \kappa'^3 v\int_0^\infty dt\frac{t^3}{\sqrt{1+t^2}+\sqrt{1-v^2}}\mathrm{e}^{-\kappa'^2 t^2}, \tag{7.8}$$

$$\xi = \kappa'^2\int_0^\infty dt\frac{2t(1+\sqrt{1+t^2}\sqrt{1-v^2})}{\sqrt{1+t^2}+\sqrt{1-v^2}}\mathrm{e}^{-\kappa'^2 p^2}. \tag{7.9}$$

The left hand side of Eq. (7.7) is expressed as

$$\xi + \frac{2\eta}{mv} = \xi + \frac{2\kappa\eta}{\kappa' v} \tag{7.10}$$

$$= \kappa'^2 \int_0^\infty dt \frac{2t(\sqrt{1+t^2}\sqrt{1-v^2} + 1 + t^2)}{\sqrt{1+t^2} + \sqrt{1-v^2}} \mathrm{e}^{-\kappa'^2 t^2} \tag{7.11}$$

$$= \kappa'^2 \int_0^\infty dt \frac{2t\sqrt{1+t^2}(\sqrt{1-v^2} + \sqrt{1+t^2})}{\sqrt{1+t^2} + \sqrt{1-v^2}} \mathrm{e}^{-\kappa'^2 t^2} \tag{7.12}$$

$$= \kappa'^2 \int_0^\infty dt 2t\sqrt{1+t^2} \mathrm{e}^{-\kappa'^2 t^2} \tag{7.13}$$

$$= 1 + \frac{\sqrt{\pi}}{2\kappa'} \mathrm{e}^{\kappa'^2} \mathrm{erfc}(\kappa'). \quad \square \tag{7.14}$$

Lemma 7.1.2 (Relation between $\kappa\eta$ and ξ II)
The following inequality

$$\Theta := \frac{1 - 2\kappa^2\eta^2}{\xi} > 1 \tag{7.15}$$

holds for any $\kappa > 0$ and $0 < v \leq 1$.

Proof

Let us define T by

$$T = 1 - 2\kappa^2\eta^2 - \xi. \tag{7.16}$$

Then, $T > 0$ is a necessary and sufficient condition for $(1 - 2\kappa^2\eta^2)/\xi > 1$. Here, we show $T > 0$ instead. We calculate the first derivative of T with respect to v to see if T is a monotonically increasing function of v.

$$\frac{\partial T}{\partial v} = -4\kappa\eta \frac{\partial(\kappa\eta)}{\partial v} - \frac{\partial \xi}{\partial v}. \tag{7.17}$$

From Eq. (7.7), we obtain a relation between the first derivatives of η and ξ as follows.

$$\frac{\partial \xi}{\partial v} = \frac{2\eta}{mv^2} - \frac{2}{\kappa' v} \frac{\partial(\kappa\eta)}{\partial v}. \tag{7.18}$$

By substituting Eq. (7.18) in Eq. (7.17), we have

$$\frac{\partial T}{\partial v} = -(4\kappa\eta + \frac{2}{\kappa' v}) \frac{\partial(\kappa\eta)}{\partial v} + \frac{2\kappa\eta}{\kappa' v^2}. \tag{7.19}$$

By a straightforward calculation, we have

$$\frac{\partial(\kappa\eta)}{\partial v} = \frac{\partial}{\partial v}(\kappa'^3 v \int_0^\infty dt \frac{t^3}{\sqrt{1+t^2} + \sqrt{1-v^2}} \mathrm{e}^{-\kappa'^2 t^2}) \tag{7.20}$$

$$= -\frac{\kappa'^3 v}{\sqrt{1-v^2}} \int_0^\infty dt \frac{2t^3}{\left(\sqrt{1+t^2} + \sqrt{1-v^2}\right)^2} \mathrm{e}^{-\kappa'^2 t^2} < 0. \tag{7.21}$$

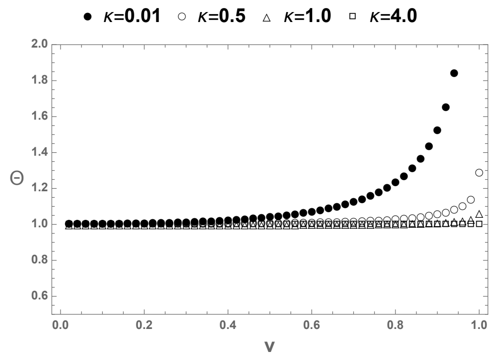

Figure 7.1: Numerically calculated $\Theta = (1 - 2\kappa^2\eta^2)/\xi$ plotted as a function of v. κ=0.01, 0.5, 1.0, 1.0, 4.0.

From Eq. (7.19), we obtain for any given $\kappa > 0$,

$$\frac{\partial T}{\partial v} > 0. \tag{7.22}$$

$T = 1 - 2\kappa^2\eta^2 - \xi$ is a monotonically increasing function of v. At $v = 0$, $\kappa\eta = 0$, and $\xi = 1$ hold.

$$T|_{v=0} = 0. \tag{7.23}$$

We show that $T > 0$ holds. Therefore,

$$\Theta = \frac{1 - 2\kappa^2\eta^2}{\xi} > 1, \tag{7.24}$$

holds. □

Figure 7.1 shows the numerically calculated Θ plotted as a function of $m\kappa$. We see that $(1 - 2\kappa^2\eta^2)/\xi > 1$. Let us remark that $\xi > 0$ for any $\kappa > 0$. From Eq. (7.15), we have

$$\frac{(1 - 2\kappa^2\eta^2)^2}{\xi^2} > 1. \tag{7.25}$$

Since $0 < \lambda^2 < 1$, we have

$$1 < \frac{(1 - 2\kappa^2\eta^2)^2}{\xi^2} < \frac{(1 - 2\kappa^2\eta^2)^2}{\lambda^2\xi^2} \tag{7.26}$$

$$\Longleftrightarrow (1 - 2\kappa^2\eta^2)^2 - \lambda^2\xi^2 > 0. \tag{7.27}$$

From Eq. (7.28), $\det(J_\lambda) > 0$ is expressed as

$$\det(J_\lambda) = \frac{4}{\kappa^2}\frac{(1 - 2\kappa^2\eta^2)^2 - \lambda^2\xi^2}{(1 - \lambda^2)(1 - \lambda^2\xi^2)}. \tag{7.28}$$

Therefore, we have

$$\det(J_\lambda) > 0. \tag{7.29}$$

Next, let us confirm that the λLD is equal to the SLD at the limit of $\lambda \to 0$.

$$\lim_{\lambda\to 0} J_\lambda = \frac{2}{\kappa^2}\begin{pmatrix} 1-2\kappa^2\eta^2 & 0 \\ 0 & 1-2\kappa^2\eta^2 \end{pmatrix} = \frac{2(1-2\kappa^2\eta^2)}{\kappa^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = J_{\mathrm{S}}. \tag{7.30}$$

Its inverse J_λ^{-1} is, then, given by

$$J_\lambda^{-1} = \frac{\kappa^2}{2}\frac{1-\lambda^2}{(1-2\kappa^2\eta^2)^2-\lambda^2\xi^2}\begin{pmatrix} 1-2\kappa^2\eta^2-\lambda^2\xi^2 & 2\mathrm{i}\kappa^2\eta^2\lambda\xi \\ -2\mathrm{i}\kappa^2\eta^2\lambda\xi & 1-2\kappa^2\eta^2-\lambda^2\xi^2 \end{pmatrix}. \tag{7.31}$$

At the limit of $\lambda \to 0$, J_λ^{-1} approaches J_{S}^{-1}.

$$\lim_{\lambda\to 0} J_\lambda^{-1} = \frac{\kappa^2}{2}\frac{1}{(1-2\kappa^2\eta^2)^2}\begin{pmatrix} 1-2\kappa^2\eta^2 & 0 \\ 0 & 1-2\kappa^2\eta^2 \end{pmatrix} = \frac{\kappa^2}{2}\frac{1}{1-2\kappa^2\eta^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = J_{\mathrm{S}}^{-1}. \tag{7.32}$$

As for the range of λ to be considered, it is enough to set the range as $0 < \lambda < 1$ because $\lambda = 0$ makes $J_\lambda = J_{\mathrm{S}}$ and J_λ diverges at $\lambda = 1$. Furthermore, the diagonal components of J_λ^{-1} and the $|J_{\lambda\,12}^{-1}|^2 = |J_{\lambda\,21}^{-1}|^2$ include λ^2. We exclude the case of $v = 0$ because $v = 0$ means the observer is in the rest frame.

7.2 Condition for trade-off existence

By using the Δ defined in Section 3.7, the condition for trade-off relation to exist is given by Eq. (3.72), i.e.,

$$\Delta = |\mathrm{Im} J_{\lambda\,12}^{-1}|^2 - (J_{\mathrm{S}\,11}^{-1} - J_{\lambda\,11}^{-1})(J_{\mathrm{S}\,22}^{-1} - J_{\lambda\,22}^{-1}) > 0. \tag{7.33}$$

Let us define ΔJ^{-1} by

$$\Delta J^{-1} = J_{\mathrm{S}}^{-1} - J_\lambda^{-1}. \tag{7.34}$$

The condition for the trede-off relation to exist is expressed as

Theorem 7.2.1 (Condition for the existence of the trade-off relation)

$$\det(\Delta J^{-1}) < 0. \tag{7.35}$$

Proof

Since the J_{S}^{-1} is a diagonal matrix, we have

$$[\Delta J^{-1}]_{12} = -J_{\lambda\,12}^{-1}, \tag{7.36}$$

$$[\Delta J^{-1}]_{21} = -J_{\lambda\,11}^{-1} = -(J_{\lambda\,12}^{-1})^*. \tag{7.37}$$

We remark $J_{\lambda\,12}^{-1}$ is pure imaginary as shown in Eq. (7.31). Therefore, $[\Delta J^{-1}]_{12}$ and $[\Delta J^{-1}]_{21}$ is pure imaginary as well.

$$[\Delta J^{-1}]_{11} = J_{\mathrm{S}\,11}^{-1} - J_{\lambda\,11}^{-1} = [\Delta J^{-1}]_{22}. \tag{7.38}$$

By using these, the Δ is written as

$$\Delta = |\mathrm{Im} J^{-1}_{\lambda\,12}|^2 - (J^{-1}_{\mathrm{S}\,11} - J^{-1}_{\lambda\,11})(J^{-1}_{\mathrm{S}\,22} - J^{-1}_{\lambda\,22}) \tag{7.39}$$

$$= [\Delta J^{-1}]_{12}[\Delta J^{-1}]_{21} - [\Delta J^{-1}]_{11}[\Delta J^{-1}]_{22} \tag{7.40}$$

$$= -\det(\Delta J^{-1}). \tag{7.41}$$

The condition for the trade-off to exist is $\Delta > 0$. Therefore, the trade-off relation exists if and only if $\det(\Delta J^{-1}) < 0$. $\square$.

We give explicit expression of $\det(\Delta J^{-1})$ below. From Eqs. (7.31, 7.32), the components of $\Delta J^{-1} = J_{\mathrm{S}}^{-1} - J_{\lambda}^{-1}$ are explicitly written as follows.

$$[\Delta J^{-1}]_{11} = \frac{\kappa^2\lambda^2}{2}\frac{(1-2\kappa^2\eta^2)^2 - [2\kappa^2\eta^2 + \lambda^2(1-2\kappa^2\eta^2)]\xi^2}{(1-2\kappa^2\eta^2)[(1-2\kappa^2\eta^2)^2 - \lambda^2\xi^2]} = [\Delta J^{-1}]_{22}, \tag{7.42}$$

$$[\Delta J^{-1}]_{12} = \frac{\kappa^2}{2}\frac{(1-\lambda^2)(2\mathrm{i}\kappa^2\eta^2\lambda\xi)}{(1-2\kappa^2\eta^2)^2 - \lambda^2\xi^2} = -[\Delta J^{-1}]_{21}. \tag{7.43}$$

Using these, we obtain $\det(\Delta J^{-1})$ as

$$\det(\Delta J^{-1}) = \frac{\kappa^4\lambda^2}{4(1-2\kappa^2\eta^2)^2}\frac{\lambda^2(1-2\kappa^2\eta^2)^2 - \xi^2[\lambda^2 + 2\kappa^2\eta^2(1-\lambda^2)]^2}{(1-2\kappa^2\eta^2)^2 - \lambda^2\xi^2}. \tag{7.44}$$

7.2.1 Strength of trade-off relation

We investigate the indicators of the strength of the trade-off relation $\Omega_1^{\mathrm{Tradeoff}}$ and $\Omega_2^{\mathrm{Tradeoff}}$ defined in Section 3.8 for this model. In this model, the strength of trade-off relation $\Omega_1^{\mathrm{Tradeoff}}$ is equal to $\Omega_2^{\mathrm{Tradeoff}}$, because $J^{-1}_{\mathrm{S}\,11} = J^{-1}_{\mathrm{S}\,22}$ and $J^{-1}_{\lambda\,11} = J^{-1}_{\lambda\,22}$ holds. As the strength of the trade-off relation, we use $\Omega^{\mathrm{Tradeoff}}(\lambda,\kappa,\nu) = \Omega_1^{\mathrm{Tradeoff}}(\lambda,\kappa,\nu) = \Omega_2^{\mathrm{Tradeoff}}(\lambda,\kappa,\nu)$.

Theorem 7.2.2 (Condition for the existence of the trade-off relation)

$$\Omega^{\mathrm{Tradeoff}}(\lambda,\kappa,\nu) = -\frac{2\det(\Delta J^{-1})}{\mathrm{tr}(\Delta J^{-1})}, \tag{7.45}$$

if $\det(\Delta J^{-1}) < 0$ *holds.*

Proof

By its definition, Eqs.(3.83, 3.84), the $\Omega^{\mathrm{Tradeoff}}(\lambda,\kappa,\nu)$ is written as

$$\Omega^{\mathrm{Tradeoff}}(\lambda,\kappa,\nu) = \begin{cases} \mathsf{V}_{22}|_{\mathsf{V}_{11}=J^{-1}_{\mathrm{S}\,11}} - J^{-1}_{\mathrm{S}\,22} & \text{if } \Delta > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7.46}$$

If $\det(\Delta J^{-1}) = -\Delta < 0$ holds , the $\Omega^{\text{Tradeoff}}(\lambda, \kappa, \nu)$ is expressed as

$$
\begin{aligned}
\Omega^{\text{Tradeoff}}(\lambda, \kappa, \nu) &= \Omega_2^{\text{Tradeoff}}(\lambda, \kappa, \nu) && (7.47)\\
&= \mathsf{V}_{22}|_{\mathsf{V}_{11}=J^{-1}_{\mathrm{S}\,11}} - J^{-1}_{\mathrm{S}\,22} && (7.48)\\
&= \frac{|\mathrm{Im} J^{-1}_{\lambda\,12}|^2}{J^{-1}_{\mathrm{S}\,11} - J^{-1}_{\lambda\,11}} + J^{-1}_{\lambda\,22} - J^{-1}_{\mathrm{S}\,22} && (7.49)\\
&= \frac{|\mathrm{Im} J^{-1}_{\lambda\,12}|^2 - (J^{-1}_{\mathrm{S}\,11} - J^{-1}_{\lambda\,11})(J^{-1}_{\mathrm{S}\,22} - J^{-1}_{\lambda\,22})}{J^{-1}_{\mathrm{S}\,11} - J^{-1}_{\lambda\,11}} && (7.50)\\
&= -\frac{2\det(\Delta J^{-1})}{\mathrm{tr}(\Delta J^{-1})}. \quad \square && (7.51)
\end{aligned}
$$

We show below that $\mathrm{tr}(\Delta J^{-1}) > 0$. Because of the positivity of $\mathrm{tr}(\Delta J^{-1}) > 0$, $\Omega^{\text{Tradeoff}}(\lambda, \kappa, \nu)$ is positive if $\det(\Delta J^{-1}) < 0$.

Lemma 7.2.3 (Positivity of $\mathrm{tr}(\Delta J^{-1})$**)**

$$\mathrm{tr}(\Delta J^{-1}) > 0 \tag{7.52}$$

for any given $\kappa > 0$*,* $0 < \nu \leq 1$ *and* $0 < \lambda < 1$.

Proof

From Eqs. (7.31, 7.32), the diagonal components of $\Delta J^{-1} = J_{\mathrm{S}}^{-1} - J_{\lambda}^{-1}$ are explicitly written as follows.

$$[\Delta J^{-1}]_{11} = \frac{\kappa^2\lambda^2}{2}\frac{(1-2\kappa^2\eta^2)^2 - [2\kappa^2\eta^2 + \lambda^2(1-2\kappa^2\eta^2)]\xi^2}{(1-2\kappa^2\eta^2)[(1-2\kappa^2\eta^2)^2 - \lambda^2\xi^2]} = [\Delta J^{-1}]_{22}. \tag{7.53}$$

$$\tag{7.54}$$

Using these, we obtain $\mathrm{tr}(\Delta J^{-1})$ and $\det(\Delta J^{-1})$ as

$$
\begin{aligned}
\mathrm{tr}(\Delta J^{-1}) &= \frac{\kappa^2\lambda^2}{1-2\kappa^2\eta^2}\frac{(1-2\kappa^2\eta^2)^2 - [2\kappa^2\eta^2 + \lambda^2(1-2\kappa^2\eta^2)]\xi^2}{(1-2\kappa^2\eta^2)^2 - \lambda^2\xi^2} && (7.55)\\
&= \frac{\kappa^2\lambda^2}{1-2\kappa^2\eta^2}\frac{\frac{(1-2\kappa^2\eta^2)^2}{\xi^2} - [2\kappa^2\eta^2 + \lambda^2(1-2\kappa^2\eta^2)]}{\frac{(1-2\kappa^2\eta^2)^2}{\xi^2} - \lambda^2}. && (7.56)
\end{aligned}
$$

Since $(1-2\kappa^2\eta^2)^2\xi^{-2} > 1$ and $\lambda^2 < 1$, $\mathrm{tr}(\Delta J^{-1})$ is positive if and only if

$$2\kappa^2\eta^2 + \lambda^2(1-2\kappa^2\eta^2) < 1, \tag{7.57}$$

holds. Both $2\kappa^2\eta^2$ and $1-2\kappa^2\eta^2$ are positive. The left hand side of Eq. (7.57) is a monotonically increasing function of λ when $0 < \lambda < 1$. We see the maximum of the left hand side of Eq. (7.57) at $\lambda = 1$, which is 1. Since we exclude $\lambda = 1$,

$$\mathrm{tr}(\Delta J^{-1}) > 0, \tag{7.58}$$

always holds. □

We remark that this relation when including equality holds in general, because the following inequality holds [43].

$$\mathrm{tr}(J_{\mathrm{S}}^{-1}) \geq \mathrm{Re}(J_{\lambda}^{-1}), \tag{7.59}$$

where $\mathrm{Re}(X) = \dfrac{1}{2}(X + X^*)$. X^* has complex conjugate of the all components of X as its components, i.e., $[X^*]_{ij} = ([X]_{ij})^*$.

Below, we give an explicit expression of $-2\det(\Delta J^{-1})/\mathrm{tr}(\Delta J^{-1})$. The trace and the determinant of ΔJ^{-1} are given by

$$\mathrm{tr}(\Delta J^{-1}) = \frac{\kappa^2\lambda^2}{1 - 2\kappa^2\eta^2}\frac{(1 - 2\kappa^2\eta^2)^2 - [2\kappa^2\eta^2 + \lambda^2(1 - 2\kappa^2\eta^2)]\xi^2}{(1 - 2\kappa^2\eta^2)^2 - \lambda^2\xi^2}, \tag{7.60}$$

$$\det(\Delta J^{-1}) = \frac{\kappa^4\lambda^2}{4(1 - 2\kappa^2\eta^2)^2}\frac{\lambda^2(1 - 2\kappa^2\eta^2)^2 - \xi^2[\lambda^2 + 2\kappa^2\eta^2(1 - \lambda^2)]^2}{(1 - 2\kappa^2\eta^2)^2 - \lambda^2\xi^2}. \tag{7.61}$$

Therefore

$$-\frac{2\det(\Delta J^{-1})}{\mathrm{tr}(\Delta J^{-1})} = \frac{\kappa^2}{2(1 - 2\kappa^2\eta^2)}\frac{\lambda^2(1 - 2\eta^2\kappa^2)^2 - \xi^2[\lambda^2(1 - 2\kappa^2\eta^2) + 2\kappa^2\eta^2]^2}{\xi^2[\lambda^2(1 - 2\kappa^2\eta^2) + 2\kappa^2\eta^2] - (1 - 2\eta^2\kappa^2)^2}. \tag{7.62}$$

To investigate the behavior of $-2\det(\Delta J^{-1})/\mathrm{tr}(\Delta J^{-1})$ as a function of $\lambda \in (0, 1)$ for any $\kappa > 0$ and for any $0 < \nu \leq 1$, we define $\omega(\lambda, \kappa, \nu)$ by

$$\begin{aligned}\omega(\lambda, \kappa, \nu) &:= -\frac{2\det(\Delta J^{-1})}{\mathrm{tr}(\Delta J^{-1})} \\ &= \frac{\kappa^2}{2(1 - 2\kappa^2\eta^2)}\frac{\lambda^2(1 - 2\eta^2\kappa^2)^2 - \xi^2[\lambda^2(1 - 2\kappa^2\eta^2) + 2\kappa^2\eta^2]^2}{\xi^2[\lambda^2(1 - 2\kappa^2\eta^2) + 2\kappa^2\eta^2] - (1 - 2\eta^2\kappa^2)^2}.\end{aligned} \tag{7.63}$$

With this $\omega(\lambda, \kappa, \nu)$, $\Omega^{\mathrm{Tradeoff}}(\lambda, \kappa, \nu)$ is expressed as

$$\Omega^{\mathrm{Tradeoff}}(\lambda, \kappa, \nu) = \begin{cases} \omega(\lambda, \kappa, \nu) & \text{if } \det(\Delta J^{-1}) < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7.64}$$

We set the range of λ as $0 < \lambda < 1$. $\lambda = 0$ is excluded. Fig. 7.2 shows the $\Omega^{\mathrm{Tradeoff}}(\lambda, \kappa, \nu)$ and $\omega(\lambda, \kappa, \nu)$ as a function of λ at $\kappa = 1$ and $\nu = 1$.

Regarding $\omega(\lambda, \kappa, \nu)$, we can show the next lemma.

Lemma 7.2.4 ($\omega(\lambda, \kappa, \nu)$)
$\omega(\lambda, \kappa, \nu)$ is a monotonically decreasing function of λ for any $\kappa > 0$ and any $0 < \nu \leq 1$.

Proof

For any given $\kappa > 0$ and for any given $0 < \nu \leq 1$, the first derivative of the right hand side of Eq. (7.62) with respect to λ is calculated as

$$\frac{\partial\omega(\lambda, \kappa, \nu)}{\partial\lambda} = -\frac{\lambda(a\lambda^4 + b\lambda^2 + c)}{\{4\kappa^4\eta^4 - 2\kappa^2\eta^2[(1 - \lambda^2)\xi^2 + 2] - \lambda^2\xi^2 + 1\}^2}, \tag{7.65}$$

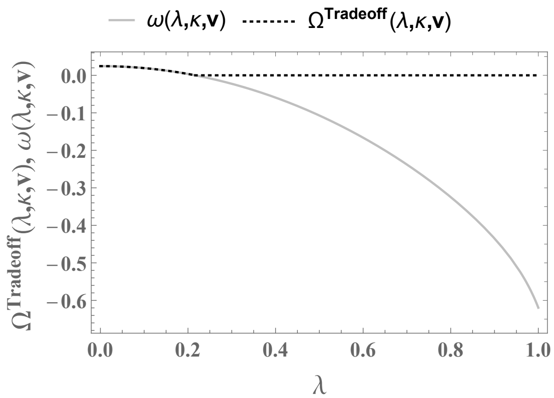

Figure 7.2: Numerically calculated strength of the trade-off $\Omega^{\mathrm{Tradeoff}}(\lambda, \kappa, v)$ (dotted line) and $\omega(\lambda, \kappa, v)$ (gray line) as a function of λ. $\kappa = 1$ and $v = 1$.

where

$$a = \xi^4 \left(1 - 2\kappa^2\eta^2\right)^2, \tag{7.66}$$

$$b = -\{2\xi^2 \left(1 - 2\kappa^2\eta^2\right)[(1 - 2\kappa^2\eta^2)^2 - 2\xi^2\kappa^2\eta^2]\}, \tag{7.67}$$

$$c = -8\eta^6\kappa^6 \left(2\xi^2 + 1\right) + 4\kappa^4\eta^4 \left(\xi^4 + 5\xi^2 + 3\right) - 6\kappa^2\eta^2 \left(\xi^2 + 1\right) + 1. \tag{7.68}$$

From these, we are to show the first derivative of $\Omega^{\mathrm{Tradeoff}}(\lambda, \kappa, v)$ with respect to λ is negative. We just need to check if $a\lambda^4 + b\lambda^2 + c > 0$ holds.

We show here that the inequality $as^2 + bs + c > 0$, where $s = \lambda^2$. Since $a > 0$, $as^2 + bs + c > 0$ holds if and only if its discriminant $D < 0$. The discriminant D is

$$D = -8\eta^2\kappa^2\xi^4(1 - 2\kappa^2\eta^2)^3[(1 - 2\kappa^2\eta^2)^2 - \xi^2]. \tag{7.69}$$

From $1 - 2\kappa^2\eta^2 > 0$ and $(1 - 2\kappa^2\eta^2)/\xi > 1$, $D < 0$ holds. Since D is negative and $a > 0$, $a\lambda^4 + b\lambda^2 + c > 0$ holds. Therefore, the first derivative of $\omega(\lambda, \kappa, v)$ with respect to λ, the left hand side of Eq. (7.65) is always negative. $\omega(\lambda, \kappa, v)$ is a strictly monotonically decreasing function of λ for any given $\kappa > 0$ and for any given $0 < v \leq 1$. □

Let us define a limit $\omega(0, \kappa, v)$ by

$$\omega(0, \kappa, v) = \lim_{\lambda \to 0} \omega(\lambda, \kappa, v). \tag{7.70}$$

We can show the following.

Lemma 7.2.5 ($\omega(0, \kappa, v) > 0$)

$$\omega(0, \kappa, v) > 0, \tag{7.71}$$

for any $\kappa > 0$ and $0 < v \leq 1$.

Proof

We have

$$\omega(0,\kappa,v) = \frac{2\kappa^6\eta^4\xi^2}{(1-2\kappa^2\eta^2)[(1-2\kappa^2\eta^2)^2 - 2\kappa^2\eta^2\xi^2]} = \frac{2\kappa^6\eta^4}{(1-2\kappa^2\eta^2)(\Theta^2 - 2\kappa^2\eta^2)}, \tag{7.72}$$

where

$$\Theta = \frac{1-2\kappa^2\eta^2}{\xi}. \tag{7.73}$$

From Eq. (7.15), $\Theta > 1$, and from $1 - 2\kappa^2\eta^2 > 0$ and $\Theta > 1$, $\Theta - 2\kappa^2\eta^2 > 0$. Since $\Theta > 1$, $\Theta^2 - 2\kappa^2\eta^2 > \Theta - 2\kappa^2\eta^2 > 0$.

$$\omega(0,\kappa,v) > 0, \tag{7.74}$$

holds for any $\kappa > 0$. As for the observer's velocity v, $\omega(0,\kappa,v) = 0$ holds at $v = 0$ only, because $\kappa\eta = 0$ at $v = 0$. We exclude the case of $v = 0$ because in that case the observer would be in the rest frame. □

Let us remark the positivity of $\omega(0,\kappa,v)$. As we can see in Eqs. (7.55, 7.61), both $\det(\Delta J^{-1})$ and $\mathrm{tr}(\Delta J^{-1})$ are proportional to λ^2. The $\det(\Delta J^{-1})$ divided by the $\mathrm{tr}(\Delta J^{-1})$ cancels out the λ^2s in both of $\det(\Delta J^{-1})$ and $\mathrm{tr}(\Delta J^{-1})$. As a result the limit $\omega(0,\kappa,v)$ has a non-zero value at the limit of $\lambda \to 0$.

Let us define a limit $\omega(1,\kappa,v)$ by

$$\omega(1,\kappa,v) = \lim_{\lambda\to 1} \omega(\lambda,\kappa,v). \tag{7.75}$$

Lemma 7.2.6 ($\omega(1,\kappa,v) < 0$)

$$\omega(1,\kappa,v) < 0, \tag{7.76}$$

for any $\kappa > 0$ *and* $0 < v \leq 1$.

From the explicit expression of $\omega(\lambda,\kappa,v)$, i.e., Eq. (7.62), $\omega(1,\kappa,v)$ is expressed as

$$\omega(1,\kappa,v) = \frac{\kappa^2}{2(1-2\kappa^2\eta^2)} \frac{(1-2\kappa^2\eta^2)^2 - \xi^2[(1-2\kappa^2\eta^2)+2\kappa^2\eta^2]^2}{\xi^2[(1-2\kappa^2\eta^2)+2\kappa^2\eta^2] - (1-2\eta^2\kappa^2)^2}, \tag{7.77}$$

$$= \frac{\kappa^2}{2(1-2\kappa^2\eta^2)} \frac{(1-2\eta^2\kappa^2)^2 - \xi^2}{\xi^2 - (1-2\eta^2\kappa^2)^2} \tag{7.78}$$

$$= -\frac{\kappa^2}{2(1-2\kappa^2\eta^2)} < 0 \quad \square \tag{7.79}$$

Theorem 7.2.7 (Existence of solution of $\omega(\lambda^*, \kappa, v) = 0$)
For any $\kappa > 0$ and any $0 < v \leq 1$, there exists a $\lambda^* \in (0, 1)$ that is a unique solution of $\omega(\lambda^*, \kappa, v) = 0$. The unique solution λ^* is expressed as

$$\lambda^*_\pm = \frac{1}{2\xi}\left(1 \pm \sqrt{1 - \frac{8\xi^2\kappa^2\eta^2}{1 - 2\kappa^2\eta^2}}\right). \tag{7.80}$$

Proof

$\omega(0, \kappa, v) > 0$ and $\omega(1, \kappa, v) < 0$ hold. Furthermore, $\omega(\lambda, \kappa, v)$ is a monotonically decreasing function with respect to λ for any given $\kappa > 0$ and for any given $0 < v \leq 1$. There always exists a unique solution, $\lambda^* \in (0, 1)$ of $\omega(\lambda^*, \kappa, v) = 0$. □

Since we show $\omega(\lambda, \kappa, v) = 0$ has a unique solution for for any $\kappa > 0$ and $0 < v \leq 1$. We derive the solution. From Eq. (7.62), $\omega(\lambda, \kappa, v)$ is written as

$$\omega(\lambda, \kappa, v) = \frac{\kappa^2}{2(1 - 2\kappa^2\eta^2)} \frac{\lambda^2(1 - 2\kappa^2\eta^2)^2 - \xi^2[\lambda^2(1 - 2\kappa^2\eta^2) + 2\kappa^2\eta^2]^2}{\xi^2[\lambda^2(1 - 2\kappa^2\eta^2) + 2\kappa^2\eta^2] - (1 - 2\eta^2\kappa^2)^2}. \tag{7.81}$$

By factoring the numerator, $\omega(\lambda^*, \kappa, v) = 0$, we obtain an equivalent condition

$$\{(1-2\kappa^2\eta^2)\lambda^* + \xi[(\lambda^*)^2(1-2\kappa^2\eta^2) + 2\kappa^2\eta^2]\}\{(1-2\kappa^2\eta^2)\lambda^* - \xi[(\lambda^*)^2(1-2\kappa^2\eta^2) + 2\kappa^2\eta^2]\} = 0 \tag{7.82}$$

For simplicity, we consider the case of $\lambda^* > 0$ because $\omega(\lambda^*, \kappa, v)$ includes the quadratic and the quartic terms of λ^*. From $1 - 2\kappa^2\eta^2 > 0$ and $0 < \xi \leq 1$ for any $\kappa > 0$ and $0 < v \leq 1$, the following inequality holds.

$$(1 - 2\kappa^2\eta^2)\lambda^* + \xi[(\lambda^*)^2(1 - 2\kappa^2\eta^2) + 2\kappa^2\eta^2] > 0. \tag{7.83}$$

Therefore, Eq (7.82) reduces to

$$(1 - 2\kappa^2\eta^2)\lambda^* - \xi[(\lambda^*)^2(1 - 2\kappa^2\eta^2) + 2\kappa^2\eta^2] = 0. \tag{7.84}$$

This λ^* is given by the smaller solution of $y(\lambda^*) = 0$ which is expressed as

$$\lambda^*_\pm = \frac{1 - 2\kappa^2\eta^2 \pm \sqrt{(1 - 2\kappa^2\eta^2)^2 - 8\xi^2\kappa^2\eta^2(1 - 2\kappa^2\eta^2)}}{2(1 - 2\kappa^2\eta^2)\xi} \tag{7.85}$$

$$= \frac{1}{2\xi}\left(1 \pm \sqrt{1 - \frac{8\xi^2\kappa^2\eta^2}{1 - 2\kappa^2\eta^2}}\right). \tag{7.86}$$

In the limit of $\kappa \to 0$ when $v = 1$, ξ approaches zero. Hence, $\lambda^*_+|_{v=1}$ diverges. We take the smaller solution of the equation $y(\lambda^*) = 0$.

$$\lambda^* = \frac{1}{2\xi}\left(1 - \sqrt{1 - \frac{8\xi^2\kappa^2\eta^2}{1 - 2\kappa^2\eta^2}}\right). \quad \square \tag{7.87}$$

The solution λ^* gives the range of λ where that the trade-off relation exists. For any $\lambda \in (0, \lambda^*)$, there always exists the trade-off relation. Figure 7.3 shows the result of the numerical calculation of λ^* plotted as a function of v. When κ is approximately less than 0.3, λ^* is a convex upward function of v.

In the rest frame, i.e., $v = 0$, $\lambda^* = 0$ holds at any given $\kappa > 0$ because $\kappa\eta = 0$ at $v = 0$, i.e., no trade-off relation exists. From Fig. 7.3, it seems that λ^* is close to zero at $v = 1$ when $\kappa \ll 1$. We are to show the following.

Lemma 7.2.8 (λ^* at $v = 1$ and $\kappa \to 0$)

$$\lim_{\kappa \to 0} \lambda^* = 0, \tag{7.88}$$

at the relativistic limit $v = 1$.

Proof

With the Taylor expansion, we have an alternative expression of λ^* as follows.

$$\lambda^* = \frac{1}{2\xi} \sum_{n=1}^{\infty} \frac{(2n-3)!!}{n!2^n} \left(\frac{8\xi^2\kappa^2\eta^2}{1 - 2\kappa^2\eta^2} \right)^n, \tag{7.89}$$

where [72]

$$(2n-3)!! = (2n-3)(2n-5)\cdots 3 \cdot 1, \quad (-1)!! = 1. \tag{7.90}$$

At $v = 1$ and at the limit of $\kappa \to 0$, $\kappa\eta \to \sqrt{\pi}/4$ and $\xi \to 0$. Therefore, at $v = 1$,

$$\lim_{\kappa \to 0} \lambda^* = 0. \quad \square \tag{7.91}$$

This result explains the fact that the λ^* is close to zero when $v = 1$ at $\kappa = 0.01$ shown in Fig. 7.3. At larger κs, $\kappa = 0.3, 0.5$, and 1, the λ^* appears to be a monotonically increasing function of v. As the κ is the spread of the wave function which is gaussian in the rest frame, the limit of $\kappa \to 0$ makes a gaussian function a delta function, which is a singularity. This suggest that the singularity result from this effect at $\kappa = 0.01$.

Hereafter, we discuss the case for $\omega > 0$, or the case in which the trade-off relation exists. Then, we can set $\Omega^{\text{Tradeoff}}(0, \kappa, v) = \omega(0, \kappa, v)$. When $\lambda = 0$, there exists no trade-off relation as $\det(\Delta J^{-1}) = 0$, that is, the condition for the existence of the trade-off relation does not hold. For the trade-off relation to exist, λ must be non-zero. No trade-off relation at $\lambda = 0$ is consistent with the SLD CR bound not giving a trade-off relation. The λLD Fisher information matrix coincides with the SLD Fisher information matrix by its definition. Therefore, no trade-off relation at $\lambda = 0$.

To sum up the result so far, since the first derivative of $\Omega^{\text{Tradeoff}}(\lambda, \kappa, v)$ with respect to λ is negative, $\Omega^{\text{Tradeoff}}(\lambda, \kappa, v)$ is a monotonically decreasing function of $\lambda \in (0, \lambda^*)$. It has already been shown that $\Omega^{\text{Tradeoff}}(0, \kappa, v) > 0$ holds. $\Omega^{\text{Tradeoff}}(0, \kappa, v)$ gives the upper limit of the strength of the trade-off relation. The trade-off relation exists for any $\lambda \in (0, \lambda^*)$. In the following, we investigate how significant the trade-off relation can be. In other words, we investigate how large $\Omega^{\text{Tradeoff}}(\lambda, \kappa, v)$ can be.

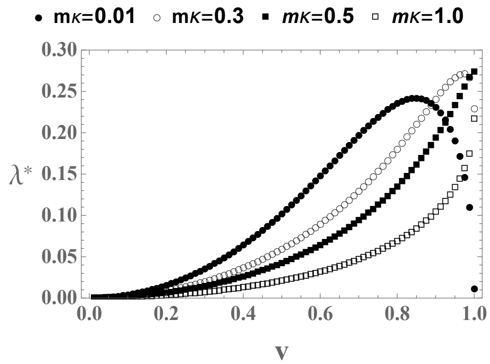

Figure 7.3: Numerically calculated λ^* plotted as a function of v

Figure 7.4 shows the numerically calculated $\Omega^{\text{Tradeoff}}(0,\kappa,v)$ plotted as a function of κ at v=0.85, 0.9, 0.95, and 1. At a given v, $\Omega^{\text{Tradeoff}}(0,\kappa,v)$ is a convex upward function of κ.

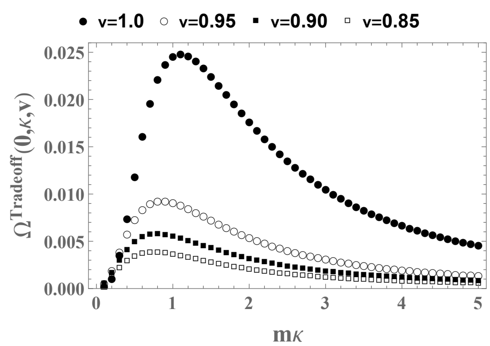

Figure 7.4: Numerically calculated $\Omega^{\text{Tradeoff}}(0,\kappa,v)$ plotted as a function of $m\kappa$ at v=0.85, 0.9, 0.95, and 1.

Figure 7.5 is plotted with a smaller range of $m\kappa$, $0 \leq m\kappa \leq 0.4$. The $\Omega^{\text{Tradeoff}}(0,\kappa,v)$ in the case of $v=1$ looks different from others. It stays low at small $m\kappa$, approximately smaller than 0.1, the strength of the trade-off relation $\Omega^{\text{Tradeoff}}(0,\kappa,v)$ is almost constant, but once $m\kappa$ goes over approximately 0.1, $\Omega^{\text{Tradeoff}}(0,\kappa,v)$ becomes an increasing with increasing v. Eventually, it takes the highest peak. From Fig 7.4, there appears to be a specific value of κ that gives the most significant trade-off relation. We define the κ that gives the maximum of $\Omega^{\text{Trade-off}}(0,\kappa,v)$ by κ^* at a given v. In other words, κ^* is expressed by

$$\kappa^* = \text{argmax}\Omega^{\text{Tradeoff}}(\lambda,\kappa,v). \tag{7.92}$$

Figure (7.6) show the numerically calculated κ^* plotted as a function of v. κ^* gives the maximum strength of the trade-off $\Omega^{\text{Tradeoff}}(0,\kappa,v)$ at a given v. The κ^* appears to be a monotonically increasing function of v. Figure 7.7 shows the numerically calculated maximum strength of the trade-off $\Omega^{\text{Tradeoff}}(0,\kappa^*,v)$ as a function of the observer's velocity v. We obtain κ^* as the κ for which the first derivative with respect to v of $\Omega^{\text{Tradeoff}}(0,\kappa^*,v)$ is zero. We also numerically calculate the $\Omega^{\text{Tradeoff}}(0,\kappa^*,v)$ which is the $\Omega^{\text{Tradeoff}}(0,\kappa,v)$ at $\kappa=\kappa^*$. The $\Omega^{\text{Tradeoff}}(0,\kappa^*,v)$ is

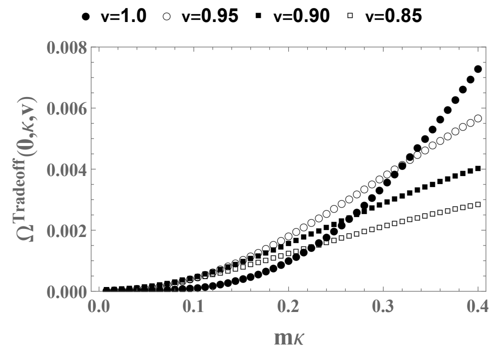

Figure 7.5: Numerically calculated $\Omega^{\mathrm{Tradeoff}}(0, \kappa, v)$ plotted as a function of κ at v=0.85, 0.9, 0.95, and 1. This figure is the same as Fig. 7.4, but the plot range of $m\kappa$ is from 0 to 0.4 so that we can see the behavior of $\Omega^{\mathrm{Tradeoff}}(0, \kappa, v)$ at low $m\kappa$.

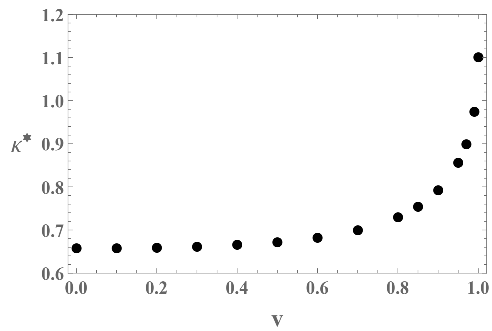

Figure 7.6: Numerically calculated κ^* plotted as a function of v. κ^* gives the maximum strength of the trade-off relation $\Omega^{\mathrm{Tradeoff}}(0, \kappa, v)$ at a given v.

also a monotonically increasing function of v. Therefore, the trade-off relation becomes more significant for larger v.

7.3 Discussion

The parametric model in the rest frame, Eq. (6.4) is defined by

$$\mathcal{M}_{\mathrm{rest}} = \{\rho_\theta \,\big|\, \theta = (\theta_1, \theta_2) \subset \mathbb{R}^2\}, \tag{7.93}$$

where

$$\rho_\theta = U(\theta)\rho_0 U^\dagger(\theta) = \mathrm{e}^{-\mathrm{i}\hat{p}^1\theta_1 - \mathrm{i}\hat{p}^2\theta_2} \left|\Psi_\downarrow\right\rangle \left\langle\Psi_\downarrow\right| \mathrm{e}^{\mathrm{i}\hat{p}^1\theta_1 + \mathrm{i}\hat{p}^2\theta_2}, \tag{7.94}$$

and

$$\left|\Psi_\downarrow\right\rangle = \frac{\kappa}{\sqrt{\pi}} \int d^3p \, \mathrm{e}^{-\frac{1}{2}\kappa^2[(p^1)^2 + (p^2)^2]} \delta(p^3) \left|\vec{p}, \downarrow\right\rangle. \tag{7.95}$$

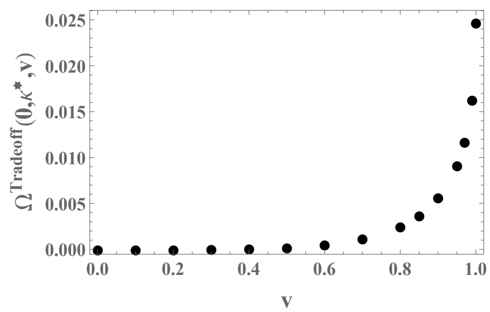

Figure 7.7: Numerically calculated maximum strength of the trade-off relation $\Omega^{\text{Tradeoff}}(0, \kappa^*, v)$ as a function of v

The parametric model in the rest frame $\mathcal{M}_{\text{rest}}$ is classical for an observer in the rest frame because the reference state is a gaussian state which is a product of two gaussian functions of p^1 and p^2. Furthermore, the generators, $\hat{p}^1$ and $\hat{p}^2$ commute, the best estimate is obtained by the measured position of each of x and y independently in the rest frame. The SLD Fisher information matrix obtained in the rest frame is the same as the classical one. On the other hand, the model in the moving frame, $\mathcal{M}^{\Lambda}$ would change from a classical to a nonclassical, non-trivial model since the wave function after the Lorentz boost changes to a more complicated form. This change results from only the observer's motion. Furthermore, this change gives rise to the trade-off relation the observer sees. We also owe λLD to see that this relationship exists since the RLD does not exist in this model, $\mathcal{M}^{\Lambda}$ because the model is not full rank.

7.3.1 Strength of trade-off relation

We find that the λLD CR bound together with the SLD CR bound gives a trade-off relation for any given spread of the wave function, $\kappa > 0$, and the velocity of the observer $0 < v \leq 1$. The trade-off relation is most substantial at the limit of $\lambda \to 0$ while at $\lambda = 0$, by definition, the λLD Fisher information matrix is equal to the SLD Fisher information matrix which gives no trade-off relation. Therefore, while the trade-off relation does not exist if λ is strictly equal to zero, the trade-off relation comes from both the SLD and the λLD is most significant in the limit of $\lambda \to 0$. The strength of the trade-off relation Ω^{Tradeoff} is defined by

$$\Omega^{\text{Tradeoff}}(\lambda, \kappa, v) = -\frac{2\det(\Delta J^{-1})}{\operatorname{tr}(\Delta J^{-1})} \tag{7.96}$$

for $\Delta > 0$. A mathematical explanation why $\Omega^{\text{Tradeoff}}(\lambda, \kappa, v) > 0$ in the limit of $\lambda \to 0$ is that both of $\det(\Delta J^{-1})$ and $\operatorname{tr}(\Delta J^{-1})$ are proportional to λ^2. Therefore, those λ^2s cancels out when $\det(\Delta J^{-1})$ is divided by $\operatorname{tr}(\Delta J^{-1})$. We obtain this non-trivial observation because we use both the SLD and the λLD to determine the trade-off relation.

7.3.2 Strength of trade-off relation and spread of wave function κ for trade-off relation

The strength of the trade-off relation can be a convex upward function of the spread of the wave function, κ as shown in Fig. 7.4. We expect that this phenomenon has something to do with the separation distance between the peaks of the spin down and the spin up wave functions. It is counter-intuitive that the trade-off can be the most significant with a specific spread of the wave function in the rest frame. We have no clue about the reason for this result.

7.3.3 λ and the strength of the trade-off relation

For any $\kappa > 0$ and any $0 < v \leq 1$, we find that there always exists some λ that gives the trade-off relation. This finding is a non-trivial result. As for the maximum λ for the trade-off relation existence, λ^*, the plot of the λ^* has a peak when the spread of the wave function κ is approximately in the range $0 < \kappa < 0.5$. When the observer's velocity is at the relativistic limit $v = 1$ and at the limit of $\kappa \to 0$, λ^* goes to 0 as shown in Eq. (7.88). We attribute this result to the wave function in the rest frame having singularity at the spread $\kappa = 0$.

7.4 Conclusion

We investigate the trade-off relation using the λLD Cramér-Rao (CR) bound in the same physical model used in chapter 6, where the trade-off relation was analyzed using the SLD CR bound. Using the λLD CR bound, we show analytically that there always exists a λ that gives a trade-off relation for any given spread of the wave function κ and any given velocity of the observer speed v. We also show analytically that the strength of the trade-off relation defined as the distance between the intersection of the λLD CR bound and the SLD CR bound is most significant in the limit where the λ is zero. We interpret this result as follows. When the λ is close to zero, the vertex of the hyperbola or the curve of the λLD CR bound, goes more upward than the asymptotes of the λLD go down as the λ becomes smaller.

Chapter 8

Summary and outlook

8.1 Summary

We have investigated whether the error trade-off relation exists in the generic two-parameter unitary models for finite dimensional systems with commuting generators. By analyzing the necessary and sufficient conditions for the SLD, RLD and λLD Crameŕ-Rao (CR) bounds to intersect each other, we obtain the necessary and sufficient conditions for the existence of a non-trivial trade-off relation based on the SLD and RLD CR bounds for the arbitrary finite-dimensional system.

By using the conditions, we show that a trade-off relation does not exist for the pure reference state and the qubit reference state with the commuting generators case. We show, however, that a trade-off relation does exist for the qutrit reference state. We offer two examples of the qutrit system with the non-trivial trade-off relation. The result of the reference state with multi-parameter indicates that the eigenvalues of the reference state be in a specific range. In the other model reference state with one-parameter, we show analytically that a non-trivial trade-off relation always exists in a specific range of the reference state parameter and that the region with the trade-off relation is up to about half of the allowed region.

We next investigate a physical model with a continuous infinite degree of freedom. We investigate the trade-off relation between x and y components of the position of one electron in a uniform magnetic field by the parameter estimation in the two-parameter unitary models. The trade-off relation between the two commuting observables, $(x,\ y)$ was investigated by the quantum estimation theory. For the mixed state (thermal state), the bound is also given by both the SLD and RLD CR bounds with the commuting generators of the unitary transformation. We confirmed that what we saw in our study regarding the finite is not exceptional, but common.

Finally, we investigate how the relativistic effects affect estimation accuracy in a similar setting as the model above. We obtain the accuracy limit for estimating the expectation value of the position of a relativistic particle for an observer moving in one direction at a constant velocity. We evaluate estimation accuracy of the position by the SLD and λLD CR bounds. Although the SLD CR bounds does not give a trade-off relation, we see that estimation accuracy is degraded by increasing the observer's velocity. We also see that this is because the spin up state appears in the moving frame while the spin down state exists in the rest frame. Furthermore, it stays finite even at the relativistic limit. Since we show that the SLD CR bound is not achievable, it is not guaranteed that a tight CR bound gives a finite bound at the relativistic limit. However, since the Wigner rotation can be expressed as a rotation matrix that acts on a state vector, we expect that any divergent behavior will not arise from applying the Wigner rotation to a state vector with a finite spread.

8.2 Outlook

The reason we applied the λLD instead of the RLD to a relativistic particle was that this model happened to have no RLD. It was quite unexpected that such non-trivial results could be obtained with the combination of λLD and SLD Cramér-Rao bounds. Therefore, it is worthwhile to conduct the research we did previously using the λLD.

The model used for relativistic particles in this study is the method used in previous studies, for example, in [23]. In this way, the model only considers a positive energy particle (an electron). Our model does not include a negative energy particle (a positron) that appears in the solution of the Dirac equation. This is an issue we need to look at.

Appendix A

Supplemental materials for Chapter 2 λLD CR inequality

A.1 Canonical projection

- Setting
  1. Given a vector space V ($\mathrm{dim}V = d$) and consider a linear subspace T spanned by l_i ($i = 1, 2, \cdots, n$).
$$T = \mathrm{span}\{l_i\}_{i=1}^{n} \subset V \quad (n \leq d). \tag{A.1}$$
  2. Let $\langle \cdot, \cdot \rangle$ be a inner product on V.

Definition: Canonical (orthogonal) projection

$\Pi_T \colon V \to T$ is said to be the canonical projection onto T with respect to $\langle \cdot, \cdot \rangle$, if

$$\forall v \in V,\ \forall t \in T,\ \langle t, v \rangle = \langle t, \Pi_T(V) \rangle \tag{A.2}$$

hold.

For $T = \mathrm{span}\{l_i\}_{i=1}^{n}$, $\Pi_{\mathrm{T}}(V)$ can be constructed as follows.

$$\Pi_{\mathrm{T}}(V) = \sum_{i=1}^{n} \langle l^i, v \rangle l_i, \tag{A.3}$$

where $l^i = \sum_{j=1} g_{ji}^{-1} l_j$ and g_{ij}^{-1} is the (i, j) component of inverse of the gram matrix $[G]_{ij} = g_{ij}$, $g_{ij} := \langle l_i, l_j \rangle$.

Proof

Since $t \in T$, by sing l_i, t is expressed as

$$t = \sum_{j=1}^{d} t_j l_j \tag{A.4}$$

Lemma A.1.1 (Lemma: Canonical projection)
For any given $v \in V$, $\|v\| \geq \|\Pi_{\mathrm{T}}\|$, *where* $\|\cdot\| := (\langle\cdot,\cdot\rangle)^{\frac{1}{2}}$. *Equality holds if and only if* $v \in T$.

A.2 CR like inequality

Here are the definition of the terms used in this section.

- Setting
  W: a vector space with inner product $\langle\cdot.\cdot\rangle$.
  Given a set of vectors $\vec{X} = (X^i)(i = 1, 2, \cdots n)$, $(X^i \in W)$ such that $\vec{X}$ is locally unbiased which is defined by
  $$\forall i, j\ \langle X^i, l_j\rangle = \delta^i{}_j. \tag{A.5}$$
- Definition The MSE matrix about $\vec{X}$ is defined by
  $$\mathsf{V}[\vec{X}] := \mathsf{V}_{ij}[\vec{X}], \quad \mathsf{V}_{ij}[\vec{X}] := \langle X^i, X^j\rangle. \tag{A.6}$$
- l_i is a score function such that $\langle \mathrm{I}, l_j\rangle = 0\ \ \forall i$ and $T := \mathrm{span}\{l_i\}_{i=1}^n$.

Theorem A.2.1 (Theorem: CR like inequality)
For any locally unbiased $\vec{X}$ *at* θ, *its MSE matrix is bounded by*

$$\mathsf{V}_\theta[\vec{X}] \geq J^{-1}, \tag{A.7}$$

where

$$J := J_{ij}, \ \ J_{ij} := \langle l_i, l_j\rangle. \tag{A.8}$$

Proof

We prove

$$\forall c \in \mathbb{C}^n, \quad c^\dagger \mathsf{V}[\vec{X}]c \geq c^\dagger J^{-1} c. \tag{A.9}$$

By definition

$$c^\dagger \mathsf{V}[\vec{X}]c = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i^\dagger \mathsf{V}_{ij}[\vec{X}]c_j \tag{A.10}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i^\dagger \langle X^i, X^j\rangle c_j \tag{A.11}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i^\dagger \langle X^i, X^j\rangle c_j, \tag{A.12}$$

where

$$X_c := \sum_i^n c_i X^i \in W. \tag{A.13}$$

Therefore, we have

$$c^\dagger \mathsf{V}[\vec{X}]c = \|X_c\|^2 \geq \|\Pi_{\mathrm{T}}(X_c)\|^2. \tag{A.14}$$

We use Lemma A1.1. Here we calculate $\Pi_{\mathrm{T}}(X_c)$

$$\Pi_{\mathrm{T}}(X_c) = \sum_{i=1}^n \langle X_c, l^i \rangle l_i \tag{A.15}$$

$$= \sum_{i=1}^n \sum_{j=1}^n c_j^* \langle X^j, l^i \rangle l_i \quad (\because \text{ definition of } X_c) \tag{A.16}$$

$$= \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n c_j^* (J^{-1})^{ki} \langle X^j, l_k \rangle l_i \quad (\because \text{ definition of } l^i) \tag{A.17}$$

$$= \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n c_j^* (J^{-1})^{ki} \delta^j_k l_i \quad (\because \text{locally unbiasedness}) \tag{A.18}$$

$$= \sum_{i=1}^n \sum_{j=1}^n c_j^* (J^{-1})^{ji} l_i. \tag{A.19}$$

Therefore, we have

$$\|\Pi_{\mathrm{T}}(X_c)\|^2 = \langle \Pi_{\mathrm{T}}(X_c), \Pi_{\mathrm{T}}(X_c) \rangle \tag{A.20}$$

$$= \sum_{i,j=1}^n \sum_{i',j'=1}^n c_j (J^{-1\,ji})^* c_{j'}^* (J^{-1\,j'i'}) \langle l_i l_{i'} \rangle \tag{A.21}$$

$$= \sum_{i,j=1}^n \sum_{i',j'=1}^n c_j c_{j'}^* (J^{-1\,ji})^* (J^{-1\,j'i'}) \langle l_i l_{i'} \rangle \tag{A.22}$$

$$= \sum_{i,i'=1}^n \sum_{j,j'=1}^n c_j c_{j'}^* (J^{-1\,ji})^* (J^{-1\,j'i'}) J_{ii'}. \tag{A.23}$$

The expression

$$\Pi_{\mathrm{T}}(\mathsf{V}) = \sum_i^n \langle \mathsf{V}, l^i \rangle, \tag{A.24}$$

is true only for a real vector. Here, we prove the case of real vector spaces. Then, $\|\Pi_{\mathrm{T}}(X_c)\|^2$ is calculated as

$$\|\Pi_{\mathrm{T}}(X_c)\|^2 = \sum_{j,j'=1}^{n} c_j c_{j'}^* \sum_{i,i'=1}^{n} ({J^{-1}}^{ji})^* ({J^{-1}}^{j'i'}) J_{ii'} \tag{A.25}$$

$$= \sum_{j,j'=1}^{n} c_j c_{j'}^* \sum_{i,i'=1}^{n} ({J^{-1}}^{ji})^* J_{ii'} ({J^{-1}}^{j'i'}) \quad (\because J_{ii'} = J_{i'i}) \tag{A.26}$$

$$= \sum_{j,j'=1}^{n} c_j c_{j'}^* \sum_{i'=1}^{n} \delta^{j}_{i'} ({J^{-1}}^{j'i'}) \tag{A.27}$$

$$= \sum_{j,j'=1}^{n} c_j c_{j'}^* ({J^{-1}}^{jj'}) \tag{A.28}$$

$$= c^{\mathrm{T}} J^{-1} c. \tag{A.29}$$

Therefore we have

$$c^{\mathrm{T}} \mathsf{V}[\vec{X}] c = c^{\mathrm{T}} J^{-1} c. \quad \square \tag{A.30}$$

Lemma A.2.2 (Lemma: Holevo)
For any POVM Π_x, $f : \mathcal{X} \to \mathbb{C}$, define

$$F := \sum_{x \in \mathcal{X}} f(x) \Pi_x \in \mathcal{L}(\mathcal{H}). \tag{A.31}$$

Then, we have

$$\sum_{x \in \mathcal{X}} \mathrm{tr}(|f(x)|^2 \rho \Pi_x) \geq \mathrm{tr}(\rho F F^\dagger). \tag{A.32}$$

We remark $F^\dagger \neq F$ in general.

Proof

Since $\Pi_x \geq 0$,

$$(f(x)I - F)\Pi_x (f(x)I - F)^\dagger \geq 0 \text{ for } \forall x \in \mathcal{X}. \tag{A.33}$$

Eq. (A.33) can be shown as follows.
For $\forall A \geq 0,\ B \in \mathcal{L}(\mathcal{H})$

$$A \geq 0 \Longrightarrow BAB^\dagger \geq 0, \tag{A.34}$$

$$A \geq 0 \Longleftrightarrow \forall c \in \mathbb{C},\ c^\dagger A c \geq 0, \tag{A.35}$$

then for any $c' \in \mathbb{C}$,

$$c'^\dagger BAB^\dagger c' = (B^\dagger c')^\dagger A. \tag{A.36}$$

From Eq. (A.33), the summation over x gives

$$\sum_{x\in\mathcal{X}}(f(x)I - F)\Pi_x(f(x)I - F)^\dagger \geq 0 \quad (\because A,\ B \geq 0 \Longrightarrow A + B \geq 0), \tag{A.37}$$

$$\Longleftrightarrow \sum_{x\in\mathcal{X}}[f(x)\Pi_x f^*(x) - F\Pi_x f^*(x) - f(x)\Pi_x F^\dagger + F\Pi_x F^\dagger] \geq 0, \tag{A.38}$$

$$\Longleftrightarrow \sum_{x\in\mathcal{X}} |f(x)|^2\Pi_x - FF^\dagger - FF^\dagger IFIF^\dagger \geq 0, \tag{A.39}$$

$$\Longleftrightarrow \sum_{x\in\mathcal{X}} |f(x)|^2\Pi_x \geq FF^\dagger. \tag{A.40}$$

If $A \geq B$, $\mathrm{tr}(A\rho) \geq \mathrm{tr}(B\rho)$ holds. Therefore, from Eq. (A.40), we have

$$\sum_{x\in\mathcal{X}} \mathrm{tr}(|f(x)|^2\rho\Pi_x) \geq \mathrm{tr}(\rho FF^\dagger). \quad \square \tag{A.41}$$

A.3 λLD CR inequality

Theorem A.3.1 (Theorem: λLD Cramér-Rao inequality)
For any locally unbiased estimator $(\Pi, \hat{\theta})$ at θ, its MSE matrix satisfies

$$\mathsf{V}_\theta[\Pi, \hat{\theta}] \geq J_{\lambda\,\theta}^{-1}, \tag{A.42}$$

where $J_{\lambda\,\theta}$ is the λLD Fisher information matrix.

Proof

From locally unbiased condition, we have the followings.

$$\begin{aligned}
&\partial_j \sum_{x\in\mathcal{X}} \hat{\theta}_i \mathrm{tr}(\rho_\theta\Pi_x) = \delta_{ij}, \\
\Longleftrightarrow &\sum_{x\in\mathcal{X}}(\hat{\theta}_i - \theta_i)\mathrm{tr}(\partial_j\rho_\theta\Pi_x) = \delta_{ij} \quad \left(\because \sum_{x\in\mathcal{X}} \theta_i \mathrm{tr}(\rho_\theta\Pi_x) = 0\right), \\
\Longleftrightarrow &\mathrm{tr}[\partial_j\rho_\theta(\sum_{x\in\mathcal{X}} \xi_i(x)\Pi_x)] = \delta_{ij} \quad \text{where} \quad \xi_i(x) := \hat{\theta}_i - \theta_i, \\
\Longleftrightarrow &\mathrm{tr}(\partial_j\rho_\theta X_i) = \delta_{ij} \quad \text{where} \quad X_i := \sum_{x\in\mathcal{X}} \xi_i(x)\Pi_x, \quad X_i \text{ is Hermitian}, \\
\Longleftrightarrow &\langle X_i, \partial_j\rho_\theta\rangle_{\mathrm{HS}} = \delta_{ij}, \\
\Longleftrightarrow &\langle X_i, L_{\lambda\,\theta,i}\rangle^{\lambda}_{\rho_\theta} = \delta_{ij}.
\end{aligned} \tag{A.43}$$

In the last line, we use the following relation.

$$\begin{aligned}\because\ \langle \partial_j \rho_\theta X_i \rangle_{\mathrm{HS}} &= \langle X_i, \frac{1+\lambda}{2}\rho_\theta L_{\lambda\,\theta,\,i} + \frac{1-\lambda}{2} L_{\lambda\,\theta,\,i}\rho_\theta \rangle_{\mathrm{HS}} \\ &= \frac{1+\lambda}{2}\mathrm{tr}(\rho_\theta L_{\lambda\,\theta,\,i} X_i) + \frac{1-\lambda}{2}\mathrm{tr}(\rho_\theta X_i^\dagger L_{\lambda\,\theta,\,i}) \\ &= \langle X_i, L_{\lambda\,\theta,\,i}\rangle^\lambda_{\rho_\theta}.\end{aligned} \tag{A.44}$$

Then, for any $c \in \mathbb{C}$, the MSE matrix is expressed as

$$c^\dagger \mathsf{V}_\theta[\Pi, \hat{\theta}]c = \sum_{i,j=1}^{n} \sum_{x\in\mathcal{X}} c_i^* \xi^i(x)\mathrm{tr}(\rho_\theta \Pi_x)\xi^j(x)c_i \tag{A.45}$$

$$= \sum_{x\in\mathcal{X}} \xi_c(x)\mathrm{tr}(\rho_\theta \Pi_x)\xi_c(x), \tag{A.46}$$

where

$$\xi_c(x) = \sum_{i=1}^{n} c_i \xi^i(x). \tag{A.47}$$

By applying Holevo's lemma to Eq. (A.46), Lemma A.2.2, we have

$$\sum_{x\in\mathcal{X}} \xi_c(x)\mathrm{tr}(\rho_\theta \Pi_x)\xi_c(x) \geq \langle X_c, X_c\rangle^\lambda_{\rho_\theta}. \tag{A.48}$$

The term $\langle X_c, X_c\rangle^\lambda_{\rho_\theta}$ is expressed as

$$\langle X_c, X_c\rangle^\lambda_{\rho_\theta} = c^\dagger Z_{\lambda\,\theta}[X]c, \tag{A.49}$$

where

$$Z_{\lambda\,\theta}[X]c = \langle X^i, X^j\rangle^\lambda_{\rho_\theta}, \tag{A.50}$$

with

$$X^i = \sum_{x\in\mathcal{X}} \xi^i(x)\Pi_x. \tag{A.51}$$

Therefore, we have

$$\mathsf{V}_\theta[\Pi, \hat{\theta}] = Z_{\lambda\,\theta}[X]. \tag{A.52}$$

The X^i is locally unbiased at θ. Then, applying the CR like inequality to Eq. (A.50) gives

$$\mathsf{V}_\theta[\Pi, \hat{\theta}] \geq (J_{\lambda\,\theta})^{-1}. \quad \square \tag{A.53}$$

Appendix B

Supplemental materials for Chapter 3

B.1 A derivation of $\det(\mathsf{V} - A) \geq \det B$, Eq. (3.34)

First, we parametrize the weight matrix W as follows.

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} = \begin{pmatrix} w_{11} & \sqrt{w_{11}w_{22}}\epsilon \\ \sqrt{w_{11}w_{22}}\epsilon & w_{22} \end{pmatrix}. \tag{B.1}$$

Since W is a real symmetric matrix, $\det W > 0$ and $\mathrm{tr}W > 0$ are the necessary and sufficient conditions for $W > 0$.

$$\det W = w_{11}w_{22}(1 - \epsilon^2) > 0 \text{ and } \mathrm{tr}W = w_{11} + w_{22} > 0 \tag{B.2}$$

$$\iff |\epsilon| < 1, w_{11} > 0, \text{ and } w_{22} > 0. \tag{B.3}$$

From Eq. (3.27), we have

$$\mathrm{tr}[W(V - A)] \geq \mathrm{tr}|\sqrt{W}B\sqrt{W}|, \tag{B.4}$$

where

$$A = \mathrm{Re}(J_{\mathrm{Q}\,\theta}^{-1}), \tag{B.5}$$

$$B = \mathrm{Im}(J_{\mathrm{Q}\,\theta}^{-1})B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}. \tag{B.6}$$

By a direct calculation, the eigenvalues of $\sqrt{W}B\sqrt{W}$, $\lambda_\pm$ are obtained as

$$\lambda_\pm = \pm\mathrm{i}\sqrt{w_{11}w_{22}(1 - \epsilon^2)}|b| = \pm\sqrt{\det W \det B}. \tag{B.7}$$

Since $Tr|X|$ is a sum of the absolute values of the eigenvalues of X, $\mathrm{tr}|\sqrt{W}B\sqrt{W}|$ is obtained as

$$\mathrm{tr}|\sqrt{W}B\sqrt{W}| = 2\sqrt{w_{11}w_{22}(1 - \epsilon^2)}|b| = 2\sqrt{\det W \det B}. \tag{B.8}$$

By substituting Eq. (B.1) in an alternative expression of the CR inequality Eq. (3.33) which is

$$\mathrm{tr}W(\mathsf{V} - A) \geq 2\sqrt{\det W \det B}, \tag{B.9}$$

A direct calculation gives the left and right hand side of the inequality above as

$$\mathrm{tr}[W(\mathsf{V}-A)] = (\mathsf{V}_{11}-A_{11})w_{11} + (\mathsf{V}_{22}-A_{22})w_{22} + 2(\mathsf{V}_{12}-A_{12})\sqrt{w_{11}w_{22}}\epsilon, \tag{B.10}$$

and

$$2\sqrt{\det W\ \det B} = 2\sqrt{w_{11}w_{22}(1-\epsilon^2)}\sqrt{\det B} \tag{B.11}$$

respectively. By dividing both sides by w_{11}, we have

$$(\mathsf{V}_{11}-A_{11}) + (\mathsf{V}_{22}-A_{22})\frac{w_{22}}{w_{11}} + 2(\mathsf{V}_{12}-A_{12})\sqrt{\frac{w_{22}}{w_{11}}}\epsilon \geq 2\sqrt{\frac{w_{22}}{w_{11}}(1-\epsilon^2)} \tag{B.12}$$

We have

$$\mathsf{V}_{11}-A_{11} \geq -2\delta\epsilon(\mathsf{V}_{12}-A_{12}) - \delta^2(\mathsf{V}_{22}-A_{22}) + 2\sqrt{\det B}\sqrt{(1-\epsilon^2)}\,\delta, \tag{B.13}$$

where $\delta = \sqrt{w_{22}/w_{11}}$. We define $\tilde{g}(\delta,\epsilon)$ by

$$\tilde{g}(\delta,\epsilon) = -a\delta^2 + h(\epsilon)\delta, \tag{B.14}$$

where $a = \mathsf{V}_{22} - A_{22}$ and $h(\epsilon) = -2\epsilon(\mathsf{V}_{12}-A_{12}) + 2\sqrt{\det B}\sqrt{(1-\epsilon^2)}$. Then, (B.13) is written as

$$\mathsf{V}_{11}-A_{11} \geq \tilde{g}(\delta,\epsilon). \tag{B.15}$$

We maximized $\tilde{g}(\delta,\epsilon)$ under the conditions $\delta > 0$ and $|\epsilon| < 1$. We impose another condition $a > 0$, because $\tilde{g}(\delta,\epsilon)$ has its maximum if and only if $a > 0$. Then, we have

$$\tilde{g}(\delta,\epsilon) = -a\delta^2 + h(\epsilon)\delta = -a(\delta - \frac{h(\epsilon)}{2a})^2 + \frac{h(\epsilon)^2}{4a}. \tag{B.16}$$

Therefore, $\tilde{g}(\delta,\epsilon)$ takes its maximum $\tilde{g}_{\max}(\delta,\epsilon)$at $\delta = \dfrac{h(\epsilon)}{2a}$. Since $h(\epsilon) = 2a\delta$, we have

$$\tilde{g}_{\max}(\delta,\epsilon) = a\delta^2. \tag{B.17}$$

At the extremum, the relation between ϵ and δ is given by

$$\frac{\partial\tilde{g}(\delta,\epsilon)}{\partial\delta} = -2a\delta + h(\epsilon) = 0, \tag{B.18}$$

$$\frac{\partial\tilde{g}(\delta,\epsilon)}{\partial\epsilon} = -2(\mathsf{V}_{12}-A_{12}) + 2\sqrt{\det B}\frac{\epsilon}{\sqrt{(1-\epsilon^2)}} = 0. \tag{B.19}$$

As the solutions, we have

$$\delta = \frac{h(\epsilon)}{2a} \tag{B.20}$$

$$\epsilon = \pm\frac{\mathsf{V}_{12}-A_{12}}{\sqrt{(\mathsf{V}_{12}-A_{12})^2 + \det B}} \tag{B.21}$$

By using this expression of ϵ, we have $1-\epsilon^2$ as

$$1-\epsilon^2 = \frac{\det B}{(\mathsf{V}_{12}-A_{12})^2+\det B} > 0. \tag{B.22}$$

Therefore, $|\epsilon| < 1$ holds.

When $\epsilon = \dfrac{\mathsf{V}_{12}-A_{12}}{(\mathsf{V}_{12}-A_{12})^2+\det B}$,

$$a\delta_+^2 = \frac{\det B + (\mathsf{V}_{12}-A_{12})^2}{\mathsf{V}_{22}-A_{22}} \tag{B.23}$$

When $\epsilon = -\dfrac{\mathsf{V}_{12}-A_{12}}{(\mathsf{V}_{12}-A_{12})^2+\det B}$,

$$a\delta_-^2 = \frac{[\det B - (\mathsf{V}_{12}-A_{12})^2]^2}{[\det B + (\mathsf{V}_{12}-A_{12})^2](\mathsf{V}_{22}-A_{22})} \tag{B.24}$$

Since we see $a\delta_+^2 > a\delta_-^2$, the maximum of $\tilde{g}(\delta,\epsilon)$, $\tilde{g}_{\max}(\delta,\epsilon)$ is obtained as

$$\tilde{g}_{\max}(\delta,\epsilon) = a\delta_-^2 = \frac{\det B + (\mathsf{V}_{12}-A_{12})^2}{\mathsf{V}_{22}-A_{22}}. \tag{B.25}$$

From $\mathsf{V}_{11}-A_{11} > \tilde{g}(\delta,\epsilon)$,

$$\mathsf{V}_{11}-A_{11} \geq \frac{\det B + (\mathsf{V}_{12}-A_{12})^2}{\mathsf{V}_{22}-A_{22}}. \tag{B.26}$$

We have

$$\det(\mathsf{V}-A) \geq \det B, \tag{B.27}$$

for $W \in \mathcal{W}$. This is the same as $\det(\mathsf{V}-A) \geq \det B$, Eq. (3.34).

When $W \in \tilde{\mathcal{W}}$, $\epsilon = 0$. Then,

$$\tilde{g}(\delta,0) = -(\mathsf{V}_{22}-A_{22})\delta^2 + 2\sqrt{\det B}. \tag{B.28}$$

From $\dfrac{d\tilde{g}(\delta,0)}{d\delta} = 0$, we have $\delta = \sqrt{\det B}/(\mathsf{V}_{22}-A_{22})$. $\tilde{g}_{\max}(\delta,0)$ is obtained as

$$\tilde{g}_{\max}(\delta,0) = (\mathsf{V}_{22}-A_{22})\delta^2 = \frac{\det B}{\mathsf{V}_{22}-A_{22}}. \tag{B.29}$$

We obtain

$$(\mathsf{V}_{11}-A_{11})(\mathsf{V}_{22}-A_{22}) \geq \det B, \tag{B.30}$$

for $W \in \tilde{\mathcal{W}}$. $\square$

Appendix C

Supplemental materials for Chapter 4

C.1 Derivation of $[L_{S\theta 1}, L_{S\theta 2}] |\psi_\theta\rangle = 4 \langle\psi_\theta| [X, Y] |\psi_\theta\rangle |\psi_\theta\rangle$

The SLD $L_{S\theta i}$ are given by [48]

$$L_{S\theta 1} = 2\partial_1(|\psi_\theta\rangle\langle\psi_\theta|) = -2\mathrm{i}X\rho_\theta + 2\mathrm{i}\rho_\theta X = 2\mathrm{i}[\rho_\theta, X], \tag{C.1}$$
$$L_{S\theta 2} = 2\partial_2(|\psi_\theta\rangle\langle\psi_\theta|) = -2\mathrm{i}Y\rho_\theta + 2\mathrm{i}\rho_\theta Y = 2\mathrm{i}[\rho_\theta, Y]. \tag{C.2}$$

By using these, $[L_{S\theta 1}, L_{S\theta 2}] |\psi_\theta\rangle$ is written as

$$\begin{aligned}[L_{S\theta 1}, L_{S\theta 2}] |\psi_\theta\rangle &= -4[[\rho_\theta, X], [\rho_\theta, Y]] |\psi_\theta\rangle \\ &= -4[(\rho_\theta X - X\rho_\theta), (\rho_\theta Y - Y\rho_\theta)] |\psi_\theta\rangle \\ &= -4[\rho_\theta X, (\rho_\theta Y - Y\rho_\theta)] |\psi_\theta\rangle + 4[X\rho_\theta, (\rho_\theta Y - Y\rho_\theta)] |\psi_\theta\rangle .\end{aligned} \tag{C.3}$$

The first term of the right hand side is

$$\begin{aligned}-4[\rho_\theta X, (\rho_\theta Y - Y\rho_\theta)] |\psi_\theta\rangle =& - 4[\rho_\theta X, \rho_\theta Y] |\psi_\theta\rangle + 4[\rho_\theta X, Y\rho_\theta] |\psi_\theta\rangle \\ =& - 4(\rho_\theta X\rho_\theta Y - \rho_\theta Y\rho_\theta X) |\psi_\theta\rangle + 4(\rho_\theta XY\rho_\theta - Y\rho_\theta\rho_\theta X) |\psi_\theta\rangle \\ =& - 4(|\psi_\theta\rangle\langle\psi_\theta| X |\psi_\theta\rangle\langle\psi_\theta| Y |\psi_\theta\rangle - |\psi_\theta\rangle\langle\psi_\theta| Y |\psi_\theta\rangle\langle\psi_\theta| X |\psi_\theta\rangle) \\ &+ 4(|\psi_\theta\rangle\langle\psi_\theta| XY |\psi_\theta\rangle - Y |\psi_\theta\rangle\langle\psi_\theta| X |\psi_\theta\rangle) \\ =&4(\langle\psi_\theta| XY |\psi_\theta\rangle |\psi_\theta\rangle - \langle\psi_\theta| X |\psi_\theta\rangle Y |\psi_\theta\rangle).\end{aligned} \tag{C.4}$$

We use $\rho_\theta = |\psi_\theta\rangle\langle\psi_\theta|$ and $\rho_\theta^2 = |\psi_\theta\rangle\langle\psi_\theta|\psi_\theta\rangle\langle\psi_\theta| = |\psi_\theta\rangle\langle\psi_\theta| = \rho_\theta$.
The second term of the right hand side is

$$\begin{aligned}4[X\rho_\theta, (\rho_\theta Y - Y\rho_\theta)] |\psi_\theta\rangle =&4[X\rho_\theta, \rho_\theta Y] |\psi_\theta\rangle - 4[X\rho_\theta, Y\rho_\theta] |\psi_\theta\rangle \\ =&4(X\rho_\theta Y - \rho_\theta YX\rho_\theta) |\psi_\theta\rangle - 4(X\rho_\theta Y\rho_\theta - Y\rho_\theta X\rho_\theta) |\psi_\theta\rangle) \\ =&4(X |\psi_\theta\rangle\langle\psi_\theta| Y |\psi_\theta\rangle - |\psi_\theta\rangle\langle\psi_\theta| YX |\psi_\theta\rangle \\ &- X |\psi_\theta\rangle\langle\psi_\theta| Y |\psi_\theta\rangle + Y |\psi_\theta\rangle\langle\psi_\theta| X |\psi_\theta\rangle) \\ =&4(- |\psi_\theta\rangle\langle\psi_\theta| YX |\psi_\theta\rangle + \langle\psi_\theta| X |\psi_\theta\rangle Y |\psi_\theta\rangle).\end{aligned} \tag{C.5}$$

By adding the first term Eq. (C.4) and the second term Eq. (C.5), we obtain

$$\begin{aligned}[L_{S\theta 1}, L_{S\theta 2}]\,|\psi_\theta\rangle &= 4(|\psi_\theta\rangle\langle\psi_\theta|\,XY\,|\psi_\theta\rangle - |\psi_\theta\rangle\langle\psi_\theta|\,YX\,|\psi_\theta\rangle) \\ &= 4\,\langle\psi_\theta|\,[X,Y]\,|\psi_\theta\rangle\,|\psi_\theta\rangle\,. \quad \square \end{aligned} \tag{C.6}$$

C.2 Derivation of SLD and RLD Fisher information matrix J_S and J_R and their inverse matrices

C.2.1 Derivation of ${J_S}^{-1}$, Eq. (4.19)

The SLD and the SLD Fisher information matrix is defined by

$$\partial_i\rho_\theta = \frac{1}{2}(\rho_\theta L_{S\,\theta,i} + L_{S\,\theta,i}\rho_\theta), \tag{C.7}$$

$$[J_S]_{ij} = \frac{1}{2}\mathrm{tr}[\rho_\theta(L_{S\,\theta i}L_{S\,\theta j} + L_{S\,\theta,j}L_{S\,\theta,i})], \tag{C.8}$$

The SLD Fisher information matrix $[J_S]_{ij}$ can be expressed as follows.

$$\mathrm{tr}[\partial_i\rho_\theta L_{S\,\theta j}] = \frac{1}{2}\mathrm{tr}[(\rho_\theta L_{S\,\theta i} + L_{S\,\theta,i}\rho_\theta)L_{S\,\theta,j}] \tag{C.9}$$

$$= \frac{1}{2}\mathrm{tr}(\rho_\theta L_{S\,\theta i}L_{S\,\theta,j}) + \frac{1}{2}\mathrm{tr}(L_{S\,\theta,i}\rho_\theta L_{S\,\theta,j}) \tag{C.10}$$

$$= \frac{1}{2}\mathrm{tr}(\rho_\theta L_{S\,\theta i}L_{S\,\theta,j}) + \frac{1}{2}\mathrm{tr}(\rho_\theta L_{S\,\theta,j}L_{S\,\theta,i}). \tag{C.11}$$

At the end of the line, we use the cyclic property of the trace, i.e, the trace being invariant under cyclic permutation. Hence, we have

$$[J_S\,]_{ij} = \mathrm{tr}(\partial_i\rho_\theta L_{S\,\theta j}). \tag{C.12}$$

i) Evaluation of $\partial_i\rho_\theta|_{\theta=0}$
We evaluate the left hand side of the equation above, $\partial_i\rho_\theta$. By differentiating the both sides of Eq. (4.16), we have

$$\partial_1\rho_\theta|_{\theta=0} = -\mathrm{i}X\rho_0 + \mathrm{i}\rho_0 X = \mathrm{i}[\rho_0, X], \tag{C.13}$$

$$\partial_2\rho_\theta|_{\theta=0} = -\mathrm{i}Y\rho_0 + \mathrm{i}\rho_0 Y = \mathrm{i}[\rho_0, Y], \tag{C.14}$$

where $[A,B] = AB - BA$. To calculate the term $[\rho_0, X]$, we first calculate $\rho_0 X$ and $X\rho_0$. By using Eqs. (4.12, 4.17), the terms $\rho_0 X$ and $X\rho_0$ are calculated as

$$\rho_0 X = \frac{1}{2}(\mathrm{I} + \vec{s_0}\vec{\sigma})(I + \vec{x}\vec{\sigma}) = \frac{1}{2}[\mathrm{I} + \vec{x}\vec{\sigma} + \vec{s_0}\vec{\sigma} + (\vec{s_0}\vec{\sigma})(\vec{x}\vec{\sigma})], \tag{C.15}$$

$$X\rho_0 = \frac{1}{2}(I + \vec{x}\vec{\sigma})(\mathrm{I} + \vec{s_0}\vec{\sigma}) = \frac{1}{2}[\mathrm{I} + \vec{x}\vec{\sigma} + \vec{s_0}\vec{\sigma} + (\vec{x}\vec{\sigma})(\vec{s_0}\vec{\sigma})]. \tag{C.16}$$

By using

$$(\vec{s_0}\vec{\sigma})(\vec{x}\vec{\sigma}) = (\vec{s_0}\vec{x})\mathrm{I} + \mathrm{i}(\vec{s_0} \times \vec{x})\vec{\sigma}, \tag{C.17}$$

we have

$$[\rho_0, X] = \rho_0 X - X\rho_0 = \mathrm{i}(\vec{s_0} \times \vec{x})\vec{\sigma}. \tag{C.18}$$

In the same way, we obtain

$$[\rho_0, Y] = \rho_0 Y - Y\rho_0 = \mathrm{i}(\vec{s_0} \times \vec{y})\vec{\sigma}. \tag{C.19}$$

Let us define $\vec{x_1}$ and $\vec{x_2}$ by

$$\vec{x_1} = \vec{x}, \tag{C.20}$$
$$\vec{x_2} = \vec{y}. \tag{C.21}$$

Hence, we have

$$\partial_j \rho_\theta|_{\theta=0} = -(\vec{s_0} \times \vec{x_j})\vec{\sigma} = (\vec{x_j} \times \vec{s_0})\vec{\sigma} \tag{C.22}$$

ii) SLD $L_{\mathrm{S}\,\theta,\,j}$

Let us set the SLD $L_{\mathrm{S}\,\theta,\,j}$ as

$$L_{\mathrm{S}\,\theta,\,j} = 2(\vec{x_j} \times \vec{s_0})\vec{\sigma}. \tag{C.23}$$

We check if this $L_{\mathrm{S}\,\theta,\,j}$ satisfies the definition of the SLD, Eq. (C.8).

$$\begin{aligned}
\rho_0 L_{\mathrm{S}\,\theta,\,j} &= \frac{1}{2}(\mathrm{I} + \vec{s_0}\vec{\sigma})2(\vec{x_j} \times \vec{s_0})\vec{\sigma} &\text{(C.24)}\\
&= (\vec{x_j} \times \vec{s_0})\vec{\sigma} + \vec{s_0}(\vec{x_j} \times \vec{s_0})\mathrm{I} + \mathrm{i}\vec{s_0} \times (\vec{x_j} \times \vec{s_0}\vec{\sigma}) &\text{(C.25)}\\
&= (\vec{x_j} \times \vec{s_0})\vec{\sigma} + \mathrm{i}[\vec{s_0} \times (\vec{x_j} \times \vec{s_0})]\vec{\sigma}. &\text{(C.26)}
\end{aligned}$$

Here, we use

$$(\vec{s_0}\vec{\sigma})(\vec{x}\vec{\sigma}) = (\vec{s_0}\vec{x})\mathrm{I} + \mathrm{i}(\vec{s_0} \times \vec{x})\vec{\sigma}. \tag{C.27}$$

We also use the Pauli matrices being traceless.

In the same way, we have

$$\begin{aligned}
L_{\mathrm{S}\,\theta,\,j}\rho_0 &= 2(\vec{x_j} \times \vec{s_0})\vec{\sigma}\,\frac{1}{2}(\mathrm{I} + \vec{s_0}\vec{\sigma}) &\text{(C.28)}\\
&= (\vec{x_j} \times \vec{s_0})\vec{\sigma} + (\vec{x_j} \times \vec{s_0})\vec{s_0}\mathrm{I} + \mathrm{i}[(\vec{x_j} \times \vec{s_0}) \times \vec{s_0}]\vec{\sigma} &\text{(C.29)}\\
&= (\vec{x_j} \times \vec{s_0})\vec{\sigma} + \mathrm{i}[(\vec{x_j} \times \vec{s_0}) \times \vec{s_0}]\vec{\sigma} &\text{(C.30)}\\
&= (\vec{x_j} \times \vec{s_0})\vec{\sigma} - \mathrm{i}[\vec{s_0} \times (\vec{x_j} \times \vec{s_0})]\vec{\sigma}. &\text{(C.31)}
\end{aligned}$$

Hence, we have

$$\frac{1}{2}(\rho_0 L_{\mathrm{S}\,i} + \rho_0 L_{\mathrm{S}\,i}) = (\vec{x_j} \times \vec{s_0})\vec{\sigma} = \partial_i \rho_\theta|_{\theta=0}. \tag{C.32}$$

iii) Evaluation of the SLD Fisher information $[J_\mathrm{S}]_{i\,j}$ From Eq. (C.12), we have

$$\begin{aligned}
[J_\mathrm{S}]_{i\,j} &= \mathrm{tr}(\partial_i \rho_\theta|_{\theta=0}\, L_{\mathrm{S}\,j}) &\text{(C.33)}\\
&= 2\mathrm{tr}[(\vec{x_j} \times \vec{s_0})\vec{\sigma}(\vec{x_j} \times \vec{s_0})\vec{\sigma}] &\text{(C.34)}\\
&= 4(\vec{x_i} \times \vec{s_0})(\vec{x_j} \times \vec{s_0}). &\text{(C.35)}
\end{aligned}$$

At the last line, we use Eq. (C.27) and use the fact that the trace of the Pauli matrices are zero.

$$J_{\mathrm{S}} = 4\begin{pmatrix} |\vec{x}\times\vec{s}_0|^2 & (\vec{x}\times\vec{s}_0)(\vec{y}\times\vec{s}_0) \\ (\vec{x}\times\vec{s}_0)(\vec{y}\times\vec{s}_0) & |\vec{y}\times\vec{s}_0|^2 \end{pmatrix}. \tag{C.36}$$

The determinant of J_{S} is calculated as

$$\det J_{\mathrm{S}} = 16(|\vec{x}\times\vec{s}_0|^2|\vec{y}\times\vec{s}_0|^2 - |(\vec{x}\times\vec{s}_0)(\vec{y}\times\vec{s}_0)|^2) = 16\,|\vec{s}_0|^2|\vec{s}_0\cdot(\vec{x}\times\vec{y})|^2. \tag{C.37}$$

Hence, we have

$$J_{\mathrm{S}}^{-1} = \frac{4}{\det J_{\mathrm{S}}}\begin{pmatrix} |\vec{y}\times\vec{s}_0|^2 & -(\vec{x}\times\vec{s}_0)(\vec{y}\times\vec{s}_0) \\ -(\vec{x}\times\vec{s}_0)(\vec{y}\times\vec{s}_0) & |\vec{x}\times\vec{s}_0|^2 \end{pmatrix}. \quad \square \tag{C.38}$$

C.2.2 Derivation of J_{R}^{-1}, Eq. (4.20)

The RLD $L_{\mathrm{R}\,i}$ and RLD Fisher information matrix $[J_{\mathrm{R}}]_{i\,j}$ are defined by

$$\partial_i\rho_\theta = \rho_\theta L_{\mathrm{R},\,i}, \tag{C.39}$$

$$[J_{\mathrm{R}}]_{i\,j} = \mathrm{tr}(\rho_\theta L_{\mathrm{R}\,j} L_{\mathrm{R}\,i}^\dagger). \tag{C.40}$$

We omit θ of the RLD and Fisher information matrix since our model is a unitary model. By using $\partial_i\rho_\theta = L_{\mathrm{R},\,i}^\dagger \rho_\theta^{-1}$, we have We show the following holds.

$$[J_{\mathrm{R}}]_{i\,j} = \mathrm{tr}(\partial_j\rho_\theta\,\partial_i\rho_\theta\,\rho_\theta^{-1}). \tag{C.41}$$

Derivation of Eq. (C.41)

The RLD is defined by

$$\partial_i\rho_\theta = \rho_\theta L_{\mathrm{R}\,j} \tag{C.42}$$

$$\Longleftrightarrow \partial_i\rho_\theta^\dagger = L_{\mathrm{R}\,j}^\dagger \rho_\theta^\dagger \tag{C.43}$$

$$\Longleftrightarrow \partial_i\rho_\theta = L_{\mathrm{R}\,j}^\dagger \rho_\theta \tag{C.44}$$

$$\Longleftrightarrow \partial_i\rho_\theta\rho_\theta^{-1} = L_{\mathrm{R}\,j}^\dagger \tag{C.45}$$

By using the definition of RLD $\partial_i\rho_\theta = \rho_\theta L_{\mathrm{R}\,j}$ and Eq. (C.45), we have

$$[J_{\mathrm{R}}]_{i\,j} = \mathrm{tr}(\rho_\theta L_{\mathrm{R}\,j} L_{\mathrm{R}\,i}^\dagger) = \mathrm{tr}(\partial_j\rho_\theta\,\partial_i\rho_\theta\,\rho_\theta^{-1}). \quad \square \tag{C.46}$$

Since our model is a unitary model, we can set $\theta = 0$.

$$[J_{\mathrm{R}}]_{i\,j} = \mathrm{tr}(\partial_j\rho_\theta|_{\theta=0}\,\partial_i\rho_\theta|_{\theta=0}\,\rho_0^{-1}). \tag{C.47}$$

By differentiating the both sides of Eq. (4.16), we have

$$\partial_1\rho_\theta|_{\theta=0} = -\mathrm{i}X\rho_0 + \mathrm{i}\rho_0 X = \mathrm{i}[\rho_0, X], \tag{C.48}$$

$$\partial_2\rho_\theta|_{\theta=0} = -\mathrm{i}Y\rho_0 + \mathrm{i}\rho_0 Y = \mathrm{i}[\rho_0, Y]. \tag{C.49}$$

By substituting above in Eq. (C.47), we obtain

$$\begin{aligned}
[J_{\mathrm{R}}]_{11} &= -\mathrm{tr}([\rho_0, X]\,[\rho_0, X]\,\rho_0^{-1}),\\
[J_{\mathrm{R}}]_{12} &= -\mathrm{tr}([\rho_0, Y]\,[\rho_0, X]\,\rho_0^{-1}),\\
[J_{\mathrm{R}}]_{21} &= -\mathrm{tr}([\rho_0, X]\,[\rho_0, Y]\,\rho_0^{-1}),\\
[J_{\mathrm{R}}]_{22} &= -\mathrm{tr}([\rho_0, Y]\,[\rho_0, Y]\,\rho_0^{-1}),
\end{aligned} \tag{C.50}$$

To calculate the term $[\rho_0, X]$, we first calculate $\rho_0 X$ and $X\rho_0$. By using Eqs. (4.12, 4.17), the term $\rho_0 X$ and $X\rho_0$ are calculated as

$$\rho_0 X = \frac{1}{2}(\mathrm{I} + \vec{s_0}\vec{\sigma})(I + \vec{x}\vec{\sigma}) = \frac{1}{2}[\mathrm{I} + \vec{x}\vec{\sigma} + \vec{s_0}\vec{\sigma} + (\vec{s_0}\vec{\sigma})(\vec{x}\vec{\sigma})], \tag{C.51}$$

$$\rho_0 X = \frac{1}{2}(I + \vec{x}\vec{\sigma})(\mathrm{I} + \vec{s_0}\vec{\sigma}) = \frac{1}{2}[\mathrm{I} + \vec{x}\vec{\sigma} + \vec{s_0}\vec{\sigma} + (\vec{x}\vec{\sigma})(\vec{s_0}\vec{\sigma})]. \tag{C.52}$$

By using

$$(\vec{s_0}\vec{\sigma})(\vec{x}\vec{\sigma}) = (\vec{s_0}\vec{x})\mathrm{I} + \mathrm{i}(\vec{s_0} \times \vec{x})\vec{\sigma}, \tag{C.53}$$

we have

$$[\rho_0, X] = \mathrm{i}(\vec{s_0} \times \vec{x})\vec{\sigma}. \tag{C.54}$$

In the same way, we obtain

$$[\rho_0, Y] = \mathrm{i}(\vec{s_0} \times \vec{y})\vec{\sigma}. \tag{C.55}$$

By a straightforward calculation, ρ_0^{-1} is obtained as

$$\rho_0^{-1} = \frac{2}{1 - |\vec{s_0}|^2}(\mathrm{I} - \vec{s_0}\vec{\sigma}). \tag{C.56}$$

By substituting $[\rho_0, X]$, $[\rho_0, Y]$, and ρ_0^{-1} in Eq. (C.50) and using $\vec{x_1} = \vec{x}$ and $\vec{x_2} = \vec{y}$, we have

$$[J_{\mathrm{R}}]_{ij} = \frac{2}{1 - |\vec{s_0}|^2}\mathrm{tr}\{[(\vec{x_j} \times \vec{s_0})\vec{\sigma}][(\vec{x_i} \times \vec{s_0})\vec{\sigma}](\mathrm{I} - \vec{s_0})\vec{\sigma}\} \tag{C.57}$$

$$= \frac{2}{1 - |\vec{s_0}|^2}\mathrm{tr}\{[(\vec{x_j} \times \vec{s_0})\vec{\sigma}][(\vec{x_i} \times \vec{s_0})\vec{\sigma}]\} - \frac{2}{1 - |\vec{s_0}|^2}\mathrm{tr}\{[(\vec{x_j} \times \vec{s_0})\vec{\sigma}][(\vec{x_i} \times \vec{s_0})\vec{\sigma})]\vec{s_0}\vec{\sigma}\}. \tag{C.58}$$

The first term of the right hand side is calculated as follows.

$$\frac{2}{1 - |\vec{s_0}|^2}\mathrm{tr}[(\vec{x_j} \times \vec{s_0})\vec{\sigma}][(\vec{x_i} \times \vec{s_0})\vec{\sigma}] = \frac{2}{1 - |\vec{s_0}|^2}\mathrm{tr}[(\vec{x_j} \times \vec{s_0})(\vec{x_i} \times \vec{s_0})\mathrm{I} + \mathrm{i}(\vec{x_j} \times \vec{s_0}) \times (\vec{x_i} \times \vec{s_0})\vec{\sigma}] \tag{C.59}$$

$$= \frac{2}{1 - |\vec{s_0}|^2}\mathrm{tr}[(\vec{x_j} \times \vec{s_0})(\vec{x_i} \times \vec{s_0})\mathrm{I}] \tag{C.60}$$

$$= \frac{4}{1 - |\vec{s_0}|^2}(\vec{x_j} \times \vec{s_0})(\vec{x_i} \times \vec{s_0}). \tag{C.61}$$

Let us define $\vec{a}$ and $\vec{b}$ by

$$\vec{a} = \vec{x}_j \times \vec{s}_0, \tag{C.62}$$

$$\vec{b} = \vec{x}_i \times \vec{s}_0. \tag{C.63}$$

The second term is calculated as follows.

$$-\frac{2}{1-|\vec{s}_0|^2}\mathrm{tr}[(\vec{a}\vec{\sigma})(\vec{b}\vec{\sigma})(\vec{s}_0\vec{\sigma})] = -\frac{2}{1-|\vec{s}_0|^2}\mathrm{tr}\{[\vec{a}\vec{b}\mathrm{I} + \mathrm{i}(\vec{a}\times\vec{b})\vec{\sigma}]\vec{s}_0\vec{\sigma}\} \tag{C.64}$$

$$= -\frac{2}{1-|\vec{s}_0|^2}\mathrm{tr}\{(\vec{a}\vec{b})\vec{s}_0\vec{\sigma} - \mathrm{i}[(\vec{a}\times\vec{b})\vec{\sigma}]\vec{s}_0\vec{\sigma}\} \tag{C.65}$$

$$= -\mathrm{i}\frac{4}{1-|\vec{s}_0|^2}\vec{s}_0(\vec{a}\times\vec{b}) \tag{C.66}$$

$$= \mathrm{i}\frac{4}{1-|\vec{s}_0|^2}\vec{s}_0(\vec{b}\times\vec{a}) \tag{C.67}$$

With the formula

$$(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = [\vec{A}(\vec{C}\times\vec{D})]\vec{B} - [\vec{B}(\vec{C}\times\vec{D})]\vec{A}, \tag{C.68}$$

$\vec{b}\times\vec{a}$ is expressed by

$$\vec{b}\times\vec{a} = (\vec{x}_i\times\vec{s}_0)(\vec{x}_j\times\vec{s}_0) \tag{C.69}$$

$$= [\vec{x}_i(\vec{x}_j\times\vec{s}_0)]\vec{s}_0 - [\vec{s}_0(\vec{x}_j\times\vec{s}_0)]\vec{x}_i \tag{C.70}$$

$$= [\vec{x}_i(\vec{x}_j\times\vec{s}_0)]\vec{s}_0 - [\vec{x}_j(\vec{s}_0\times\vec{s}_0)]\vec{x}_i \tag{C.71}$$

$$= [\vec{x}_i(\vec{x}_j\times\vec{s}_0)]\vec{s}_0 \tag{C.72}$$

$$= [\vec{s}_0(\vec{x}_i\times\vec{x}_j)]\vec{s}_0. \tag{C.73}$$

For the second term, we have

$$\frac{2}{1-|\vec{s}_0|^2}\mathrm{tr}[(\vec{a}\vec{\sigma})(\vec{b}\vec{\sigma})(\vec{s}_0\vec{\sigma})] = \mathrm{i}\frac{4}{1-|\vec{s}_0|^2}|\vec{s}_0|^2[\vec{s}_0(\vec{x}_i\times\vec{x}_j)]]. \tag{C.74}$$

Hence, the RLD Fisher information matrix J_{R} is given by

$$J_{\mathrm{R}} = \frac{4}{1-|\vec{s}_0|^2}\begin{pmatrix} |\vec{x}\times\vec{s}_0|^2 & (\vec{x}\times\vec{s}_0)(\vec{y}\times\vec{s}_0) + \mathrm{i}|\vec{s}_0|^2[\vec{s}_0(\vec{x}\times\vec{y})] \\ (\vec{x}\times\vec{s}_0)(\vec{y}\times\vec{s}_0) - \mathrm{i}|\vec{s}_0|^2[\vec{s}_0(\vec{x}\times\vec{y})] & |\vec{y}\times\vec{s}_0|^2 \end{pmatrix}, \tag{C.75}$$

$$J_{\mathrm{R}}^{-1} = J_{\mathrm{S}}^{-1} + \frac{4}{\det J_{\mathrm{S}}}\begin{pmatrix} 0 & -\mathrm{i}\vec{s}_0^{\,2}[\vec{s}_0\cdot(\vec{x}\times\vec{y})] \\ \mathrm{i}\vec{s}_0^{\,2}[\vec{s}_0\cdot(\vec{x}\times\vec{y})] & 0 \end{pmatrix}, \tag{C.76}$$

where $\det J_{\mathrm{S}}$ is the determinant of J_{S}, and it is

$$\det J_{\mathrm{S}} = 16\,|\vec{s}_0|^2|\vec{s}_0\cdot(\vec{x}\times\vec{y})|^2. \quad \square \tag{C.77}$$

C.3 Solution u_0 of $F_\zeta(u_0) = 0$

In this section, we investigate the solution s_0 of $F_\zeta(u_0) = 0$. We check up to the fourth partial derivative of $F_\zeta(u)$ with respect to s to see $F_\zeta(u)$ in the allowed range for t and s.

Let $F_\zeta^{(n)}(u) = \frac{d^n F_\zeta(u)}{\partial u^n}$. Up to the fourth partial derivative of $F_\zeta(u)$ with respect to u are as follows

$$F_\zeta^{(1)}(u) = -45u^4 - 64(1 + 7\zeta) + 72u^3(3 + 8\zeta) + 32u(9 + 61\zeta) - 9u^2(43 + 224\zeta), \tag{C.78}$$

$$F_\zeta^{(2)}(u) = 2[-90u^3 + 108u^2(3 + 8\zeta) + 16(9 + 61\zeta) - 9u(43 + 224\zeta)], \tag{C.79}$$

$$F_\zeta^{(3)}(u) = -18[43 + 30u^2 + 224\zeta - 24u(3 + 8\zeta)], \tag{C.80}$$

$$F_\zeta^{(4)}(u) = -216(-6 + 5u - 16\zeta). \tag{C.81}$$

$F_\zeta^{(3)}(u)$ is convex upward, because the coefficient of u^2 in $F_\zeta^{(3)}(u)$ is negative. Therefore, the extremum, in this case, the maximum of $F_\zeta^{(3)}(u)$ is given by $u_0^{(4)}$ which is the solution of $F_\zeta^{(4)}(u_0^{(4)}) = 0$. The solution $u_0^{(4)}$ is given by

$$u_0^{(4)} = \frac{2}{5}(8\zeta + 3). \tag{C.82}$$

$u_0^{(4)}$ which gives the maximum of $F_\zeta^{(3)}(u)$ becomes minimum at $\zeta = 0$. At $\zeta = 0$, $u_0^{(4)} = 6/5 = 1.2 > 1/3$. Because of $u_0^{(4)} > 1/3$, $F_\zeta^{(3)}(u)$ increases monotonically in the range $0 < u < 1/3$.

$F_\zeta^{(3)}(u)$ at $u = 1/3$ is

$$F_\zeta^{(3)}(\frac{1}{3}) = -6(67 + 480\zeta) < 0 \quad \text{when} \quad (0 < \zeta < \frac{1}{3}). \tag{C.83}$$

Then, we see $F_\zeta^{(3)}(u) < 0$ when $0 < u < 1/3$. Therefore, $F_\zeta^{(2)}(u)$ decreases monotonically when $0 < u < 1/3$.

$F_\zeta^{(2)}(u)$ at $u = 1/3$ is

$$F_\zeta^{(2)}(\frac{1}{3}) = \frac{286}{3} + 800\zeta > 0 \quad \text{when} \quad (0 < \zeta < \frac{1}{3}). \tag{C.84}$$

Therefore, $F_\zeta^{(1)}(u)$ increases monotonically when $0 < u < 1/3$.

$$F_\zeta^{(1)}(\frac{1}{3}) = -\frac{32}{9} < 0 \quad \text{when} \quad (0 < \zeta < \frac{1}{3}). \tag{C.85}$$

Therefore, $F_\zeta(u)$ decreases monotonically when $0 < u < 1/3$. The values of $F_\zeta(u)$ at the both ends, $u = 0$ and $u = 1/3$ are

$$F_\zeta(0) = 64\zeta, \tag{C.86}$$

$$F_\zeta(\frac{1}{3}) = -\frac{256}{27}. \tag{C.87}$$

With a given ζ in the range $0 < \zeta \leq 1/3$, there always exists only one solution u_0 that satisfies

$F_\zeta(u_0) = 0$ in the range $0 < u_0 \leq 1/3$.

Appendix D

Supplemental materials for Chapter 5

D.1 Thermal state and Gaussian state

The thermal state for a single harmonic oscillator, ρ_β is described as

$$\rho_\beta = Z_\beta^{-1} \mathrm{e}^{-\beta H}, \tag{D.1}$$

where $Z_\beta = \mathrm{tr}\,[\mathrm{e}^{-\beta H}]$ and $\beta = T^{-1}$ is the inverse temperature.

By using Hamiltonian $H = \omega\,(a^\dagger a + 1/2)$ and $a^\dagger a\,|n\rangle = n\,|n\rangle$, $\mathrm{e}^{-\beta H}$ is

$$\mathrm{e}^{-\beta H} = \sum_{n=0}^{\infty} \mathrm{e}^{-\beta H}\,|n\rangle\,\langle n| = \mathrm{e}^{-\frac{1}{2}\beta\omega} \sum_{n=0}^{\infty} \gamma^n\,|n\rangle\,\langle n|\,, \tag{D.2}$$

where $\gamma = \mathrm{e}^{-\beta\omega}$. Z_β is

$$Z_\beta = \mathrm{tr}\,[\mathrm{e}^{-\beta H}] = \frac{\mathrm{e}^{-\frac{1}{2}\beta\omega}}{1-\gamma}. \tag{D.3}$$

We obtain

$$\rho_\beta = Z_\beta^{-1} \mathrm{e}^{-\beta H} = (1-\gamma) \sum_n \gamma^n\,|n\rangle\,\langle n|\,. \tag{D.4}$$

We first calculate the matrix element of ρ_β by the basis as the coherent state, $\langle z_1|\rho_\beta|z_2\rangle$. Next, we make the same matrix element of the Gaussian state to see if they match.

$\langle z_1|\rho_\beta|z_2\rangle$ is

$$\langle z_1|\rho_\beta|z_2\rangle = (1-\gamma) \sum_n \gamma^n\,\langle z_1|n\rangle\,\langle n|z_2\rangle \tag{D.5}$$

$$= (1-\gamma)\mathrm{e}^{-\frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2 + \gamma z_1^* z_2}. \tag{D.6}$$

The Gaussian state S_κ is defined by

$$S_\kappa = \frac{1}{2\pi\kappa^2} \int \mathrm{e}^{-\frac{|z|^2}{2\kappa^2}}\,|z\rangle\,\langle z|\,d^2 z. \tag{D.7}$$

where

$$d^2 z := d(\mathrm{Re}(z))d(\mathrm{Im}(z)). \tag{D.8}$$

Then its matrix element $\langle z_1| S_\kappa |z_2\rangle$ is

$$\langle z_1| S_\kappa |z_2\rangle = \frac{1}{2\pi\kappa^2}\int \mathrm{e}^{-\frac{|z|^2}{2\kappa^2}} \langle z_1|z\rangle \langle z|z_2\rangle\, d^2z \tag{D.9}$$

$$= \frac{1}{2\pi\kappa^2}\int \mathrm{e}^{-(\frac{1}{2\kappa^2}+1)|z|^2+z_1^* z_2+z_1 z_2^*} d^2z\, \mathrm{e}^{-\frac{1}{2}|z_1|^2-\frac{1}{2}|z_2|^2}. \tag{D.10}$$

By using

$$\int \mathrm{e}^{-\alpha|z|^2+\beta z+\gamma z^*} d^2z = \frac{\pi}{\alpha}\mathrm{e}^{\frac{\beta\gamma}{\alpha}}, \tag{D.11}$$

we obtain

$$\langle z_1| \rho_\beta |z_2\rangle = \frac{1}{2\kappa^2+1}\mathrm{e}^{-\frac{1}{2}|z_1|^2-\frac{1}{2}|z_2|^2+\frac{z_1^* z_2}{\frac{1}{2\kappa^2}+1}}. \tag{D.12}$$

From (D.6) and (D.12), $\langle z_1|S_\kappa|z_2\rangle = \langle z_1| \rho_\beta |z_2\rangle$ holds iff

$$2\kappa^2 = \frac{\gamma}{1-\gamma} = \frac{1}{\mathrm{e}^{\beta\omega}-1}. \tag{D.13}$$

Therefore, we obtain

$$\rho_\beta = \frac{1}{2\pi\kappa^2}\int \mathrm{e}^{-\frac{|z'|^2}{2\kappa^2}} |z\rangle \langle z|\, d^2z. \tag{D.14}$$

where $2\kappa^2$ is given by Eq. (D.13).

D.2 Calculation: SLD, RLD Fisher information matrices, and Z matrix

D.2.1 SLD and RLD: The thermal state as the reference state

First, we briefly explain that SLD and RLD Fisher information matrices for the mixed state are independent of the parameters $\theta = (\theta_1,\, \theta_2)$ in the unitary transformation $U(\theta_1,\, \theta_2)$.

Let SLD and RLD of our m odel be $L_{\mathrm{S},\, j}(\theta)$ and $L_{R,\, j}(\theta)$, respectively. The SLD and RLD are defined by the solutions of the following equation.

$$\frac{\partial\rho_\theta}{\partial\theta_i} = \frac{1}{2}(\rho_\theta L_{\mathrm{S},\, j}(\theta) + L_{\mathrm{S},\, j}(\theta)\rho_\theta), \tag{D.15}$$

$$\frac{\partial\rho_\theta}{\partial\theta_i} = \rho_\theta L_{\mathrm{R},\, j}(\theta). \tag{D.16}$$

With using the unitary transformation $U(\theta_1,\, \theta_2) = \mathrm{e}^{-\mathrm{i}\theta_1 p_x}\mathrm{e}^{-\mathrm{i}\theta_2 p_y}$, $L_{\mathrm{S},\, j}(0)$ and $L_{\mathrm{R},\, j}(0)$ are written as

$$L_{\mathrm{S},\, j}(\theta) = U(\theta_1,\, \theta_2)\, L_{\mathrm{S},\, j}(0)\, U^\dagger(\theta_1,\, \theta_2), \tag{D.17}$$

$$L_{\mathrm{R},\, j}(\theta) = U(\theta_1,\, \theta_2)\, L_{\mathrm{R},\, j}(0)\, U^\dagger(\theta_1,\, \theta_2). \tag{D.18}$$

For the RLD Fisher information $J_{\mathrm{R}}(\theta) = [J_{\mathrm{R},\, ij}(\theta)]$, we can derive the relation below if the transformation is unitary.

$$J_{\mathrm{R},\,ij}(\theta) = \mathrm{tr}\,[\rho_\theta\, L_{\mathrm{R},\,j}(\theta) L^\dagger_{\mathrm{R},\,i}(\theta)] = \mathrm{tr}\,[\rho_0\, L_{\mathrm{R},\,j}(0) L^\dagger_{\mathrm{R},\,i}(0)] = J_{\mathrm{R},\,ij}(0). \tag{D.19}$$

If the transformation is unitary, the RLD Fisher information $J_\mathrm{R}(\theta)$ does not depend on the parameters θ_1 and θ_2. Then, we can write $J_\mathrm{R}(\theta)$ as $J_\mathrm{R} = [J_{\mathrm{R},\,ij}]$. From Eqs. (D.17, D.18), we can show that the same holds for the SLD Fisher information $G_\mathrm{S}(\theta)$. Therefore, if we have $L_{\mathrm{S},\,i}(0)$ and $L_{\mathrm{R},\,i}(0)$, it is enough to obtain the SLD and RLD Fisher information matrices.

D.2.2 Thermal state SLD: $L_{S,1}(0)$, $L_{S,2}(0)$, Z matrix, Z

The SLD is defined by,

$$\frac{\partial \rho_\theta}{\partial \theta_i} = \frac{1}{2}(\rho_\theta L_{\mathrm{S},\,j}(\theta) + L_{\mathrm{S},\,j}(\theta)\rho_\theta). \tag{D.20}$$

The unitary transformation $\mathrm{e}^{-\mathrm{i}\theta_1 p_x}\mathrm{e}^{-\mathrm{i}\theta_2 p_y}$ is expressed by

$$\mathrm{e}^{-\mathrm{i}\theta_1 p_x}\mathrm{e}^{-\mathrm{i}\theta_2 p_y} = \mathrm{e}^{\xi_1 a^\dagger - \xi_1^* a}\mathrm{e}^{\xi_2^* b^\dagger - \xi_2 b}, \tag{D.21}$$

where

$$\xi_1 = \frac{1}{2\kappa}\theta_1 - \mathrm{i}\theta_2, \quad \xi_2 = \xi_1^*. \tag{D.22}$$

By using these, we obtain the SLD as

$$L_{\mathrm{S},1}(0) = \frac{1}{\kappa(1+4\kappa_\mathrm{a}^2)}(a+a^\dagger) + \frac{1}{\kappa(1+4\kappa_\mathrm{b}^2)}(b+b^\dagger), \tag{D.23}$$

$$L_{\mathrm{S},2}(0) = \frac{\mathrm{i}}{\kappa(1+4\kappa_\mathrm{a}^2)}(a-a^\dagger) - \frac{\mathrm{i}}{\kappa(1+4\kappa_\mathrm{b}^2)}(b-b^\dagger). \tag{D.24}$$

With using p_x, p_y and x, y,

$$L_{\mathrm{S},1}(0) = (\frac{1}{1+4\kappa_\mathrm{a}^2} + \frac{1}{1+4\kappa_\mathrm{b}^2})p_y + \frac{1}{\kappa^2}(\frac{1}{1+4\kappa_\mathrm{a}^2} - \frac{1}{1+4\kappa_\mathrm{b}^2})\,x, \tag{D.25}$$

$$L_{\mathrm{S},2}(0) = -(\frac{1}{1+4\kappa_\mathrm{a}^2} - \frac{1}{1+4\kappa_\mathrm{b}^2})p_x + \frac{1}{\kappa^2}(\frac{1}{1+4\kappa_\mathrm{a}^2} + \frac{1}{1+4\kappa_\mathrm{b}^2})\,y. \tag{D.26}$$

The SLD Fisher information matrix J_S is calculated as

$$J_\mathrm{S} = \frac{1}{\kappa^2}(\frac{1}{1+4\kappa_a^2} + \frac{1}{1+4\kappa_b^2})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{2(1+\kappa_a^2+\kappa_b^2)}{\kappa^2(1+4\kappa_a^2)(1+4\kappa_b^2)}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{D.27}$$

Z matrix Z is calculated as

$$Z = \frac{\kappa^2}{1+2\kappa_\mathrm{a}^2+2\kappa_\mathrm{b}^2}\begin{pmatrix}\frac{1}{2}+2\kappa_\mathrm{a}^2+2\kappa_\mathrm{b}^2+8\kappa_\mathrm{a}^2\kappa_\mathrm{b}^2 & \mathrm{i}\,(2\kappa_\mathrm{b}^2-2\kappa_\mathrm{a}^2)\\ -\mathrm{i}\,(2\kappa_\mathrm{b}^2-2\kappa_\mathrm{a}^2) & \frac{1}{2}+2\kappa_\mathrm{a}^2+2\kappa_\mathrm{b}^2+8\kappa_\mathrm{a}^2\kappa_\mathrm{b}^2\end{pmatrix}. \tag{D.28}$$

From this expression, we have

$$Z = (J_\mathrm{R})^{-1} + \Delta g\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{D.29}$$

Since $\Delta g \neq 0$, we see that $Z \neq (J_{\mathrm{R}})^{-1}$. This implies the model is not D-invariant [68].

D.2.3 Thermal state RLD: $L_{\mathrm{R},1}(0),\ L_{\mathrm{R},2}(0)$

By using the definition of RLD,

$$\frac{\partial \rho_\theta}{\partial \theta_i} = \rho_\theta L_{\mathrm{R},j}(\theta), \tag{D.30}$$

the unitary transformation $\mathrm{e}^{\xi_1 a^\dagger - \xi_1^* a}\mathrm{e}^{\xi_2^* b^\dagger - \xi_2 b}$, and the reference state, we obtain the RLD.

$$L_{\mathrm{R},1}(0) = \frac{1}{2\kappa}(\frac{1}{1+2\kappa_{\mathrm{a}}^2}a + \frac{1}{2\kappa_{\mathrm{a}}^2}a^\dagger) + \frac{1}{2\kappa}(\frac{1}{1+2\kappa_{\mathrm{b}}^2}b + \frac{1}{2\kappa_{\mathrm{b}}^2}b^\dagger), \tag{D.31}$$

$$L_{\mathrm{R},2}(0) = -\frac{\mathrm{i}}{2\kappa}(-\frac{1}{1+2\kappa_{\mathrm{a}}^2}a + \frac{1}{2\kappa_{\mathrm{a}}^2}a^\dagger) + \frac{\mathrm{i}}{2\kappa}(-\frac{1}{1+2\kappa_{\mathrm{b}}^2}b + \frac{1}{2\kappa_{\mathrm{b}}^2}b^\dagger). \tag{D.32}$$

The RLD Fisher information matrix J_{R} is calculated as

$$J_{\mathrm{R}} = \frac{1}{4\kappa^2}\begin{pmatrix} \frac{1}{1+2\kappa_a^2} + \frac{1}{2\kappa_a^2} + \frac{1}{1+2\kappa_b^2} + \frac{1}{2\kappa_b^2} & -\mathrm{i}\,[\frac{1}{2\kappa_a^2(1+2\kappa_a^2)} - \frac{1}{2\kappa_b^2(1+2\kappa_b^2)}] \\ \mathrm{i}\,[\frac{1}{2\kappa_a^2(1+2\kappa_a^2)} - \frac{1}{2\kappa_b^2(1+2\kappa_b^2)}] & \frac{1}{1+2\kappa_a^2} + \frac{1}{2\kappa_a^2} + \frac{1}{1+2\kappa_b^2} + \frac{1}{2\kappa_b^2} \end{pmatrix}. \tag{D.33}$$

Appendix E

Supplemental materials for Chapter 6

E.1 Wigner Rotation

For a massive particle with spin-1/2, we have the relation [31, 32],

$$U(\Lambda)\,|p,\sigma\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'=\downarrow,\uparrow} D^{(\frac{1}{2})}_{\sigma',\sigma}(W(\Lambda,p))\,|\Lambda p,\sigma'\rangle\,, \tag{E.1}$$

where $W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p)$. The Lorentz boost $L(p) = [L^i{}_j(p)]$ is chosen as in [32].

$$L^i{}_j(p) = \delta_{ij} + \frac{(\sqrt{m^2+|\vec{p}\,|^2} - m)p^i p^j}{m|\vec{p}\,|^2}, \tag{E.2}$$

$$L^i{}_0(p) = \frac{p^i}{m}, \tag{E.3}$$

$$L^0{}_0(p) = \frac{\sqrt{m^2+|\vec{p}\,|^2}}{m}. \tag{E.4}$$

A direct calculation for our setting $\vec{p} = (p^1, p^2, 0)$ gives the explicit representation of the matrix $W(\Lambda, p)$ as follows.

$$[W(\Lambda,p)]^0{}_0 = 1, \tag{E.5}$$

$$[W(\Lambda,p)]^1{}_0 = [W(\Lambda,p)]^0{}_1 = 0, \tag{E.6}$$

$$[W(\Lambda,p)]^2{}_0 = [W(\Lambda,p)]^0{}_2 = 0, \tag{E.7}$$

$$[W(\Lambda,p)]^3{}_0 = [W(\Lambda,p)]^0{}_3 = 0, \tag{E.8}$$

$$[W(\Lambda,p)]^1{}_1 = [W(\Lambda,p)]^2{}_2 = \frac{p^0[m(p^1)^2 + p^0(p^2)^2]\sinh^2\chi + |\vec{p}\,|^2[(p^1)^2\cosh\chi + (p^2)^2]}{|\vec{p}\,|^2\left[(p^0)^2\sinh^2\chi + |\vec{p}\,|^2\right]}, \tag{E.9}$$

$$[W(\Lambda,p)]^2{}_1 = [W(\Lambda,p)]^1{}_2 = -\frac{p^1 p^2(\cosh\chi - 1)(p^0 - m)}{|\vec{p}|^2(p^0\cosh\chi + m)}, \tag{E.10}$$

$$[W(\Lambda,p)]^3{}_1 = -[W(\Lambda,p)]^1{}_3 = -\frac{p^1\sinh\chi}{p^0\cosh\chi + m}, \tag{E.11}$$

$$[W(\Lambda,p)]^3{}_2 = -[W(\Lambda,p)]^2{}_3 = -\frac{p^2\sinh\chi}{p^0\cosh\chi + m}, \tag{E.12}$$

$$[W(\Lambda,p)]^3{}_3 = \frac{p^0 + m\cosh\chi}{m + p^0\cosh\chi}. \tag{E.13}$$

A 3×3 real matrix $[R(\Lambda,p)]_{jk}$ defined by the spatial part of $W(\Lambda,p)$ as

$$[R(\Lambda,p)]_{jk} = [W(\Lambda,p)]^j{}_k, \quad (j,k = 1,2,3). \tag{E.14}$$

This is a real rotation matrix acting on the three-dimensional vector space. We next decompose the rotation matrix $R(\Lambda,p)$ with the Euler angles. A straightforward calculation shows that we need only two Euler angles in this case. The matrices $R_2(\alpha)$ and $R_3(\phi)$ that express a rotation by angles α and ϕ around the 2 and 3-axis, respectively [74], i.e.,

$$R(\Lambda,p) = R_3(-\phi)R_2(\alpha)R_3(\phi), \tag{E.15}$$

where

$$R_2(\alpha) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}, \tag{E.16}$$

$$R_3(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{E.17}$$

As we have the Euler rotation representation, Eq. (E.15), we obatain the 2×2 matrix representation of the rotation for the spin-1/2 particle [74], $D^{(\frac{1}{2})}(W(\Lambda,p))$ as

$$D^{(\frac{1}{2})}(W(\Lambda,p)) = \mathrm{e}^{\mathrm{i}\phi\frac{\sigma_3}{2}}\mathrm{e}^{-\mathrm{i}\alpha\frac{\sigma_2}{2}}\mathrm{e}^{-\mathrm{i}\phi\frac{\sigma_3}{2}} \tag{E.18}$$

$$= \begin{pmatrix} \cos\frac{\alpha}{2} & -\mathrm{e}^{\mathrm{i}\phi}\sin\frac{\alpha}{2} \\ \mathrm{e}^{-\mathrm{i}\phi}\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}. \tag{E.19}$$

By substituting the expression of $D^{(\frac{1}{2})}(W(\Lambda,p))$ in (E.1), we obtain Eqs. (6.28), (6.30), (6.31), and (6.32).

E.2 Inner product $\langle\psi^{\Lambda}_{\sigma}(\theta)|\psi^{\Lambda}_{\sigma'}(\theta)\rangle$

E.2.1 Spins in the same direction: $\langle\psi^{\Lambda}_{\sigma}(\theta)|\psi^{\Lambda}_{\sigma}(\theta)\rangle$

From Eq. (6.30), $\langle\psi^{\Lambda}_{\sigma}(\theta)|\psi^{\Lambda}_{\sigma}(\theta)\rangle$ is calculated as

$$\begin{aligned}
\langle\psi^{\Lambda}_{\sigma}(\theta)|\psi^{\Lambda}_{\sigma}(\theta)\rangle &= \int d^3p \int d^3p' \sqrt{\frac{(\Lambda p)^0}{p^0}} F^{*}_{\theta,\sigma}(p^1,p^2)\delta(p^3) \\
&\quad \times \sqrt{\frac{(\Lambda p')^0}{p'^0}} F_{\theta,\sigma}(p'^1,p'^2)\delta(p'^3)\,\langle\overrightarrow{\Lambda p}|\overrightarrow{\Lambda p'}\rangle \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F_{\theta,\sigma}(p^1,p^2)|^2 dp^1 dp^2.
\end{aligned} \tag{E.20}$$

We use the relation [32],

$$\langle\overrightarrow{\Lambda p}|\overrightarrow{\Lambda p'}\rangle = \frac{p^0}{(\Lambda p)^0}\langle\vec{p}|\vec{p}'\rangle = \frac{p^0}{(\Lambda p)^0}\delta(\vec{p}-\vec{p}'). \tag{E.21}$$

As given by Eqs. (6.31), (6.32), (6.35), and (6.36), the expression of $F_{\theta,\sigma}(p^1,p^2)$ $(\sigma =\downarrow,\uparrow)$ is explicitly written as

$$F_{\theta,\downarrow}(p^1,p^2) = \varphi_0(p^1)\varphi_0(p^2)\mathrm{e}^{-\mathrm{i}p^1\theta_1-\mathrm{i}p^2\theta_2}\cos\frac{\alpha(|\vec{p}|)}{2}, \tag{E.22}$$

$$F_{\theta,\uparrow}(p^1,p^2) = -\varphi_0(p^1)\varphi_0(p^2)\mathrm{e}^{-\mathrm{i}p^1\theta_1-\mathrm{i}p^2\theta_2}\mathrm{e}^{\mathrm{i}\phi(p^1,p^2)}\sin\frac{\alpha(|\vec{p}|)}{2}, \tag{E.23}$$

$$|\vec{p}| = \sqrt{(p^1)^2+(p^2)^2}, \tag{E.24}$$

$$\mathrm{e}^{\mathrm{i}\phi(p^1,p^2)} = \frac{p^1}{|\vec{p}|} + \mathrm{i}\frac{p^2}{|\vec{p}|}, \tag{E.25}$$

$$\cos\alpha(|\vec{p}|) = \frac{\sqrt{m^2+|\vec{p}|^2} + m\cosh\chi}{\sqrt{m^2+|\vec{p}|^2}\cosh\chi + m}, \tag{E.26}$$

$$\sin\alpha(|\vec{p}|) = -\frac{|\vec{p}|\sinh\chi}{\sqrt{m^2+|\vec{p}|^2}\cosh\chi + m}. \tag{E.27}$$

Therefore, $\langle\psi^{\Lambda}_{\downarrow}(\theta)|\psi^{\Lambda}_{\downarrow}(\theta)\rangle$ is calculated as follows.

$$\begin{aligned}
\langle\psi^{\Lambda}_{\downarrow}(\theta)|\psi^{\Lambda}_{\downarrow}(\theta)\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F_{\theta,\downarrow}(p^1,p^2)|^2 dp^1dp^2, \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\varphi_0(p^1)\varphi_0(p^2)|^2\cos^2\frac{\alpha(|\vec{p}|)}{2}dp^1dp^2. \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\varphi_0(p^1)\varphi_0(p^2)|^2(1+\cos\alpha(|\vec{p}|))dp^1dp^2. \\
&= \frac{1}{2}(1+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\varphi_0(p^1)\varphi_0(p^2)|^2\cos\alpha(|\vec{p}|))dp^1dp^2).
\end{aligned} \tag{E.28}$$

At the last line, we use

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi_0(p^1)\varphi_0(p^2)|^2 dp^1 dp^2 = 1. \tag{E.29}$$

As for $\langle\psi_\uparrow^\Lambda(\theta)|\psi_\uparrow^\Lambda(\theta)\rangle$,

$$\begin{aligned}
\langle\psi_\uparrow^\Lambda(\theta)|\psi_\uparrow^\Lambda(\theta)\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F_{\theta,\uparrow}(p^1,p^2)|^2 dp^1 dp^2, \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi_0(p^1)\varphi_0(p^2)|^2 \sin^2\frac{\alpha(|\vec{p}\,|)}{2} dp^1 dp^2. \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi_0(p^1)\varphi_0(p^2)|^2 (1-\cos\alpha(|\vec{p}\,|)) dp^1 dp^2. \\
&= \frac{1}{2}(1-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi_0(p^1)\varphi_0(p^2)|^2 \cos\alpha(|\vec{p}\,|) dp^1 dp^2.
\end{aligned} \tag{E.30}$$

We define ξ by

$$\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi_0(p^1)\varphi_0(p^2)|^2 \cos\alpha(|\vec{p}\,|) dp^1 dp^2. \tag{E.31}$$

By converting to the two dimensional polar coordinate and integrating over the angle, we have

$$\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi_0(p^1)\varphi_0(p^2)|^2 \cos\alpha(|\vec{p}\,|) dp^1 dp^2, \tag{E.32}$$

$$= 2\kappa^2\int_0^{\infty} dt\, t\, \mathrm{e}^{-\kappa^2 t^2}\frac{\sqrt{m^2+t^2}\sqrt{1-v^2}+m}{\sqrt{m^2+t^2}+m\sqrt{1-v^2}}. \tag{E.33}$$

Hence, we have

$$\langle\psi_\downarrow^\Lambda(\theta)|\psi_\downarrow^\Lambda(\theta)\rangle = \frac{1}{2}(1+\xi), \tag{E.34}$$

$$\langle\psi_\uparrow^\Lambda(\theta)|\psi_\uparrow^\Lambda(\theta)\rangle = \frac{1}{2}(1-\xi), \tag{E.35}$$

where

$$\xi = 2\kappa^2\int_0^{\infty} dt\, t\, \mathrm{e}^{-\kappa^2 t^2}\frac{\sqrt{m^2+t^2}\sqrt{1-v^2}+m}{\sqrt{m^2+t^2}+m\sqrt{1-v^2}}. \tag{E.36}$$

From the equation above, we see that ξ is a monotonically decreasing function of $\sqrt{1-v^2}$. Therefore, ξ is a monotonically increasing function of v. When $v = 1$, ξ takes its minimum, ξ_{rel} which is evaluated as follows.

$$\xi_{\mathrm{rel}} = 2\kappa^2\int_0^{\infty}\frac{t\,\mathrm{e}^{-\kappa^2 p^2}}{\sqrt{m^2+t^2}}dt = \sqrt{\pi}m\kappa e^{m^2\kappa^2}\mathrm{erfc}(m\kappa). \tag{E.37}$$

The $\mathrm{erfc}(m\kappa)$ is called complementary error function which is defined by

$$\mathrm{erfc}(m\kappa) := 1-\frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}dt. \tag{E.38}$$

Performing the standard gaussian integration, we see $\xi = 1$ when $v = 0$.

E.2.2 Spins in the opposite direction: $\langle\psi^\Lambda_\uparrow(\theta)|\psi^\Lambda_\downarrow(\theta)\rangle$

From Eq. (6.30), $\langle\psi^\Lambda_\uparrow(\theta)|\psi^\Lambda_\downarrow(\theta)\rangle$ is calculated as

$$\begin{aligned}\langle\psi^\Lambda_\uparrow(\theta)|\psi^\Lambda_\downarrow(\theta)\rangle &= \int d^3p \int d^3p' \sqrt{\frac{(\Lambda p)^0}{p^0}} F^*_{\theta,\uparrow}(p^1,p^2)\delta(p^3) \\ &\quad\times \sqrt{\frac{(\Lambda p')^0}{p'^0}} F_{\theta,\downarrow}(p'^1,p'^2)\delta(p'^3)\,\langle\overrightarrow{\Lambda p}|\overrightarrow{\Lambda p'}\rangle \\ &= \int_\infty^\infty \int_\infty^\infty F^*_{\theta,\uparrow}(p^1,p^2)F_{\theta,\downarrow}(p^1,p^2)dp^1dp^2.\end{aligned} \tag{E.39}$$

With Eqs. (6.31, 6.32), we have the explicit expression of the right hand side above as follows.

$$\begin{aligned}\langle\psi^\Lambda_\uparrow(\theta)|\psi^\Lambda_\downarrow(\theta)\rangle &= -\frac{\kappa^2}{\pi}\int_\infty^\infty\int_\infty^\infty \mathrm{e}^{-\kappa^2[(p^1)^2+(p^2)^2]}\mathrm{e}^{\mathrm{i}p^1\theta_1+\mathrm{i}p^2\theta_2}\cos\frac{\alpha(|\vec{p}\,|)}{2}\mathrm{e}^{-\mathrm{i}p^1\theta_1-\mathrm{i}p^2\theta_2}(\frac{p^1}{|\vec{p}\,|}+\mathrm{i}\frac{p^2}{|\vec{p}\,|})\sin\frac{\alpha(|\vec{p}\,|)}{2} \\ &= -\frac{\kappa^2}{2\pi}\int_\infty^\infty\int_\infty^\infty \mathrm{e}^{-\kappa^2|\vec{p}\,|^2}(\frac{p^1}{|\vec{p}\,|}+\mathrm{i}\frac{p^2}{|\vec{p}\,|})\sin\alpha(|\vec{p}\,|)\end{aligned} \tag{E.40}$$

where $|\vec{p}\,| = \sqrt{(p^1)^2+(p^2)^2}$. Since $\mathrm{e}^{-\kappa^2|\vec{p}\,|^2}\frac{p^1}{|\vec{p}\,|}\sin\alpha(|\vec{p}\,|)$ and $\mathrm{e}^{-\kappa^2|\vec{p}\,|^2}\frac{p^2}{|\vec{p}\,|}\sin\alpha(|\vec{p}\,|)$ are odd functions of p^1 and p^2, respectively, integral of these terms from negative infinity to infinity over p^1 and p^2 vanishes. Hence, we obtain

$$\langle\psi^\Lambda_\uparrow(\theta)|\psi^\Lambda_\downarrow(\theta)\rangle = 0. \tag{E.41}$$

E.3 Probability density of a spin-1/2 particle: coordinate representation

We define the wave function of a particle with up spin in coordinate representation $\psi^\Lambda_\uparrow(x)$ by

$$\psi^\Lambda_\uparrow(x) = \langle x|\bar{\psi}^\Lambda_\uparrow(\theta)\rangle\Big|_{\theta=0}. \tag{E.42}$$

From Eqs. (6.32) and (6.43), the wave function $\psi^\Lambda_\uparrow(x)$ is given by

$$\begin{aligned}\psi^\Lambda_\uparrow(x) &= -\sqrt{\frac{2}{1-\xi}}\int d^3p\sqrt{\frac{(\Lambda p)^0}{p^0}}\varphi_0(p^1,p^2)\mathrm{e}^{\mathrm{i}\phi(p^1,p^2)} \\ &\quad\times\sin\frac{\alpha(p)}{2}\delta(p^3)\,\langle x\,|\Lambda p\rangle\,,\end{aligned} \tag{E.43}$$

where $\varphi_0(p^1, p^2) = \varphi_0(p^1)\varphi_0(p^2)$. By a direct calculation, we have the wave function $\psi^\Lambda_\uparrow(x)$ as follows.

$$\begin{aligned}\psi^\Lambda_\uparrow(x) = &-\sqrt{\frac{2}{1-\xi}}\frac{\kappa}{(2\pi)^2}\sqrt{\cosh\chi}\\ &\times\int dp^1dp^2 \mathrm{e}^{-\kappa^2[(p^1)^2+(p^2)^2]+\mathrm{i}\phi(p^1,p^2)}\\ &\times\sin\frac{\alpha(p)}{2}\mathrm{e}^{-\mathrm{i}p^1x^1-\mathrm{i}p^2x^2-\mathrm{i}\sqrt{(p^1)^2+(p^2)^2+m^2}\sinh\chi x^3}.\end{aligned} \tag{E.44}$$

From the expression above, we see the probability density $\psi^\Lambda_\uparrow(x^1, x^2)^*\psi^\Lambda_\uparrow(x^1, x^2)$ has a property

$$\psi^\Lambda_\uparrow(x^1, x^2)^*\psi^\Lambda_\uparrow(x^1, x^2) = \psi^\Lambda_\uparrow(-x^1, x^2)^*\psi^\Lambda_\uparrow(-x^1, x^2), \tag{E.45}$$
$$\psi^\Lambda_\uparrow(x^1, x^2)^*\psi^\Lambda_\uparrow(x^1, x^2) = \psi^\Lambda_\uparrow(x^1, -x^2)^*\psi^\Lambda_\uparrow(x^1, -x^2). \tag{E.46}$$

E.4 SLD and SLD Fisher information matrix

E.4.1 SLD Fisher information matrix

The state we are considering $\rho^\Lambda(\theta)$ is written by

$$\rho^\Lambda(\theta) = \frac{1}{2}(1+\xi)\,|\bar{\psi}^\Lambda_\downarrow(\theta)\rangle\,\langle\bar{\psi}^\Lambda_\downarrow(\theta)| + \frac{1}{2}(1-\xi)\,|\bar{\psi}^\Lambda_\uparrow(\theta)\rangle\,\langle\bar{\psi}^\Lambda_\uparrow(\theta)|\,. \tag{E.47}$$

By using the formula given in Section F.1, the SLD Fisher information matrix J^Λ for the state $\rho^\Lambda(\theta)$ which is non-full rank is calculated as

$$\begin{aligned}J^\Lambda{}_{jk} = &\,2(1+\xi)[\mathrm{Re}\,\langle\partial_j\bar{\psi}^\Lambda_\downarrow(\theta)|\partial_k\bar{\psi}^\Lambda_\downarrow(\theta)\rangle - \langle\partial_j\bar{\psi}^\Lambda_\downarrow(\theta)|\bar{\psi}^\Lambda_\downarrow(\theta)\rangle\,\langle\bar{\psi}^\Lambda_\downarrow(\theta)|\partial_k\bar{\psi}^\Lambda_\downarrow(\theta)\rangle]\\ &+2(1-\xi)[\mathrm{Re}\,\langle\partial_j\bar{\psi}^\Lambda_\uparrow(\theta)|\partial_k\bar{\psi}^\Lambda_\uparrow(\theta)\rangle - \langle\partial_j\bar{\psi}^\Lambda_\uparrow(\theta)|\bar{\psi}^\Lambda_\uparrow(\theta)\rangle\,\langle\bar{\psi}^\Lambda_\uparrow(\theta)|\partial_k\bar{\psi}^\Lambda_\uparrow(\theta)\rangle]\\ &-4(1-\xi)(1+\xi)\mathrm{Re}(\langle\bar{\psi}^\Lambda_\uparrow(\theta)|\partial_j\bar{\psi}^\Lambda_\downarrow(\theta)\rangle^*\,\langle\bar{\psi}^\Lambda_\uparrow(\theta)|\partial_k\bar{\psi}^\Lambda_\downarrow(\theta)\rangle).\end{aligned} \tag{E.48}$$

Regarding the calculation, see for example [59]. The terms below appear in the second and fourth terms of Eq. (E.48) vanish, because their integrands are an odd function of p^j, i.e.,

$$\langle\partial_j\bar{\psi}^\Lambda_\sigma(\theta)|\bar{\psi}^\Lambda_\sigma(\theta)\rangle = 0, \quad (\sigma = \downarrow, \uparrow). \tag{E.49}$$

From Eqs. (6.30), (6.31), and (6.32), the inner products $\langle\partial_j\bar{\psi}^\Lambda_\sigma(\theta)|\partial_j\bar{\psi}^\Lambda_\sigma(\theta)\rangle$, $(j = 1, 2)$ are obtained as follows.

$$F_{\theta,\downarrow}(p^1,p^2) = \varphi_0(p^1)\varphi_0(p^2)\mathrm{e}^{-\mathrm{i}p^1\theta_1-\mathrm{i}p^2\theta_2}\cos\frac{\alpha(|\vec{p}\,|)}{2}, \tag{E.50}$$

$$F_{\theta,\uparrow}(p^1,p^2) = -\varphi_0(p^1)\varphi_0(p^2)\mathrm{e}^{-\mathrm{i}p^1\theta_1-\mathrm{i}p^2\theta_2}\mathrm{e}^{\mathrm{i}\phi(p^1,\,p^2)}\sin\frac{\alpha(|\vec{p}\,|)}{2}, \tag{E.51}$$

$$|\vec{p}\,| = \sqrt{(p^1)^2+(p^2)^2}, \tag{E.52}$$

$$\mathrm{e}^{\mathrm{i}\phi(p^1,\,p^2)} = \frac{p^1}{|\vec{p}\,|} + \mathrm{i}\frac{p^2}{|\vec{p}\,|}, \tag{E.53}$$

$$\cos\alpha(|\vec{p}\,|) = \frac{\sqrt{m^2+|\vec{p}\,|^2}+m\cosh\chi}{\sqrt{m^2+|\vec{p}\,|^2}\cosh\chi+m}, \tag{E.54}$$

$$\sin\alpha(|\vec{p}\,|) = -\frac{|\vec{p}\,|\sinh\chi}{\sqrt{m^2+|\vec{p}\,|^2}\cosh\chi+m}. \tag{E.55}$$

The term $\langle\partial_1\bar{\psi}^{\wedge}_{\downarrow}(\theta)|\partial_1\bar{\psi}^{\wedge}_{\downarrow}(\theta)\rangle$ is calculated as follows.

$$\begin{aligned}
\langle\partial_1\bar{\psi}^{\wedge}_{\downarrow}(\theta)|\partial_1\bar{\psi}^{\wedge}_{\downarrow}(\theta)\rangle &= \frac{2}{1+\xi}\left\langle\partial_1\psi^{\wedge}_{\downarrow}(\theta)|\partial_1\psi^{\wedge}_{\downarrow}(\theta)\right\rangle, \\
&= \frac{2}{1+\xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|(p^1)^2F_{\theta,\downarrow}(p^1,p^2)|^2dp^1dp^2, \\
&= \frac{2}{1+\xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(p^1)^2|\varphi_0(p^1)\varphi_0(p^2)|^2\cos^2\frac{\alpha(|\vec{p}\,|)}{2}dp^1dp^2, \\
&= \frac{1}{1+\xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(p^1)^2|\varphi_0(p^1)\varphi_0(p^2)|^2(1+\cos\alpha(|\vec{p}\,|))dp^1dp^2, \\
&= \frac{1}{1+\xi}(\frac{1}{2\kappa^2}+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(p^1)^2|\varphi_0(p^1)\varphi_0(p^2)|^2\cos\alpha(|\vec{p}\,|))dp^1dp^2).
\end{aligned} \tag{E.56}$$

At the last line we use,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp^1dp^2(p^1)^2[\varphi_0(p^1,p^2)]^2 = \frac{1}{2\kappa^2}. \tag{E.57}$$

We see that $\langle\partial_1\bar{\psi}^{\wedge}_{\downarrow}(\theta)|\partial_1\bar{\psi}^{\wedge}_{\downarrow}(\theta)\rangle = \langle\partial_2\bar{\psi}^{\wedge}_{\downarrow}(\theta)|\partial_2\bar{\psi}^{\wedge}_{\downarrow}(\theta)\rangle$. We define ζ by

$$\zeta := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp^1dp^2(p^1)^2[\varphi_0(p^1,p^2)]^2\cos\alpha(|\vec{p}\,|). \tag{E.58}$$

Then, we have

$$\langle\partial_j\bar{\psi}^{\wedge}_{\downarrow}(\theta)|\partial_j\bar{\psi}^{\wedge}_{\downarrow}(\theta)\rangle = \frac{(2\kappa^2)^{-1}+\zeta}{1+\xi}. \tag{E.59}$$

Therefore, $\langle \partial_1 \bar{\psi}^{\Lambda}_{\uparrow}(\theta) | \partial_1 \bar{\psi}^{\Lambda}_{\uparrow}(\theta) \rangle$ is calculated as follows.

$$
\begin{aligned}
\langle \partial_1 \bar{\psi}^{\Lambda}_{\uparrow}(\theta) | \partial_1 \bar{\psi}^{\Lambda}_{\uparrow}(\theta) \rangle &= \frac{2}{1-\xi} \langle \partial_1 \psi^{\Lambda}_{\uparrow}(\theta) | \partial_2 \psi^{\Lambda}_{\uparrow}(\theta) \rangle , \\
&= \frac{2}{1-\xi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(p^1)^2 F_{\theta,\downarrow}(p^1, p^2)|^2 dp^1 dp^2, \\
&= \frac{2}{1-\xi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (p^1)^2 |\varphi_0(p^1)\varphi_0(p^2)|^2 \sin^2 \frac{\alpha(|\vec{p}\,|)}{2} dp^1 dp^2, \\
&= \frac{1}{1-\xi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (p^1)^2 |\varphi_0(p^1)\varphi_0(p^2)|^2 (1 - \cos\alpha(|\vec{p}\,|)) dp^1 dp^2, \\
&= \frac{1}{1-\xi} (\frac{1}{2\kappa^2} - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (p^1)^2 |\varphi_0(p^1)\varphi_0(p^2)|^2 \cos\alpha(|\vec{p}\,|)) dp^1 dp^2), \\
&= \frac{1}{1-\xi} (\frac{1}{2\kappa^2} - \zeta).
\end{aligned}
\tag{E.60}
$$

$$
\therefore \langle \partial_j \bar{\psi}^{\Lambda}_{\uparrow}(\theta) | \partial_j \bar{\psi}^{\Lambda}_{\uparrow}(\theta) \rangle = \frac{(2\kappa^2)^{-1} - \zeta}{1-\xi}, \tag{E.61}
$$

We also use Eq. (6.3)

$$
\varphi_0(p^j) = \frac{\kappa^{1/2}}{\pi^{1/4}} \, \mathrm{e}^{-\frac{1}{2}\kappa^2 (p^j)^2}, \tag{E.62}
$$

and

$$
\int \int dp^1 dp^2 (p^1)^2 [\varphi_0(p^1, p^2)]^2 = \frac{1}{2\kappa^2}. \tag{E.63}
$$

As for $\langle \partial_j \bar{\psi}^{\Lambda}_{\downarrow}(\theta) | \bar{\psi}^{\Lambda}_{\uparrow}(\theta) \rangle$ $(j = 1, 2)$, a direct calculation gives

$$
\langle \partial_1 \bar{\psi}^{\Lambda}_{\downarrow}(\theta) | \bar{\psi}^{\Lambda}_{\uparrow}(\theta) \rangle = -\frac{\mathrm{i}\eta}{\sqrt{(1+\xi)(1-\xi)}}, \tag{E.64}
$$

$$
\langle \partial_2 \bar{\psi}^{\Lambda}_{\downarrow}(\theta) | \bar{\psi}^{\Lambda}_{\uparrow}(\theta) \rangle = -\frac{\eta}{\sqrt{(1+\xi)(1-\xi)}}, \tag{E.65}
$$

where

$$
\eta = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dp^1 dp^2 \frac{(p^j)^2}{|\vec{p}\,|} [\varphi_0(p^1, p^2)]^2 \sin\alpha(|\vec{p}\,|). \tag{E.66}
$$

The SLD Fisher information matrix J^{Λ} is expressed as follows.

$$
J^{\Lambda} = 2(\kappa^{-2} - 2\eta^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{E.67}
$$

It turns out the ζ has no effect on the SLD Fisher information.

E.4.2 SLD

By using the formula given in Section F.1, we have $L^{\Lambda}(\theta)_j$ ($j = 1, 2$) which are expressed by

$$\begin{aligned} L^{\Lambda}{}_1(\theta) =& \frac{4}{1+\xi}\partial_1(|\bar{\psi}^{\Lambda}_{\downarrow}(\theta)\rangle)\,\langle\bar{\psi}^{\Lambda}_{\downarrow}(\theta)|) \\ &+ \frac{4}{1-\xi}\partial_1(|\bar{\psi}^{\Lambda}_{\uparrow}(\theta)\rangle)\,\langle\bar{\psi}^{\Lambda}_{\uparrow}(\theta)|) \\ &+ 2\mathrm{i}\xi\eta(|\bar{\psi}^{\Lambda}_{\downarrow}(\theta)\rangle\,\langle\bar{\psi}^{\Lambda}_{\uparrow}(\theta)| - |\bar{\psi}^{\Lambda}_{\uparrow}(\theta)\rangle\,\langle\bar{\psi}^{\Lambda}_{\downarrow}(\theta)|), \end{aligned} \tag{E.68}$$

$$\begin{aligned} L^{\Lambda}{}_2(\theta) =& \frac{4}{1+\xi}\partial_2(|\bar{\psi}^{\Lambda}_{\downarrow}(\theta)\rangle)\,\langle\bar{\psi}^{\Lambda}_{\downarrow}(\theta)|) \\ &+ \frac{4}{1-\xi}\partial_2(|\bar{\psi}^{\Lambda}_{\uparrow}(\theta)\rangle)\,\langle\bar{\psi}^{\Lambda}_{\uparrow}(\theta)|) \\ &+ 2\xi\eta(|\bar{\psi}^{\Lambda}_{\downarrow}(\theta)\rangle\,\langle\bar{\psi}^{\Lambda}_{\uparrow}(\theta)| - |\bar{\psi}^{\Lambda}_{\uparrow}(\theta)\rangle\,\langle\bar{\psi}^{\Lambda}_{\downarrow}(\theta)|). \end{aligned} \tag{E.69}$$

By using these, we can show that $L^{\Lambda}{}_1(\theta)$ and $L^{\Lambda}{}_2(\theta)$ do not commute, i.e., $[L^{\Lambda}{}_1(\theta), L^{\Lambda}{}_2(\theta)] \neq 0$.

Furthermore, by a direct calculation, we can evaluate the weak commutativity condition as

$$\mathrm{tr}(\rho^{\Lambda}(\theta)[L^{\Lambda}{}_1(\theta), L^{\Lambda}{}_2(\theta)]) = 8\mathrm{i}\xi\eta^2 \neq 0. \tag{E.70}$$

This shows that the SLD CR bound is not achievable even in the asymptotic setting.

E.5 Maximum and minimum of $\kappa\eta$

From Eq. (6.53), η is expressed as

$$\eta = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp^1 dp^2 \frac{(p^1)^2}{|\vec{p}\,|}[\varphi_0(p^1)\varphi_0(p^2)]^2 \sin\alpha(|\vec{p}\,|). \tag{E.71}$$

where

$$\varphi_0(p^j) = \frac{\kappa^{1/2}}{\pi^{1/4}}\,\mathrm{e}^{-\frac{1}{2}\kappa^2(p^j)^2}, \tag{E.72}$$

$$\sin\alpha(|\vec{p}\,|) = -\frac{|\vec{p}\,|\sinh\chi}{\sqrt{m^2+|\vec{p}\,|^2}\cosh\chi + m} = -\frac{v|\vec{p}\,|}{\sqrt{m^2+|\vec{p}\,|^2} + m\sqrt{1-v^2}} \tag{E.73}$$

Then, $\kappa\eta$ is written as

$$\begin{aligned} \kappa\eta &= \kappa\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp^1 dp^2 \frac{\kappa^2}{\pi}\frac{(p^1)^2}{|\vec{p}\,|}\mathrm{e}^{-\kappa^2[(p^1)^2+(p^2)^2]}\frac{v|\vec{p}\,|}{\sqrt{m^2+|\vec{p}\,|^2} + m\sqrt{1-v^2}} && \text{(E.74)} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp^1 dp^2 \frac{\kappa^3 v}{\pi}\frac{(p^1)^2}{|\vec{p}\,|}\mathrm{e}^{-\kappa^2|\vec{p}\,|^2}\frac{|\vec{p}\,|}{\sqrt{m^2+|\vec{p}\,|^2} + m\sqrt{1-v^2}} && \text{(E.75)} \\ & && \text{(E.76)} \end{aligned}$$

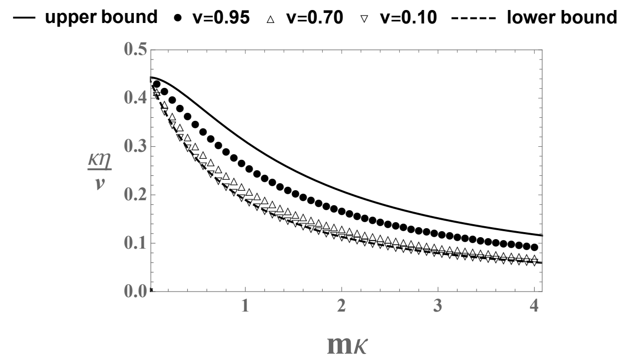

Figure E.1: $\kappa\eta/V$ as a function of $m\kappa$ at $v = 0.95, 0.7$, and 0.1.

We convert to the two dimensional polar coordinate system, i.e.,

$$p^1 = t\cos\theta, p^2 = t\cos\theta. \tag{E.77}$$

Then,

$$\kappa\eta = \int_0^{2\pi} d\theta \int_0^\infty tdt \frac{\kappa^3 v}{\pi} \frac{(t)^2\cos^2\theta}{t} \mathrm{e}^{-\kappa^2 t^2} \frac{t}{\sqrt{m^2+t^2} + m\sqrt{1-v^2}}, \tag{E.78}$$

$$= \frac{\kappa^3 v}{\pi} \int_0^{2\pi} d\theta \cos^2\theta \int_0^\infty dt \frac{t^3 \mathrm{e}^{-\kappa^2 t^2}}{\sqrt{m^2+t^2} + m\sqrt{1-v^2}}, \tag{E.79}$$

$$= v \int_0^\infty dt \frac{\kappa^3 t^3 \mathrm{e}^{-\kappa^2 t^2}}{\sqrt{m^2+t^2} + m\sqrt{1-v^2}}, \tag{E.80}$$

$$= v \int_0^\infty dt \frac{\kappa'^3 t^3 \mathrm{e}^{-\kappa'^2 t^2}}{\sqrt{1+t^2} + \sqrt{1-v^2}}. \tag{E.81}$$

where $\kappa' = m\kappa$. By the velocity dependence of the integrand, we have an upper bound with $v = 1$, and the lower bound with $v = 0$. We obtain the following inequality for $\kappa\eta/v$.

$$\int_0^\infty dt \frac{\kappa'^3 t^3 \mathrm{e}^{-\kappa'^2 t^2}}{\sqrt{1+t^2}+1} \le \frac{\kappa\eta}{v} \le \int_0^\infty dt \frac{\kappa'^3 t^3 \mathrm{e}^{-\kappa'^2 t^2}}{\sqrt{1+t^2}}. \tag{E.82}$$

These integrations are explicitly written as

$$\int_0^\infty dt \frac{\kappa'^3 t^3 \mathrm{e}^{-\kappa'^2 t^2}}{\sqrt{1+t^2}+1} = \frac{\sqrt{\pi}}{4} \mathrm{e}^{\kappa'^2} \mathrm{erfc}(\kappa'), \tag{E.83}$$

$$\int_0^\infty dt \frac{\kappa'^3 t^3 \mathrm{e}^{-\kappa'^2 t^2}}{\sqrt{1+t^2}} = \frac{\kappa'}{2} + \frac{\sqrt{\pi}}{4} \mathrm{e}^{\kappa'^2} (1 - 2\kappa'^2)\, \mathrm{erfc}(\kappa'). \tag{E.84}$$

The right hand sides of Eqs. (E.83), and (E.84) are monotonically decreasing functions of κ', or $m\kappa$. Their maxima at the limit of $\kappa \to 0$ for both are $\sqrt{\pi}V/4$, i.e., $\kappa\eta < \sqrt{\pi}V/4$ for any $\kappa > 0$. Figure E.1 shows numerically calculated $|\kappa\eta|/V$ together with the upper and lower bounds.

By using the asymptotic expansion of the complimentary error function erfc(x),

$$\mathrm{erfc}(x) = \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}x}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{2^n x^{2n}}, \tag{E.85}$$

for $\kappa' \gg 1$, we have

$$\int_0^{\infty} dt \frac{\kappa'^3 t^3 \mathrm{e}^{-\kappa'^2 t^2}}{\sqrt{1+t^2}+1} \simeq \frac{1}{4\kappa'}, \tag{E.86}$$

$$\int_0^{\infty} dt \frac{\kappa'^3 t^3 \mathrm{e}^{-\kappa'^2 t^2}}{\sqrt{1+t^2}} \simeq \frac{1}{2\kappa'}. \tag{E.87}$$

Therefore, when $\kappa = m\kappa \gg 1$, $\kappa\eta$ decreases as $(m\kappa)^{-1}$ with increasing $m\kappa$. Then, $\Delta(v)$ is approximately expressed as

$$1 + \frac{v^2}{8\kappa'^2} \leq \Delta(v) \leq 1 + \frac{v^2}{2\kappa'^2}. \tag{E.88}$$

For $\kappa' \ll 1$, by the Taylor expansion, we have

$$\frac{1}{1-\frac{\pi v^2}{8}}\left(1 - \frac{\pi}{4}\frac{V^4\kappa'^2}{1-\frac{\pi v^2}{8}}\right) \leq \Delta(v) \leq \frac{1}{1-\frac{\pi v^2}{8}}\left(1 - \frac{\sqrt{\pi}}{2}\frac{v^2\kappa'}{1-\frac{\pi v^2}{8}}\right). \tag{E.89}$$

Appendix F

Supplemental materials for Chapter 7

F.1 λLD for an n-parameter non-full model

Given a n-parameter model which is not full-rank:

$$\mathcal{M} = \{\rho_\theta | \theta = (\theta_1, \theta_2, \cdots, \theta_n) \in \Theta\}, \tag{F.1}$$

where rank $\rho_\theta = r \geq d = \dim \mathcal{H}$ for all $\theta \in \Theta$. We drive λLD Fisher information matrix. Here is the index convention.

- $\alpha, \beta, \gamma, \cdots$ for $\{1, 2, \cdots, d\}$: All indices
- $i, j, k, \cdots$ for $\{1, 2, \cdots, r\}$: Support of ρ
- $a, b, c, \cdots$ for $\{r+1, \cdots, d\}$: Kernel of ρ

In the following, we drop θ. A state at θ is decomposed by an orthogonal normalized basis as

$$\rho = \sum_{i=1}^{r} \rho_i \left| e_i \right\rangle \left\langle e_i \right| . \tag{F.2}$$

If we use $\rho_\alpha = 0$ for the kernel space of ρ, we can also write

$$\rho = \sum_{\alpha=1}^{d} \rho_\alpha \left| e_\alpha \right\rangle \left\langle e_\alpha \right| . \tag{F.3}$$

The λLD is defined by a solution to

$$\partial_j \rho = \frac{1+\lambda}{2} \rho L_j + \frac{1-\lambda}{2} L_j \rho, \tag{F.4}$$

where

$$\partial_j = \frac{\partial}{\partial \theta_j}. \tag{F.5}$$

We expand L in the $\left| e_\alpha \right\rangle$ basis as

$$L = \sum_{\alpha,\beta=1}^{d} l_{\alpha,\beta} \left| e_\alpha \right\rangle \left\langle e_\beta \right| . \tag{F.6}$$

Here we omit the subscription of the direction of the theta because we fix the direction. The coefficients $l_{\alpha,\beta}$ are determined by

$$\langle e_\alpha|\partial\rho|e_\beta\rangle = \left[\frac{1+\lambda}{2}\rho_\alpha + \frac{1-\lambda}{2}\rho_\beta\right] l_{\alpha,\beta} \tag{F.7}$$

This equation determines $l_{\alpha,\beta}$ for $\alpha,\beta \notin \{r+1,\cdots,d\}$ only.

$$l_{\alpha,\beta} = \begin{cases} \frac{1+\lambda}{2}\rho_\alpha + \frac{1-\lambda}{2}\rho_\beta & \text{for } \alpha,\beta \notin \{r+1,\cdots,d\}, \\ \text{indetermined} & \text{otherwise.} \end{cases} \tag{F.8}$$

For convenience, we denote $\lambda_i^\pm = \frac{1\pm\lambda}{2}\rho_i$. Therefore,

$$\lambda_{i,a} = \lambda_i^+, \tag{F.9}$$

$$\lambda_{a,i} = \lambda_i^-. \tag{F.10}$$

The λLD is obtained as

$$L = \sum\nolimits'_{\alpha,\beta} l_{\alpha,\beta} \left|e_\alpha\right\rangle \left\langle e_\alpha|\partial\rho|e_\beta\right\rangle \left\langle e_\beta\right|, \tag{F.11}$$

where the prime indicates summing over $\alpha,\beta \notin \{r+1,\cdots,d\}$. By using the projectors $P_i = |e_i\rangle\langle e_i|$, we can express

$$L = \sum_{i=1}^{r}(\lambda_i^+)^{-1}P_i\partial\rho + \sum_{i=1}^{r}(\lambda_j^-)^{-1}\partial\rho P_j + \sum_{i,j=1}\left[(\lambda_{i,j})^{-1} - (\lambda_i^+)^{-1} - (\lambda_j^-)^{-1}\right]P_i\partial\rho P_j \tag{F.12}$$

By substituting Eq. (F.2), we obtain an alternative expression.

$$\begin{aligned} L = \sum_{i=1}^{r}\frac{\partial\rho_i}{\rho_i} + \sum_{i=1}^{r}(\lambda_i^+)^{-1}\left|e_i\right\rangle\left\langle\partial e_i\right| + \sum_{i=1}^{r}(\lambda_i^-)^{-1}\left|\partial e_i\right\rangle\left\langle e_i\right| + \sum_{i,j=1}\left[(\lambda_{i,j})^{-1} - (\lambda_i^+)^{-1}\right]\rho_i\left\langle\partial e_i|e_j\right\rangle\left|e_i\right\rangle\left\langle e_j\right| \\ + \sum_{i,j=1}\left[(\lambda_{i,j})^{-1} - (\lambda_i^-)^{-1}\right]\rho_j\left\langle e_i|\partial e_j\right\rangle\left|e_i\right\rangle\left\langle e_j\right| \end{aligned} \tag{F.13}$$

The λLD Fisher information matrix J_λ is calculated by

$$J_\lambda = \mathrm{tr}(\partial\rho L') \tag{F.14}$$

where L' denotes the λLD with respect to ∂'. To get the usual expression, we need to replace ∂ and ∂' by ∂_m and ∂_n, respectively. Therefore, the $(m,\ n)$ component of J_λ is expressed as

$$J_\lambda = \mathrm{tr}(\partial\rho L') \Longleftrightarrow [J_\lambda]_{mn} = \mathrm{tr}(\partial_m\rho L_n). \tag{F.15}$$

The final expression for J_λ is

$$J_\lambda = \sum_{i=1}^{r} (\lambda_i^+)^{-1} \langle e_i | \partial' \rho \partial \rho | e_i \rangle + \sum_{i=1}^{r} (\lambda_i^-)^{-1} \langle e_i | \partial \rho \partial' \rho | e_i \rangle$$
$$+ \sum_{i,j=1} \left[(\lambda_{i,j})^{-1} - (\lambda_i^+)^{-1} - (\lambda_j^-)^{-1} \right] \langle e_i | \partial' \rho | e_j \rangle \langle e_j | \partial \rho | e_i \rangle . \tag{F.16}$$

With further calculation, we have

$$J_\lambda = \sum_{i=1}^{r} \frac{\partial' \rho_i \partial \rho_i}{\rho_i} + \sum_{i=1}^{r} (\lambda_i^+)^{-1} (\rho_i)^2 \langle \partial' e_i | \partial e_i \rangle + \sum_{i=1}^{r} (\lambda_i^-)^{-1} (\rho_i)^2 \langle \partial e_i | \partial' e_i \rangle$$
$$+ \sum_{i,j=1} \left[(\lambda_{i,j})^{-1} (\rho_i - \rho_j)^2 - (\lambda_i^+)^{-1} (\rho_i)^2 - (\lambda_j^-)^{-1} (\rho_j)^2 \right] \langle e_i | \partial' e_j \rangle \langle e_j | \partial e_i \rangle . \tag{F.17}$$

Taking the SLD limit ($\lambda = 0$) gives $\lambda_{i,j} = (\rho_i + \rho_j)/2$ and $\lambda_i^\pm = \rho_i$. These lead to the following expression of the SLD Fisher information matrix J_S.

$$J_\mathrm{S} = \sum_{i=1}^{r} \frac{\partial' \rho_i \partial \rho_i}{\rho_i} + 2 \sum_{i=1}^{r} \rho_i \langle \partial' e_i | \partial e_i \rangle + 2 \sum_{i=1}^{r} \rho_i \langle \partial e_i | \partial' e_i \rangle + 2 \sum_{i,j=1} \left[\frac{(\rho_i - \rho_j)^2}{\rho_1 + \rho_j} - \rho_i - \rho_j \right] \langle e_i | \partial' e_j \rangle \langle e_j | \partial e_i \rangle , \tag{F.18}$$

$$J_\mathrm{S} = \sum_{i=1}^{r} \frac{\partial' \rho_i \partial \rho_i}{\rho_i} + 2 \sum_{i=1}^{r} \rho_i \langle \partial' e_i | \partial e_i \rangle + 2 \sum_{i=1}^{r} \rho_i \langle \partial e_i | \partial' e_i \rangle - 8 \sum_{i,j=1} \frac{\rho_i \rho_j}{\rho_1 + \rho_j} \langle e_i | \partial' e_j \rangle \langle e_j | \partial e_i \rangle . \tag{F.19}$$

F.2 Evaluation of λLD Fisher information matrix J_λ

As given by Eq. (6.48), the reference state we are considering is $\rho^\Lambda(\theta)$ which is written as

$$\rho^\Lambda(\theta) = \frac{1}{2}(1+\xi) |\bar{\psi}_\downarrow^\Lambda(\theta)\rangle \langle \bar{\psi}_\downarrow^\Lambda(\theta)| + \frac{1}{2}(1-\xi) |\bar{\psi}_\uparrow^\Lambda(\theta)\rangle \langle \bar{\psi}_\uparrow^\Lambda(\theta)| . \tag{F.20}$$

The state vectors $|\bar{\psi}_\downarrow^\Lambda(\theta)\rangle$ and $|\bar{\psi}_\uparrow^\Lambda(\theta)\rangle$ are orthogonal. This is two rank model. The ρ_1 and ρ_2 in Eq. (F.17) are written as

$$\rho_1 = \frac{1}{2}(1+\xi), \quad \rho_2 = \frac{1}{2}(1+\xi). \tag{F.21}$$

$|e_1\rangle$ and $|e_2\rangle$ are written as

$$|e_1\rangle = |\bar{\psi}_\downarrow^\Lambda(\theta)\rangle , \quad |e_2\rangle = |\bar{\psi}_\uparrow^\Lambda(\theta)\rangle . \tag{F.22}$$

F.2.1 First term in $J_{\lambda, mn}$

The first term is as follows.

$$\sum_i (\lambda_i^+)^{-1}(\rho_i)^2 \langle \partial_n e_i | \partial_m e_i \rangle = \frac{(\rho_1)^2}{\lambda_1^+} \langle \partial_n e_1 | \partial_m e_1 \rangle + \frac{(\rho_2)^2}{\lambda_2^+} \langle \partial_n e_2 | \partial_m e_2 \rangle$$
$$= \frac{2}{1+\lambda}(\rho_1 \langle \partial_n e_1 | \partial_m e_1 \rangle + \rho_2 \langle \partial_n e_2 | \partial_m e_2 \rangle). \tag{F.23}$$

We use $\lambda_i^{\pm} = \dfrac{1 \pm \lambda}{2}\rho_i$. In the following, we evaluate the terms that appear in the equation above.

Proof of $\langle \partial_1 e_1 | \partial_2 e_1 \rangle = 0$

The term $\langle \partial_1 e_1 | \partial_2 e_1 \rangle = 0$ is explicitly written as

$$\langle \partial_1 e_1 | \partial_2 e_1 \rangle = \frac{1}{\rho_1} \int d^3p \int d^3p' (\mathrm{i}p^1) \sqrt{\frac{(\Lambda p)^0}{p^0}} F_{\theta,\downarrow}(p^1, p^2)\delta(p^3) \langle \overrightarrow{\Lambda p} \, | \overrightarrow{\Lambda p'} \rangle \tag{F.24}$$
$$\times (-\mathrm{i}p'^2) \sqrt{\frac{(\Lambda p')^0}{p'^0}} F_{\theta,\downarrow}(p'^1, p'^2)\delta(p'^3) \tag{F.25}$$
$$= \frac{1}{\rho_1} \int dp^1 \int dp^2 (p^1)(p^2) |F_{\theta,\downarrow}(p^1, p^2)|^2 \tag{F.26}$$
$$= \frac{1}{\rho_1} \int dp^1 \int dp^2 (p^1)(p^2) |\varphi_0(p^1, p^2)|^2 \cos^2 \frac{\alpha(|\vec{p}|)}{2} \tag{F.27}$$
$$= \frac{1}{2\rho_1} \int dp^1 \int dp^2 (p^1)(p^2) |\varphi_0(p^1, p^2)|^2 (1 + \cos\alpha(|\vec{p}|)). \tag{F.28}$$

From the first line to the second line, we use

$$\langle \overrightarrow{\Lambda p} \, | \overrightarrow{\Lambda p'} \rangle = \sqrt{\frac{p^0}{(\Lambda p)^0}} \, \delta(\vec{p} - \vec{p}'). \tag{F.29}$$

$\varphi_0(p^1, p^2)$ is a gaussian function of p^1 and p^2. Therefore, it is a function of $|\vec{p}| = \sqrt{(p^1)^2 + (p^2)^2}$. We execute the integration in two dimensional polar coordinate.

$$\langle \partial_1 e_1 | \partial_2 e_1 \rangle = \frac{1}{2\rho_1} \int \int d|\vec{p}| \, |\varphi_0(|\vec{p}|)|^2 |\vec{p}|^3 [1 + \cos\alpha(|\vec{p}|)] \int_0^{2\pi} d\theta \sin\theta \cos\theta. \tag{F.30}$$

The integration with respect to θ vanishes. Therefore, we have

$$\langle \partial_1 e_1 | \partial_2 e_1 \rangle = 0. \tag{F.31}$$

We can show $\langle \partial_1 e_2 | \partial_2 e_2 \rangle = 0$ in the same way.

Calculation of $\langle \partial_i e_1 | \partial_i e_1 \rangle$, $(i = 1, 2)$

When $i = 1$, we have

$$\langle \partial_1 e_1 | \partial_1 e_1 \rangle = \frac{1}{\rho_1} \int d^3 p \int d^3 p' (\mathrm{i}p^1) \sqrt{\frac{(\Lambda p)^0}{p^0}} F_{\theta,\downarrow}(p^1, p^2) \delta(p^3) \langle \overrightarrow{\Lambda p} \, | \overrightarrow{\Lambda p'} \rangle \tag{F.32}$$

$$\times (\mathrm{i}p'^1) \sqrt{\frac{(\Lambda p')^0}{p'^0}} F_{\theta,\downarrow}(p'^1, p'^2) \delta(p'^3) \tag{F.33}$$

$$= \frac{1}{\rho_1} \int_{-\infty}^{\infty} dp^1 \int_{-\infty}^{\infty} dp^2 (p^1)^2 |F_{\theta,\downarrow}(p^1, p^2)|^2 \tag{F.34}$$

$$= \frac{1}{\rho_1} \int_{-\infty}^{\infty} dp^1 \int_{-\infty}^{\infty} (p^1)^2 |\varphi_0(p^1, p^2)|^2 \cos^2 \frac{\alpha(|\vec{p}|)}{2} \tag{F.35}$$

$$= \frac{1}{2\rho_1} \int_{-\infty}^{\infty} dp^1 \int_{-\infty}^{\infty} (p^1)^2 |\varphi_0(p^1, p^2)|^2 (1 + \cos\alpha(|\vec{p}|)). \tag{F.36}$$

We defined ζ by

$$\zeta = \int_{-\infty}^{\infty} dp^1 \int_{-\infty}^{\infty} (p^1)^2 |\varphi_0(p^1, p^2)|^2 \cos\alpha(|\vec{p}|). \tag{F.37}$$

By using the explicit expression of $\varphi_0(p^1, p^2)$, we have

$$\int_{-\infty}^{\infty} dp^1 \int_{-\infty}^{\infty} (p^1)^2 |\varphi_0(p^1, p^2)|^2 = \int_{-\infty}^{\infty} dp^1 \int_{-\infty}^{\infty} (p^1)^2 \frac{\kappa}{\sqrt{\pi}} \mathrm{e}^{-\kappa^2[(p^1)^2+(p^2)^2]} = \frac{1}{2\pi\kappa^2}. \tag{F.38}$$

By substituting Eqs. (F.38, F.37) in Eq. (F.36), we obtain

$$\langle \partial_1 e_1 | \partial_1 e_1 \rangle = \frac{1}{\rho_1} (\frac{1}{2\kappa^2} + \zeta). \tag{F.39}$$

When $i = 2$, we can calculate in the same way and have

$$\langle \partial_2 e_1 | \partial_2 e_1 \rangle = \frac{1}{2\rho_2} \int_0^{\infty} dp (p^1)^2 |\varphi_0(p^1, p^2)|^2 [1 - \cos\alpha(|\vec{p}|)] \int_0^{2\pi} d\theta \cos^2\theta \tag{F.40}$$

$$= \frac{\pi}{2\rho_2} \int_0^{\infty} dp (p^1)^2 |\varphi_0(p^1, p^2)|^2 [1 - \cos\alpha(|\vec{p}|)]. \tag{F.41}$$

Then, we obtain

$$\langle \partial_2 e_1 | \partial_2 e_1 \rangle = \frac{1}{\rho_2} (\frac{1}{2\kappa^2} - \zeta) \tag{F.42}$$

We finally obtain the first term as follows.

$$\sum_i (\lambda_i^+)^{-1} (\rho_i)^2 \langle \partial_n e_i | \partial_m e_i \rangle = \frac{2}{1+\lambda} (\rho_1 \langle \partial_n e_1 | \partial_m e_1 \rangle + \rho_2 \langle \partial_n e_2 | \partial_m e_2 \rangle) = \frac{1}{(1+\lambda)\kappa^2}. \tag{F.43}$$

F.2.2 Second term in $J_{\lambda,mn}$

The second term is as follows.

$$\sum_i (\lambda_i^-)^{-1}(\rho_i)^2 \langle \partial_n e_i | \partial_m e_i \rangle = \frac{(\rho_1)^2}{\lambda_1^-} \langle \partial_n e_1 | \partial_m e_1 \rangle + \frac{(\rho_2)^2}{\lambda_2^-} \langle \partial_n e_2 | \partial_m e_2 \rangle$$
$$= \frac{2}{1-\lambda}(\rho_1 \langle \partial_n e_1 | \partial_m e_1 \rangle + \rho_2 \langle \partial_n e_2 | \partial_m e_2 \rangle). \tag{F.44}$$

Therefore, we have

$$\sum_i (\lambda_i^-)^{-1}(\rho_i)^2 \langle \partial_n e_i | \partial_m e_i \rangle = \frac{2}{1-\lambda}(\rho_1 \langle \partial_n e_1 | \partial_m e_1 \rangle + \rho_2 \langle \partial_n e_2 | \partial_m e_2 \rangle) = \frac{1}{(1-\lambda)\kappa^2}. \tag{F.45}$$

F.2.3 Third term in $[J_\lambda]_{mn}$

The third term is as follows.

$$\sum_{i,j} [(\lambda_{i,j})^{-1}(\rho_i - \rho_m)^2 - \lambda_i^+(\rho_i)^2 - \lambda_i^-(\rho_i)^2] \langle e_i | \partial_n e_j \rangle \langle \partial_m e_j | e_i \rangle . \tag{F.46}$$

Since $\langle e_i | \partial_n e_i \rangle = 0$ holds, there is no contribution from the terms with $i = j$. We need to calculate the case of $i = 1, j = 2$ and $i = 2, j = 1$ only.

$i = 1, j = 2$:

$$[\frac{1}{\lambda_{1,2}}(\rho_1 - \rho_2)^2 - \frac{2\rho_1}{1+\lambda} - \frac{2\rho_2}{1-\lambda}] \langle e_1 | \partial_n e_2 \rangle \langle \partial_m e_2 | e_1 \rangle . \tag{F.47}$$

By the definition of $\lambda_{i,j}$, we have $\lambda_{1,2} = \frac{1+\lambda}{2}\rho_1 + \frac{1-\lambda}{2}\rho_2$. Then, the third term can be written as

$$[\frac{2(\rho_1 - \rho_2)^2}{(1+\lambda)\rho_1 + (1-\lambda)\rho_2} - \frac{2\rho_1}{1+\lambda} - \frac{2\rho_2}{1-\lambda}] \langle e_1 | \partial_n e_2 \rangle \langle \partial_m e_2 | e_1 \rangle \tag{F.48}$$
$$= -\frac{8\rho_1\rho_2}{(1-\lambda)(1+\lambda)[\rho_1 + \rho_2 + \lambda(\rho_1 - \rho_2)]} \langle e_1 | \partial_n e_2 \rangle \langle \partial_m e_2 | e_1 \rangle \tag{F.49}$$
$$= -\frac{8\rho_1\rho_2}{(1-\lambda^2)[1 + \lambda(\rho_1 - \rho_2)]} \langle e_1 | \partial_n e_2 \rangle \langle \partial_m e_2 | e_1 \rangle \tag{F.50}$$

At he last line, we use $\rho_1 + \rho_2 = 1$. Furthermore, by the definition of ξ is expressed by

$$\rho_1 = \frac{1}{2}(1+\xi), \tag{F.51}$$
$$\rho_2 = \frac{1}{2}(1-\xi). \tag{F.52}$$

Then, we have $\rho_1 - \rho_2 = \xi$. The term for $i = 1, j = 2$ is expressed as

$$[\frac{2(\rho_1 - \rho_2)^2}{(1+\lambda)\rho_1 + (1-\lambda)\rho_2} - \frac{2\rho_1}{1+\lambda} - \frac{2\rho_2}{1-\lambda}] \langle e_1 | \partial_n e_2 \rangle \langle \partial_m e_2 | e_1 \rangle = -\frac{8\rho_1\rho_2}{(1-\lambda^2)(1+\lambda\xi)} \langle e_1 | \partial_n e_2 \rangle \langle \partial_m e_2 | e_1 \rangle \tag{F.53}$$

$i = 2, j = 1$

The term for $i = 2, j = 1$: is

$$[\frac{1}{\lambda_{2,1}}(\rho_2 - \rho_1)^2 - \frac{2\rho_2}{1+\lambda} - \frac{2\rho_1}{1-\lambda}] \langle e_2|\partial_n e_1\rangle \langle \partial_m e_1|e_2\rangle \,. \tag{F.54}$$

From $\lambda_{2,1} = \frac{1+\lambda}{2}\rho_2 + \frac{1-\lambda}{2}\rho_1$, the third term can be written as

$$[\frac{2(\rho_1-\rho_2)^2}{(1+\lambda)\rho_2 + (1-\lambda)\rho_1} - \frac{2\rho_2}{1+\lambda} - \frac{2\rho_1}{1-\lambda}] \langle e_2|\partial_n e_1\rangle \langle \partial_m e_1|e_2\rangle \tag{F.55}$$

$$= -\frac{8\rho_1\rho_2}{(1-\lambda^2)(1-\lambda\xi)} \langle e_2|\partial_n e_1\rangle \langle \partial_m e_1|e_2\rangle \,. \tag{F.56}$$

We use $\rho_1 + \rho_2 = 1$ and $\rho_1 - \rho_2 = \xi$ here also.

F.2.4 Evaluation of $J_{\lambda,\,11}$

The contribution from the third term to $[J_\lambda]_{11}$ is obtained by substituting $m,\ n = 1$ in Eqs. (F.2.5, F.56).

$$-\frac{8\rho_1\rho_2}{(1-\lambda^2)}(\frac{\langle e_1|\partial_1 e_2\rangle \langle \partial_1 e_2|e_1\rangle}{1+\lambda^2\xi^2} + \frac{\langle e_2|\partial_1 e_1\rangle \langle \partial_1 e_1|e_2\rangle}{1-\lambda^2\xi^2}) = -\frac{16\rho_1\rho_2}{(1-\lambda^2)(1-\lambda^2\xi^2)} |\langle e_1|\partial_1 e_2\rangle|^2 \tag{F.57}$$

We use $\langle e_1|\partial_1 e_2\rangle = -\langle \partial_1 e_1|e_2\rangle$. We also have

$$\langle \partial_1 e_1|e_2\rangle = -\frac{\mathrm{i}\eta}{2\sqrt{\rho_1\rho_2}}, \tag{F.58}$$

$$\langle \partial_2 e_1|e_2\rangle = \frac{\eta}{2\sqrt{\rho_1\rho_2}}. \tag{F.59}$$

A detailed explanation about their derivation is at the end of this section. The contribution from the third term is, then

$$-\frac{4\eta^2}{(1-\lambda^2)(1-\lambda^2\xi^2)}. \tag{F.60}$$

By adding the contributions of the first and the second terms, we obtain $[J_\lambda]_{11}$ as

$$[J_\lambda]_{11} = \frac{1}{(1+\lambda)\kappa^2} + \frac{1}{(1-\lambda)\kappa^2} - \frac{4\eta^2}{(1-\lambda^2)(1-\lambda^2\xi^2)} \tag{F.61}$$

$$= \frac{2}{(1-\lambda^2)\kappa^2} - \frac{4\eta^2}{(1-\lambda^2)(1-\lambda^2\xi^2)} \tag{F.62}$$

$$= \frac{2}{\kappa^2}\frac{1-2\kappa^2\eta^2-\lambda^2\xi^2}{(1-\lambda^2)(1-\lambda^2\xi^2)}. \tag{F.63}$$

Calculation of $\langle\partial_1 e_1|e_2\rangle$

$$\langle\partial_1 e_1|e_2\rangle = \frac{1}{\sqrt{\rho_1\rho_2}}\int d^3p\int d^3p'(\mathrm{i}p^1)\sqrt{\frac{(\Lambda p)^0}{p^0}}F_{\theta,\downarrow}(p^1,p^2)\delta(p^3)\,\langle\overrightarrow{\Lambda p}\,|\overrightarrow{\Lambda p'}\rangle \tag{F.64}$$

$$\times\sqrt{\frac{(\Lambda p')^0}{p'^0}}F_{\theta,\uparrow}(p'^1,p'^2)\delta(p'^3) \tag{F.65}$$

$$= \frac{1}{\sqrt{\rho_1\rho_2}}\int_{-\infty}^{\infty}dp^1\int_{-\infty}^{\infty}dp^2(\mathrm{i}p^1)F^*_{\theta,\downarrow}(p^1,p^2)F_{\theta,\uparrow}(p^1,p^2) \tag{F.66}$$

$$= \frac{1}{\sqrt{\rho_1\rho_2}}\int_{-\infty}^{\infty}dp^1\int_{-\infty}^{\infty}dp^2(\mathrm{i}p^1)\left(\frac{p^1}{|\vec{p}|}+\mathrm{i}\frac{p^2}{|\vec{p}|}\right)|\varphi_0(p^1,p^2)|^2\cos\frac{\alpha(|\vec{p}|)}{2}\sin\frac{\alpha(|\vec{p}|)}{2} \tag{F.67}$$

$$= \frac{1}{2\sqrt{\rho_1\rho_2}}\int_{-\infty}^{\infty}dp^1\int_{-\infty}^{\infty}dp^2\frac{\mathrm{i}(p^1)^2}{|\vec{p}|}|\varphi_0(p^1,p^2)|^2\sin\alpha(|\vec{p}|). \tag{F.68}$$

We define η by

$$\eta = -\int_{-\infty}^{\infty}dp^1\int_{-\infty}^{\infty}dp^2\frac{(p^1)^2}{|\vec{p}|}|\varphi_0(p^1,p^2)|^2\sin\alpha(|\vec{p}|). \tag{F.69}$$

By using η, we can write

$$\langle\partial_1 e_1|e_2\rangle = -\frac{\mathrm{i}\eta}{2\sqrt{\rho_1\rho_2}}. \tag{F.70}$$

Calculation of $\langle\partial_2 e_1|e_2\rangle$

We can calculate $\langle\partial_2 e_1|e_2\rangle$ in the same way we use above.

$$\langle\partial_2 e_1|e_2\rangle = \frac{1}{\sqrt{\rho_1\rho_2}}\int_{-\infty}^{\infty}dp^1\int_{-\infty}^{\infty}dp^2(\mathrm{i}p^2)\left(\frac{p^1}{|\vec{p}|}+\mathrm{i}\frac{p^2}{|\vec{p}|}\right)|\varphi_0(p^1,p^2)|^2\cos\frac{\alpha(|\vec{p}|)}{2}\sin\frac{\alpha(|\vec{p}|)}{2} \tag{F.71}$$

$$= -\frac{1}{2\sqrt{\rho_1\rho_2}}\int_{-\infty}^{\infty}dp^1\int_{-\infty}^{\infty}dp^2\frac{(p^2)^2}{|\vec{p}|}|\varphi_0(p^1,p^2)|^2\sin\alpha(|\vec{p}|) \tag{F.72}$$

$$= \frac{\eta}{2\sqrt{\rho_1\rho_2}}. \tag{F.73}$$

F.2.5 Evaluation of $J_{\lambda,12}$

Since $\langle\partial_n e_i|\partial_m e_i\rangle$ vanishes, there is no contribution from the first and the second term. The contribution from the third term is a sum of Eqs. (F.2.5, F.56). From Eq. ,

$$[J_\lambda]_{12} = -\frac{8\rho_1\rho_2}{(1-\lambda^2)(1+\lambda\xi)}\langle e_1|\partial_2 e_2\rangle\langle\partial_1 e_2|e_1\rangle - \frac{8\rho_1\rho_2}{(1-\lambda^2)(1-\lambda\xi)}\langle e_2|\partial_2 e_1\rangle\langle\partial_1 e_1|e_2\rangle\,. \tag{F.74}$$

By using,

$$\langle \partial_1 e_1 | e_2 \rangle = -\frac{\mathrm{i}\eta}{2\sqrt{\rho_1 \rho_2}}, \tag{F.75}$$

$$\langle \partial_2 e_1 | e_2 \rangle = \frac{\eta}{2\sqrt{\rho_1 \rho_2}}. \tag{F.76}$$

we obtain

$$[J_\lambda]_{12} = \frac{2\mathrm{i}}{(1-\lambda^2)(1+\lambda\xi)} - \frac{2\mathrm{i}}{(1-\lambda^2)(1-\lambda\xi)} = -\frac{4\mathrm{i}\eta^2\lambda\xi}{(1-\lambda^2)(1-\lambda^2\xi^2)}. \tag{F.77}$$

The λLD information matrix J_λ is written as follows.

$$J_\lambda = \frac{2}{\kappa^2}\frac{1}{(1-\lambda^2)(1-\lambda^2\xi^2)}\begin{pmatrix} 1-2\kappa^2\eta^2-\lambda^2\xi^2 & -2\mathrm{i}\kappa^2\eta^2\lambda\xi \\ 2\mathrm{i}\kappa^2\eta^2\lambda\xi & 1-2\kappa^2\eta^2-\lambda^2\xi^2 \end{pmatrix}. \tag{F.78}$$

Its inverse Its inverse J_λ^{-1} is given by

$$J_\lambda^{-1} = \frac{\kappa^2}{2}\frac{1-\lambda^2}{(1-2\kappa^2\eta^2)^2-\lambda^2\xi^2}\begin{pmatrix} 1-2\kappa^2\eta^2-\lambda^2\xi^2 & 2\mathrm{i}\kappa^2\eta^2\lambda\xi \\ -2\mathrm{i}\kappa^2\eta^2\lambda\xi & 1-2\kappa^2\eta^2-\lambda^2\xi^2 \end{pmatrix}. \tag{F.79}$$

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